A GENERALIZATION OF INNER PRODUCT

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The purpose of this note is twofold: first, to show that if B is a real Banach space whose conjugate space B* is uniformly convex [2], it is possible to define a real-valued function on $B \times B$ which for every $x \in B$ is a linear functional in y, for every $y \in B$ is a continuous (although not necessarily linear) function of x, and which is the usual inner product when B is a unitary (or Hilbert) space; second, to show that if B* is strictly convex, a necessary and sufficient condition that B be unitary is that if $x \in B$, $y \in B$, and $\|x\| = \|y\| = 1$, then

$$\frac{d}{dt}\|x + ty\|\Big]_{t=0} \quad \text{and} \quad \frac{d}{dt}\|y + tx\|\Big]_{t=0}$$

exist and are equal. This condition is not new; James proved a similar result for Banach spaces of three or more dimensions [5, p. 283]. His proof depends on a generalization of orthogonality, ours on a generalization of inner product; even this generalization appears in a disguised form in James' paper; the elementary nature of our argument is the justification for its presentation here. We recall two definitions: A Banach space is *strictly convex* if $\|x + y\| = \|x\| + \|y\|$ implies that there exists a $\lambda \geq 0$ such that $x = \lambda y$; a Banach space is *uniformly convex* if

$$\|x_n\| = \|y_n\| = 1,$$

together with $\lim \|(x_n + y_n)/2\| = 1$, implies that $\lim \|x_n - y_n\| = 0$. This definition of uniform convexity is given by Hille [3, p. 11], and it can be shown to be equivalent to the definition originally given by Clarkson.

THEOREM. *Let* B *be a real Banach space whose conjugate space* B* *is strictly convex. Then if* $x \in B$ $(x \neq 0)$, *there exists a unique* $\phi_x \in B^*$ *such that* $\phi_x(x) = 1$ *and* $\|\phi_x\| = 1/\|x\|$. *If* B* *is uniformly convex, the mapping* $x \to \phi_x$ *is continuous.*

Proof. A slightly different version of the first statement has been proved by Pettis [7]. Briefly, if $x^*(x) = y^*(x) = 1$ and $\|x^*\| = \|y^*\| = 1/\|x\|$, then

$$\frac{1}{2}(x^* + y^*)(x) = 1;$$

thus $1/\|x\| \leq \|(x^* + y^*)/2\| \leq 1/\|x\|$, which contradicts the strict convexity of B*, unless $x^* = y^*$.

To prove the second statement, we first observe that a uniformly convex space is reflexive [7] and that bounded sets are therefore sequentially compact [8]. Second, we prove that if $\{x_n^*\}$ is a sequence in B* converging weakly to x_o^*, and if $\|x_n^*\|$ converges to $\|x_o^*\|$, then x_n^* converges uniformly to x_o^*. Since $x_n^*/\|x_n^*\|$ converges weakly to $x_o^*/\|x_o^*\|$, we can assume that $\|x_n^*\| = \|x_o^*\| = 1$. It remains to show that $\|(x_n^* + x_o^*)/2\|$ converges to 1.

Clearly $\|(x_n^* + x_o^*)/2\| \leq 1$, for all n. Let

$$x^{**} \in B^{**}, \quad \|x^{**}\| = 1, \quad x^{**}(x_o^*) = 1.$$

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Then

$$|x^{**}(x_n^*) + 1| = |x^{**}(x_n^* + x_o^*)| \leq \|x_n^* + x_o^*\|.$$

If $\varepsilon > 0$, the weak convergence of $\{x_n^*\}$ guarantees that if n is sufficient $\|x_n^* + x_o^*\| \geq 2 - \varepsilon$, so that $\lim \| (x_n^* + x_o^*)/2 \| = 1$.

To complete the proof we have only to show that if $\lim x_n = x_o \neq 0$ $(x_n$ then a subsequence of $\{\phi_{x_n}\}$ converges weakly to ϕ_{x_n}. But since $\{\phi_{x_n}\}$ i formly bounded, it contains a subsequence which converges weakly to some For simplicity, and without loss of generality, we shall suppose that this su is $\{\phi_{x_n}\}$. Now

$$x_o^*(x_o) = (x_o^* - \phi_{x_n})(x_o) + \phi_{x_n}(x_o - x_n) + \phi_{x_n}(x_n).$$

Therefore $x_o^*(x_o) = 1$ and $\|x_o^*\| \geq 1/\|x_o\|$.

Further, $|x_o^*(x)| = \lim |\phi_{x_n}(x)| \leq \lim\sup \|\phi_{x_n}\| \, \|x\| \leq \|x\|/\|x_o\|$, for so that $\|x_o^*\| \leq 1/\|x_o\|$. This, together with the uniqueness of ϕ_{x_o}, prove $x_o^* = \phi_{x_o}$.

COROLLARY.

$$\langle x, y\rangle = \begin{cases} \|x\|^2 \phi_x(y) & (x \neq 0), \\ 0 & (x = 0) \end{cases}$$

is a continuous linear functional in y *for every* $x \in B$, *and it is a continuou of* x *for every* $y \in B$.

THEORÈM (Mazur [6]). *If* $\|x\| = \|y\| = 1$, *then*

$$\phi_x(y) = \frac{d}{dt}\|x + ty\|\Big]_{t=0}.$$

Proof. If $y = \pm x$, then

$$\phi_x(y) = \pm 1 \quad \text{and} \quad \frac{d}{dt}\|x + ty\| = \pm 1$$

for all t. Otherwise, y is independent of x. Reconstructing the proof of tl Banach extension theorem, [1, p. 28], and extending the linear functional ϕ_x from the subspace generated by x to that generated by x and y we have, fı uniqueness of $\phi_x(y)$,

$$\sup_\lambda (-\|\lambda x + y\| - \lambda) = \phi_x(y) = \inf_\lambda (\|\lambda x + y\| - \lambda).$$

Now if $\lambda_1 > \lambda_2$, then

$$-\|\lambda_1 x + y\| + \|\lambda_2 x + y\| \leq \|(\lambda_2 - \lambda_1)x\| = \lambda_1 - \lambda_2,$$

so that

$$-\|\lambda_1 x + y\| - \lambda_1 \leq -\|\lambda_2 + y\| - \lambda_2 .$$

Therefore

$$\phi_x(y) = \lim_{\lambda \to -\infty} (-\|\lambda x + y\| - \lambda) = \lim_{t \to 0-} (-\|x/t + y\| - 1/t)$$

$$= \lim_{t \to 0-} (\|x + ty\| - \|x\|)/t.$$

Similarly, $\phi_x(y) = \lim_{t \to 0+} (\|x + ty\| - \|x\|)/t$.

COROLLARY. *If* $\|x\| = \|y\| = 1$, *then* $\phi_x(y) = \phi_y(x)$ *if and only if*

$$\frac{d}{dt}\|x + ty\|\Big]_{t=0} \quad \textit{and} \quad \frac{d}{dt}\|y + tx\|\Big]_{t=0}$$

exist and are equal. (We notice that if these derivatives exist for all x and y, the uniqueness of ϕ_x is assured without regard to strict convexity of B^*.)

THEOREM. *The three conditions*

(a) *if* $\|x\| = \|y\| = 1$, *then* $\phi_x(y) = \phi_y(x)$,

(b) $\|x\|^2\phi_x(y) = \|y\|^2\phi_y(x)$, *for all* $x \in B$, $y \in B$,

(c) $<x, y>$ *is an inner product for* B

are equivalent.

Proof. That (b) implies (a) is obvious. To prove that (a) implies (b), observe that $\phi_{\lambda x} = (1/\lambda)\phi_x$ for $\lambda \neq 0$. Thus

$$\|x\|^2 \phi_x(y) = \|x\| \; \|y\| \phi_{x/\|x\|}(y/\|y\|)$$

$$= \|x\| \; \|y\| \phi_{y/\|y\|}(x/\|x\|)$$

$$= \|y\|^2\phi_y(x).$$

If (b) holds, then $<x, y>$ is linear in x, $<x, y> = <y, x>$, and $<x, x> = \|x\|^2$, so that (c) is proved. Finally, if $<x, y>$ is an inner product for B, then $B = B^*$, and the unique function $\phi_x(y)$ is given by the formula

$$\phi_x(y) = \frac{<x,y>}{\|x\|^2},$$

which is readily seen to imply (b).

Example. If $B = L^p(E)$, where E is a measure space and $p > 1$, then if $x \in B$, $\phi_x(y) = \int y|x|^{p-1} \operatorname{sig} x/\|x\|^p \, dE$. If $p = 2$, $<x, y>$ is the usual inner product $\int xy \, dE$.

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