ON A CONJECTURE OF LUSIN

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1. INTRODUCTION

A point $e^{i\theta}$ will be called a Lusin point of the function f(z) provided f is holomorphic in $|z| < 1$ and maps every disc $|z - te^{i\theta}| < 1 - t$ $(0 < t < 1)$ upon a region (possibly many-sheeted) of infinite area. With this terminology, a conjecture of Lusin [2] may be stated as follows: *There exists a bounded function for which every point of* $|z| = 1$ *is a Lusin point.* Recently, Kufarev and Semukhina [1] have shown that the set of Lusin points of a bounded function can be everywhere dense on $|z| = 1$. In Section 2, we shall prove that there exists a function which is continuous in $|z| \leq 1$ and for which every point $e^{i\theta}$ of $|z| = 1$ is a Lusin point. Our method consists in proving that the function

$$\Sigma a_k z^{n_k} \quad (a_k \neq 0;\ k = 1, 2, \cdots) \tag{1}$$

has every point $e^{i\theta}$ as a Lusin point, provided $n_k \to \infty$ rapidly enough; in this statement, the expression "rapidly enough" must of course be interpreted in terms of the sequence $\{a_k\}$. The result is somewhat related to theorems of Salem and Zygmund [3] and of Schaeffer [4], who showed that if the series $\Sigma |a_k|$ converges slowly enough and $n_k \to \infty$ rapidly enough, the function (1) maps the circle $|z| = 1$ into a Peano curve. Intuitively, this proposition is suggested by the fact that, for $|a| < 1$ and large n, the polynomial $z + az^n$ maps the unit circle into a curve which consists of $n - 1$ nearly circular loops, of radius $|a|$, whose "moving center" lies on the unit circle.

In Section 3 we show that a function f, holomorphic in $|z| < 1$, continuous in $|z| \leq 1$, and mapping the unit disc onto a region of infinite area, need not possess any Lusin points at all. Our proof is based on the construction of a function which maps the unit disc upon a roughly circular disc to which many small discs are attached. The function has the further property that it takes no value infinitely often, for $|z| \leq 1$. (In a conversation, Professor K. Noshiro had raised the question whether there exists a function $\Sigma a_n z^n$, continuous in $|z| \leq 1$, with $\Sigma n |a_n|^2 = \infty$, and taking no value infinitely often in $|z| \leq 1$. The referee has pointed out a very simple alternate construction of such a function: let the Riemann surface R consist of a ribbon which covers

once	the annulus	$0 < \lvert w \rvert < 1/2$,
4 times	the annulus	$0 < \lvert w - 1/2 \rvert < 1/4$,
16 times	the annulus	$0 < \lvert w - 3/4 \rvert < 1/8$,

and so forth; and let f map the unit disc upon R.)

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2. LACUNARY TAYLOR SERIES

THEOREM 1. *Let* $\{a_k\}$ *be a sequence of complex numbers different f*
Then every point $e^{i\theta}$ *is a Lusin point for the function* $\Sigma a_k z^{n_k}$, *provided* n
rapidly enough.

The theorem can be restated as follows: If $a_k \neq 0$ $(k = 1, 2, \cdots)$, there
set of functions

$$N_1 = N_1(a_1),\ N_2 = N_2(a_1, a_2; n_1), \cdots, \quad N_k = N_k(a_1, a_2, \cdots, a_k; n_1, n_2, \cdots,$$

such that every point $e^{i\theta}$ is a Lusin point of the function (1) provided $n_k >$
$(k = 1, 2, \cdots)$. The sense of the expression "provided $n_k \to \infty$ rapidly enoug
thus made precise; for the sake of brevity, we shall henceforth write: if n
(rap).

Suppose now that $\{a_k\}$ is a sequence of constants, all different from ze
clear that the function (1) is holomorphic in $|z| < 1$ if $n_k \to \infty$ (rap), for ex
$n_k > k + |a_k|$. To deal with the question of Lusin points we write, for $0 <$
$0 \leq \theta < 2\pi$,

$$f(z) = \sum_j b_j(te^{i\theta})(z - te^{i\theta})^j \quad (|z - te^{i\theta}| < 1 - t);$$

that is, we let $b_j(te^{i\theta})$ denote the jth coefficient in the Taylor expansion of
point $te^{i\theta}$; and we make use of the fact that the area of the surface upon wh
function $\Sigma c_j z^j$ maps the disc $|z| < r$ is $\pi \Sigma j |c_j|^2 r^{2j}$.

With each exponent n_k we associate the special radius

$$t_k = 1 - \frac{\log n_k}{n_k},$$

and we proceed to prove that for each θ the sequence of finite sums

$$\sum_{j=1+n_{k-1}}^{n_k} j\,|b_j(t_k e^{i\theta})|^2 (1 - t_k)^{2j}$$

is unbounded if $n_k \to \infty$ (rap). Since the discs $|z - t_k e^{i\theta}| < 1 - t_k$ (θ fixed,
$k = 1, 2, \cdots$) form a nested sequence, it will then follow that $e^{i\theta}$ is a Lusin
for each θ.

From the identity

$$z^n = \sum_{j=0}^{n} \binom{n}{j} (te^{i\theta})^{n-j} (z - te^{i\theta})^j$$

it follows that, for $n_{k-1} < j \leq n_k$,

$$b_j(t_k e^{i\theta}) = a_k \binom{n_k}{j} (t_k e^{i\theta})^{n_k - j} + \sum_{p>k} a_p \binom{n_p}{j} (t_k e^{i\theta})^{n_p - j}.$$

The term of index p under the summation sign has modulus less than

$$|a_p|\, n_p^j t_k^{n_p-j} = |a_p|\, (n_p/t_k)^j t_k^{n_p} \leq |a_p|\, (n_p/t_k)^{n_k} t_k^{n_p} .$$

Once n_k (and hence t_k) is chosen, the last quantity can be made arbitrarily small for each p by choosing n_p sufficiently large. It follows that

$$|b_j(t_k e^{i\theta})| \geq \frac{1}{2}|a_k| \binom{n_k}{j} t_k^{n_k-j}$$

for $n_{k-1} < j \leq n_k$ $(k = 2, 3, \cdots)$ and all θ, if $n_k \to \infty$ (rap). Therefore it will be sufficient to prove that the sequence of sums

$$a_k^2 \sum_{j=1+n_{k-1}}^{n_k} j\left[\binom{n_k}{j} t_k^{n_k-j}(1 - t_k)^j\right]^2$$

is unbounded if $n_k \to \infty$ (rap). The quantity in brackets has its greatest value when $j = [(1 - t_k)n_k - t_k] = \log n_k + O(1)$. We therefore require that $\log n_k > n_{k-1}$; Stirling's formula then gives the result that some of the terms under the summation sign have a modulus of approximately $1/2\pi$. Moreover, the number of terms with modulus at least $1/3\pi$ is greater than $k/|a_k|^2$, if n_k is sufficiently large. This completes the proof of the theorem.

COROLLARY. *There exists a function* f, *continuous in* $|z| \leq 1$, *for which every point* $e^{i\theta}$ *is a Lusin point.*

This follows at once from Theorem 1, since the sequence $\{a_k\}$ can be taken so that $\Sigma\, |a_k| < \infty$.

3. A FUNCTION WITHOUT LUSIN POINTS

THEOREM 2. *There exists a function* $f(z) = \Sigma\, a_n z^n$, *holomorphic in* $|z| < 1$, *continuous in* $|z| \leq 1$, *and such that*

i) $\Sigma\, n\, |a_n|^2 = \infty$;

ii) f *has no Lusin points on* $|z| = 1$;

iii) *no value is taken infinitely often by* f *in* $|z| \leq 1$.

The example to be constructed will be of the form

$$f(z) = z + \frac{1}{2}\sum_{k=1}^{\infty} \frac{z_k c_k k^{-1/2}}{1 + c_k - z/z_k}, \tag{2}$$

where $z_k = e^{ik^{-1/4}}$ and $\{c_k\}$ is an appropriate sequence of small positive numbers. An intuitive discussion of the mapping function will make the motivation of the proof obvious.

If the constants c and d are positive and fairly small, the function

$$w_1 = z + \frac{cd}{1 + c - z}$$

maps the unit disc upon a region which consists roughly of the unit disc witl lar disc of diameter d attached near $z = 1$; the smaller the constant c, the is the isthmus joining the two discs. This follows from the identity

$$z + \frac{cd}{1 + c - z} = z + d/2 + \frac{c - (1 - z)}{c + (1 - z)} \frac{d}{2};$$

for as z moves around the unit circle in the positive direction, the point

$$w_2 = \frac{c - (1 - z)}{c + (1 - z)}$$

moves similarly around the circle whose diameter is the segment [-(2 - c), on the real axis; and if c is small, the image point w_2 describes almost al circular path while z moves along a short arc through $z = 1$.

Similar considerations show that the function $w = f(z)$ defined by (2) ma unit disc upon a roughly circular disc to which circular discs of radius 1/2 $(k = 1, 2, \cdots)$ are attached at $\arg w = k^{-1/4}$. Naturally, the attached discs The area of the image of $|z| < 1$ is infinite; but the mapping function is ve at $z = 1$, since the diameter of the kth disc is small compared with the dis tween its point of attachment and the point $w = f(1)$.

Let A_k denote the circular disc with center at $z = z_k(1 + c_k)$ and with $1/(k + 3)!$; then no two of the discs A_k overlap. Since the modulus of the k of the series (2) is arbitrarily small outside of A_k, for c_k small enough, a bounded by $k^{-1/2}$ on $|z| = 1$, it follows that if $c_k \to 0$ (rap), the function f morphic in the extended plane, except for simple poles at the points $z = z_k$ and an essential singularity at $z = 1$, and that it is continuous in $|z| \leq 1$.

To establish property (i), we observe that

$$\frac{c_k k^{-1/2}}{1 + c_k - z/z_k} = \frac{c_k k^{-1/2}}{1 + c_k} \sum_{j=0}^{\infty} \left(\frac{z}{z_k(1 + c_k)}\right)^j = \sum_{j=0}^{\infty} b_{kj} z^j;$$

because

$$\sum_{j=0}^{\infty} j\,|b_{kj}|^2 = c_k^2 k^{-1} \sum_{j=0}^{\infty} j\,(1 + c_k)^{-(2j+2)} = k^{-1}(2 + c_k)^{-2},$$

it is an elementary exercise to show that $\Sigma\, n\,|a_n|^2 = \infty$ if $c_k \to 0$ (rap).

Since $z = 1$ is the only singularity of f on $|z| = 1$, it is the only possib point on $|z| = 1$. To show that it is not a Lusin point, it is sufficient to prc every open disc $|z - t| < 1 - t$ $(0 < t < 1)$ the derivative f' is bounded. N

$$\left|\frac{d}{dz}\left(\frac{z_k c_k k^{-1/2}}{1 + c_k - z/z_k}\right)\right| = \frac{c_k k^{-1/2}}{|(1 + c_k)z_k - z|^2} < \frac{c_k k^{-1/2}}{|z_k - z|^2}.$$

In every disc $|z - t| < 1 - t$,

$$|z - z_k| > B(t) \arg^2 z_k = B(t) k^{-1/2},$$

where $B(t)$ is independent of k. It follows that

$$|f'| < 1 + \sum_{k=1}^{\infty} c_k k^{1/2} / B^2(t)$$

throughout the disc. The right member is finite for all t, if $c_k \to 0$ (rap).

Finally, suppose that f takes the same value w at the distinct points $z = t_n$ ($|t_n| \leq 1$, $n = 1, 2, \cdots$). Since $z = 1$ is the only singular point of f on $|z| = 1$, the sequence $\{t_n\}$ converges to 1; and since f is continuous, $w = f(1)$. But from (2) it follows that

$$(3) \qquad f(z) - f(1) = (z - 1)\left\{1 + \frac{1}{2}\sum_{k=1}^{\infty} \frac{c_k k^{-1/2}}{(1 + c_k - z/z_k)(1 + c_k - 1/z_k)}\right\}.$$

For each index k and each z in $|z| \leq 1$,

$$|1 + c_k - z/z_k| \geq 1 + c_k - |z/z_k| \geq c_k.$$

Moreover, except when

$$(k + 1/2)^{-1/4} < \arg z < (k - 1/2)^{-1/4},$$

we have the stronger inequality

$$|1 + c_k - z/z_k| \geq |z_k - z| \geq B k^{-5/4},$$

where B is a positive constant. Therefore, if $|z| \leq 1$,

$$\frac{c_k}{|1 + c_k - z/z_k|} \leq 1$$

for every k, and

$$\frac{c_k}{|1 + c_k - z/z_k|} < \frac{c_k k^{5/4}}{B}$$

for every k except one. Since

$$\frac{k^{-1/2}}{|1 + c_k - 1/z_k|} < \frac{k^{-1/2}}{|1 - 1/z_k|} = \frac{k^{-1/2}}{2 \sin \frac{1}{2} k^{-1/4}} < \frac{4}{3} k^{-1/4},$$

the kth term under the summation sign has modulus less than $\frac{4}{3}k^{-1/4} \leq 4/3$, and in every case except one it has modulus less than $4k c_k/3B$. It follows that the sum of the series in (3) has modulus less than $3/2$ if $c_k \to 0$ (rap). The value $f(1)$ is then not taken by f in $|z| \leq 1$, except at $z = 1$. This implies that f takes no value infinitely often in $|z| \leq 1$. The proof is complete.

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