A REMARK ON DIFFERENTIABLE MAPPINGS

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1. We consider a function $y = y(x)$ which maps the sphere $|x| < R$ $(R \leq \infty)$ of the n-dimensional euclidean space L_n $(n \geq 2)$ onto a domain G of the same space. Let G_r be the image of the sphere $|x| \leq r$ $(r < R)$. If $V(r)$ denotes the volume of the domain G_r and $A(r)$ the measure of its boundary Γ_r, then we have, by the isoperimetric inequality,

$$n^n v_1 V^{n-1} \leq A^n, \tag{1}$$

where v_1 denotes the volume of the unit sphere (equal to $\pi^k/k!$ for $n = 2k$, and to $2(2\pi)^{k-1}/1\cdot 3 \cdots (2k-1)$ for $n = 2k - 1$).

2. We suppose that the derivative operator $y'(x) \equiv \frac{dy(x)}{dx}$ is continuous. The jacobian $\Delta(x)$ of the function $y(x)$ is then also continuous. Let $d\sigma_x$ be the measure of an $(n-1)$-dimensional element of the sphere $|x| = r$, and $d\sigma_y$ the measure of its image. By the Hölder inequality, we have

$$A^n = \left(\int_{|x|=r} \left(\frac{d\sigma_y}{d\sigma_x} \Delta^{\frac{1}{n}-1} \right) \Delta^{1-\frac{1}{n}} d\sigma_x \right)^n$$

$$\leq \int_{|x|=r} \left(\frac{d\sigma_y}{d\sigma_x} \right)^n \Delta^{1-n} d\sigma_x \cdot \left(\int_{|x|=r} \Delta\, d\sigma_x \right)^{n-1}.$$

It follows now from (1) that

$$n^n v_1 V^{n-1} \leq \int_{|x|=r} \left(\frac{d\sigma_y}{d\sigma_x} \right)^n \Delta^{1-n} d\sigma_x \left(\int_{|x|=r} \Delta\, d\sigma_x \right)^{n-1}. \tag{2}$$

3. Here obviously

$$\int_{|x|=r} \Delta\, d\sigma_x = \frac{dV}{dr}. \tag{3}$$

The first right-hand integral in (2) has the following geometrical meaning. Let $|x - x_0| \leq r$ be an infinitesimal n-dimensional sphere and E_n the corresponding ellipsoid. If E_n has the semi-axes $a_1 \leq a_2 \leq \cdots \leq a_n$, then the jacobian takes the form

$$\Delta = \frac{a_1 \cdots a_n}{r^n}.$$

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Consider now an $(n-1)$-dimensional plane section S_{n-1} of the infinitesimal $|x - x_0| \leq r$ $(x_0 \in S_{n-1})$. Under the mapping $y = y(x)$, the corresponding set $(n-1)$-dimensional ellipsoid E_{n-1} with the semi-axes $b_1 \leq b_2 \leq \cdots \leq b_{n-1}$ this notation,

$$\frac{d\sigma_y}{d\sigma_x} = \frac{b_1 \cdots b_{n-1}}{r^{n-1}},$$

and it follows that

$$\left(\frac{d\sigma_y}{d\sigma_x}\right)^n \Delta^{1-n} = \frac{(b_1 \cdots b_{n-1})^n}{(a_1 \cdots a_n)^{n-1}}.$$

Now let $l = (l_1, \cdots, l_n)$ be the unit vector normal to the plane S_{n-1}. The

$$\frac{(b_1 \cdots b_{n-1})^n}{(a_1 \cdots a_n)^{n-1}} = a_1 \cdots a_n \left(\sum_{i=1}^{n} \frac{l_i^2}{a_i^2}\right)^{n/2} \leq \delta(x_0), \tag{4}$$

where

$$\delta = \frac{a_2}{a_1} \cdots \frac{a_n}{a_1} \tag{4)'$$

is the "dilatation quotient" of the map $x \to y$. This expression is the ratio l the volumes of the infinitesimal ellipsoid E_n and of the inscribed sphere.

4. We introduce now the mean value $L(r)$, defined by

$$L^{n-1} = \frac{1}{n v_1 r^{n-1}} \int_{|x|=r} \left(\frac{d\sigma_y}{d\sigma_x}\right)^n \Delta^{1-n} d\sigma_x. \tag{5}$$

It follows from (4) and (4)' that

$$L^{n-1} \leq \frac{1}{n v_1 r^{n-1}} \int_{|x|=r} \delta(x)\, d\sigma_x \leq D^{n-1},$$

where D is the "maximal dilatation"

$$D(r) = \max_{|x|=r} a_n/a_1 \tag{4)''$$

on the sphere $|x| = r$.

By the relations (3) to (5), the fundamental inequality (2) becomes

$$n V(r) \leq r L(r) \frac{dV(r)}{dr} \leq r D(r) \frac{dV(r)}{dr}.$$

Hence, the relation

$$\log \frac{V(r)}{V(r_0)} \geq n \int_{r_0}^{r} \frac{dt}{tL(t)} \geq n \int_{r_0}^{r} \frac{dt}{tD(t)} \tag{6}$$

holds for $0 < r_0 < r \leq R$.

5. In order to study the behavior of the integral in (6), we first suppose that, for $x = 0$, the derivative $A(x) = \frac{dy(x)}{dx}$ is the identity operator. Because A is continuous at this point, we then have

$$dy = dx + |dx| < \varepsilon(|x|)>, \tag{7}$$

where $\varepsilon(r)$ is a positive function, vanishing for $r = 0$; the expression $<\varepsilon>$ denotes a vector in the space L_n with a norm $|<\varepsilon>| \leq \varepsilon$. (For the notations used in the following, we refer to [2].) By the relation (7),

$$1 - \varepsilon(|x|) \leq \frac{|dy|}{|dx|} \leq 1 + \varepsilon(|x|), \tag{7)'$$

and it follows that

$$D(r) \leq \frac{1}{1 - \theta(r)}, \tag{8}$$

where

$$\theta(r) = \min\left(1, \frac{2\varepsilon}{1+\varepsilon}\right). \tag{8)'$$

6. By the formula (8), the relation (6) becomes

$$\log \frac{V(r)}{V(r_0)} \geq n \log \frac{r}{r_0} - n \int_{r_0}^{r} \frac{\theta(t)}{t} dt. \tag{8)''$$

Now it follows from the development (7) that the difference

$$\log V(r_0) - n \log r_0$$

tends to the limit v_1 as $r_0 \to 0$. Hence, with the notation $v_r = r^n v_1$,

$$\log V(r) \geq \log v_r - n \int_0^r \frac{\theta(t)}{t} dt.$$

This relation is not trivial, provided that the last integral is convergent. This condition requires that the function $\varepsilon(r)$ in (7) tends to zero so rapidly that the integral

$$\int_0 \frac{\varepsilon(r)}{r} dr$$

is finite; continuity of the derivative does not by itself imply this property (compare [1]).

7. If the derivative $A = y'(0)$ is not zero and is different from the ident
operator, we consider the lower bound

$$m_0 = \inf_{|h|=1} |y'(0)h|. \tag{9}$$

The differential of the function $z(x) = A^{-1}y(x)$ is $dz = A^{-1}dy(x)$, and we con from this that $|dz| \leq m_0^{-1}|dy|$ at each point x. Let now f_y be a set of poir and denote by f_z the set (z) corresponding to f_y under the linear transfor A^{-1}. It follows that the n-dimensional measure of f_y is at least equal to th ure of f_z, multiplied by the constant m_0^n.

Now the function $z(x)$ satisfies all the assumptions of the preceding sec Hence, by the relation (8)'', we finally get the inequality

$$V(r) \geq m_0^n v_r \exp\left(-n\int_0^r \frac{\theta(t)}{t}dt\right). \tag{10}$$

This relation is valid for every value r in the interval $0 \leq r \leq R$. Her the volume of the sphere $|x| \leq r$, $V(r)$ denotes the volume of its image unc transformation $y = y(x)$, and the constant m_0 is defined by (9), while θ has lowing significance:

The differential $dy = y'(x)dx$ admits an expansion of the form

$$dy = y'(0)\left(dx + |dx| < \varepsilon(|x|) >\right),$$

where, in view of the hypothesis concerning the continuity of y', $\varepsilon(r) \to 0$ The expression θ is defined by (8)'.

If the derivative y' is continuous at $x = 0$, in the stronger sense that n $\varepsilon \to 0$ as $r \to 0$, but also

$$\int_0 \frac{\varepsilon(t)}{t}dt < \infty, \tag{11}$$

then the integral in (10) is finite for all values r $(0 \leq r < R)$.

In the particular case of a conformal mapping, the expression θ vanisl

$$m_0 = \left(\frac{|dy|}{|dx|}\right)_{x=0}.$$

For $n = 2$, this result reduces to a well-known inequality due to Bieberbac

8. The condition (11) of "strong continuity" at the point $x = 0$ is fulfill second derivative y'' of y exists or, more generally, if the first derivativ fies a Lipschitz condition

$$\left|\left(y'(x) - y'(0)\right)dx\right| \leq M_0 |x| |dx|. \tag{12}$$

In this case it can be shown (see [1]) that the image of the sphere $|x| < r =$ is a domain G_r which contains the sphere $|y - y(0)| < m_0^2/2M_0$ and has th

$$V(r) \geq v_1 \, (m_0^2/2M_0)^n .$$

Let us compare this result (which cannot be sharpened) with the assertion contained in the general inequality (10). The function $\varepsilon(r)$ now becomes $\varepsilon = M_0 r/m_0$, whence

$$\theta = \frac{2 M_0 r}{m_0 + M_0 r}$$

and

$$\int_0^r \frac{\theta \, dt}{t} = 2 \log (1 + M_0 r/m_0) = \log 4.$$

It follows thus from (10) that

$$V(r) \geq v_1 \left(\frac{m_0^2}{4 M_0} \right)^n .$$

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