A THEOREM OF FRIEDRICHS

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§1. Friedrichs [2] has given a characterization of the Lie elements among the set of noncommutative polynomials. A proof of the characterization theorem was also given by Magnus [3], who refers to other proofs by P. M. Cohn and D. Finkelstein. It is the purpose of the present paper to give a short proof of the theorem.

Let Φ be the free associative ring, over a field K of characteristic zero, of polynomials $F(x) = F(x_1, x_2, \cdots)$ in the noncommuting indeterminates $x_1, x_2, \cdots$. Let Λ be the K-submodule of Φ generated by the $x_1, x_2, \cdots$ under the operation of forming commutators $[G, H] = GH - HG$. A *Lie element* of Φ is a member of Λ.

THEOREM (Friedrichs). $F(x)$ *is a Lie polynomial if and only if the relations*

$$x_i' x_j'' = x_j'' x_i'$$

imply

(1) $$F(x' + x'') = F(x') + F(x'').$$

§2. Induction from Lie elements G, H to [G, H], together with linearity, establishes that (1) holds for every Lie element F.

For the converse, begin by introducing the left, right, and adjoint representations L, R, and A = R - L of Φ. These are defined, on the free generators x_i, and for each element u of Φ, by the relations

$$uR(x_i) = ux_i,$$
$$uL(x_i) = x_i u,$$
$$uA(x_i) = ux_i - x_i u = [u, x_i].$$

Since the $R(x_i)$ commute with the $L(x_j)$, condition (1) on F(x) implies that

$$uF(A(x)) = uF(R(x)) + uF(-L(x)).$$

Clearly $uF(R(x)) = uF(x)$, while $uF(-L(x)) = F(x)^* u$, where $F(x)^*$ is defined by the equation

$$(x_{i_1} x_{i_2} \cdots x_{i_n})^* = (-1)^n x_{i_n} \cdots x_{i_2} x_{i_1}$$

and the condition of linearity. Thus (1) gives

(2) $$uF(A(x)) = uF(x) + F(x)^* u.$$

Induction from G, H to [G, H] establishes that

(3) $$F^* = -F, \quad \textit{for each Lie element } F.$$

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This, with (2), yields the familiar property of the adjoint representation:

(4) $$uF(A(x)) = [u, F(x)], \quad \textit{for each Lie element } F.$$

§3. The operator of Dynkin, Specht, and Wever [1], [4], [5] is defined by equation

$$\{x_{i_0} x_{i_1} x_{i_2} \cdots x_{i_n}\} = x_{i_0} A(x_{i_1}) A(x_{i_2}) \cdots A(x_{i_n})$$
$$= [[\cdots[[x_{i_0}, x_{i_1}], x_{i_2}], \cdots], x_{i_n}]$$

and the condition of linearity. Henceforth, let u be a new indeterminate. T a polynomial F(x) without constant term, we have

(5) $$\{uF(x)\} = uF(A(x)).$$

If F(x) satisfies (1), then F(0) = 2F(0), so that F(x) is without constant ter (5), with (2), gives

(6) $$\{uF(x)\} = uF(x) + F(x)^* u .$$

LEMMA (Dynkin, Specht, Wever). *If* F *is a Lie element, homogeneous* $n \geq 1$, *then*

(7) $$\{F\} = nF .$$

The assertion is trivial for n = 1. We argue by induction from G, H, of d q, to [G, H]. From the definition and (4) we have $\{GH\} = \{G\} H(A(x)) = [\{G$ By the induction hypothesis, this gives $\{GH\} = [pG, H] = p[G, H]$. Similarly $-\{HG\} = -q[H, G] = q[G, H]$. Addition gives $\{[G, H]\} = (p + q)[G, H]$.

It is evident from the definition that $\{F\}$ is always a Lie element, wher

COROLLARY. *If* F *is a polynomial, homogeneous of degree* $n \geq 1$, *the*

(8) $$\{\{F\}\} = n\{F\} .$$

§4. To conclude the proof of Friedrichs' theorem, we may assume that homogeneous of degree $n \geq 1$. Applying (8) to (6) gives

$$(n + 1)\{uF\} = \{uF\} + \{F^*u\} ,$$
$$n\{uF\} = \{F^*u\} ,$$
(9) $$n\{uF\} = \{F^*\}u - u\{F^*\} ;$$

the last by virtue of the definition of $\{F^*u\}$. From (6) directly we have

(10) $$n\{uF\} = nF^*u + unF .$$

Since u is a new indeterminate, not occurring in F or F*, comparison of t terms of (9) and (10) gives

$$nF = -\{F^*\} .$$

Since $\{F^*\}$ is a Lie element, it follows that nF is a Lie element. Using the assumption that the coefficient field is of characteristic zero, we conclude that F is a Lie element, as was to be shown.

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