AN ALGEBRA RELATED TO THE ORTHOGONAL GROUP

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INTRODUCTION

I wish to express my sincere gratitude to Professor R. M. Thrall, under whose guidance the work presented here was done. I am also grateful to Professor R. Brauer, who suggested the problem, and to Professor H. Weyl. The extent to which I relied upon their earlier work will be apparent.

The question of commuting algebras of representations of classical groups over tensor space has been discussed by Brauer [2] and Weyl [5, Chapters III, IV and V]. In the case of the general linear group, the algebra concerned is the group algebra of the symmetric group. Much information concerning the representation theory of the general linear group is obtained from the corresponding theory of the symmetric group. Brauer [1] has defined a number of algebras which replace the symmetric group algebra when the general linear group is replaced by certain subgroups.

This paper is concerned with the algebra which arises in the case of the orthogonal group. Its definition by means of diagrams is taken from Brauer's paper [2]. In the first three chapters, results concerning the structure of the algebra are obtained directly. The fourth chapter makes contact with Weyl's results concerning the representation of the algebra in tensor space [5, Chapter V, Section B].

CHAPTER I

THE ALGEBRA ω_f^n

1.1. A REPRESENTATION OF $\mathfrak{S}_f$ BY DIAGRAMS

A permutation $\sigma \in \mathfrak{S}_f$, the symmetric group on f symbols, may be represented by a diagram consisting of two rows of f dots, the dots of each row being associated with the integers 1, 2, $\cdots$, f, from left to right, and dot i of the lower row being joined to dot σi of the upper row $(i = 1, 2, \cdots, f)$. Multiplication of diagrams to obtain $\tau\sigma$ is performed by placing a diagram for τ with its lower row of dots coincident with the upper row of dots of the diagram for σ. In this way a composite diagram (τ, σ) is obtained in which dot i of the lower row joins dot σi of the middle row, which in turn joins dot $\tau(\sigma i)$ of the upper row. If multiplication of permutations is performed from right to left, then the new diagram, obtained by deleting the middle row and joining dot i directly to dot $\tau(\sigma i)$, is the diagram for $\tau\sigma$. Suppose for example that $\tau = (1\,3) \in \mathfrak{S}_3$ and $\sigma = (1\,3\,2) \in \mathfrak{S}_3$;

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then the diagrams for τ and σ are shown by Figures 1a and 1b, respective Figures 1c and 1d show the composite diagram and the corresponding new (gram, respectively.

It is seen that composition of diagrams corresponds to the calculation c cycle product from right to left.

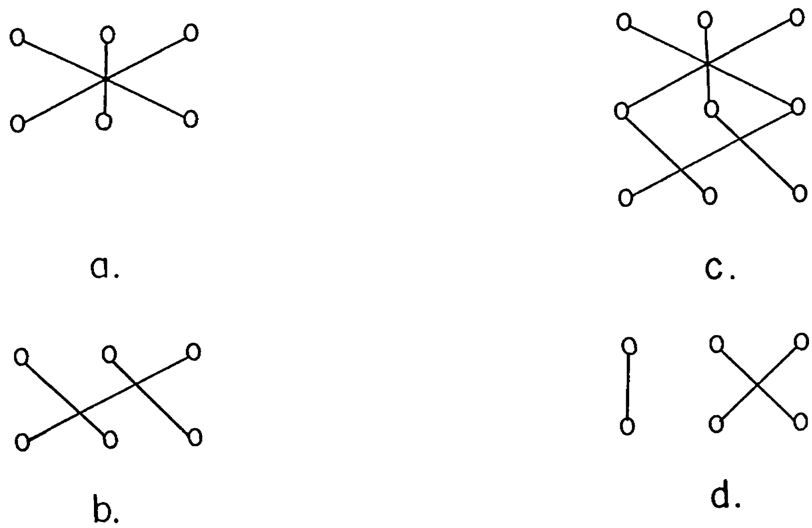

Figure 1.

1.2. A GENERALIZATION OF DIAGRAMS

Henceforth, the term 'diagram' is used to denote two rows of f dots in every dot is joined to exactly one other dot by a 'line' or a 'bar'. The word is used to denote the join of two dots of different rows, and the word 'bar' t note the join of two dots of the same row. The number of such diagrams is

$$N(f) = (2f - 1)(2f - 3) \cdots 5 \cdot 3 \cdot 1,$$

since there are $2f - 1$ ways of joining the first dot to another, then $2f - 3$ v joining the next unconnected dot, etc. Composition of two diagrams is perfc in a manner similar to that described above. A clear picture is most easil by means of an example. Let Figures 2 and 3 be 'diagrams' U and V resp Then Figures 4 and 5 are respectively the composite diagram (U, V) and th responding new diagram UV(2). The figure (2) following UV is a temporar tion. It indicates the number of "cycles" occuring in the composite from wl is derived. By a cycle is meant a sequence of dots on the middle row of a c posite diagram, say $D_1, D_2, \cdots, D_h$, such that D_1 joins D_2, D_2 joins $D_3, \cdots,$ joins D_1.

The following properties of diagrams and their composition are importa

P_1. A diagram has the same number of bars in its upper and lower rows may therefore refer to a diagram having r bars in each row as an 'r-bar d gram'.

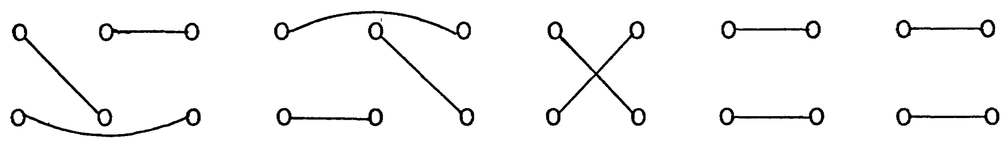

Figure 2.

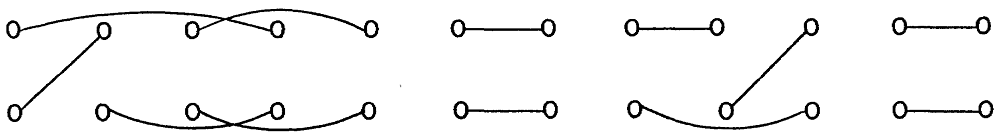

Figure 3.

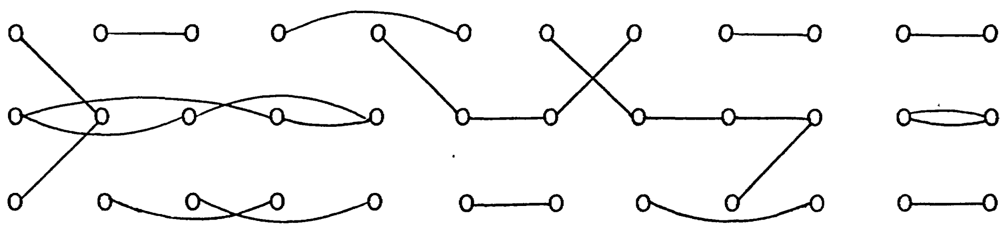

Figure 4.

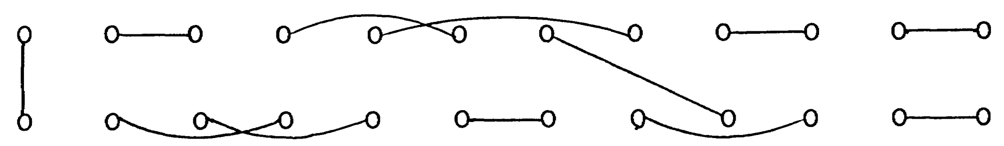

Figure 5.

P_2'. The same is true relative to lower bars of V in a product UV.

P_3. The number of cycles occuring in the composite (U, V) of two r-bar diagrams is at most r. It is exactly r when the bar structures of the lower row of U and the upper row of V are identical. Otherwise the number is less than r.

P_4. If by $\hat{U}$ we denote the diagram obtained by inverting U, then $\widehat{UV} = \hat{V}\hat{U}$.

1.3. THE DEFINITION OF ω_f^n

Let $U_1, U_2, \cdots, U_N$, $(N = N(f))$ be a basis for a vector space of dimension N(f) over a field K of characteristic zero. To each basis element assign a diagram of the type described in §1.2. The diagram assigned to U_i will now be denoted by the symbol U_i. If the composite (U_i, U_j) yields a diagram $U_k(q)$ and has q cycles, then we define the product of the basis elements to be

$$U_i U_j = n^q U_k,$$

where n is an integer. It is not quite clear that the multiplication so defined is associative. This does however follow from the origin of the definition to be found in [2], where the set of diagrams is given as a model for a set of matrices, with multiplication defined isomorphically. Multiplication is extended to arbitrary elements of the vector space in accordance with the distributive laws. In this way the vector space is given the structure of an algebra.

The algebra defined in this way is designated by ω_f^n.

1.4. SUBSPACES OF ω_f^n

The set of basis elements corresponding to r-bar diagrams spans a line subset which will be denoted by $\mathfrak{B}_r$. The algebra ω_f^n has therefore the dec sition

$$\omega_f^n = \mathfrak{B}_0 + \mathfrak{B}_1 + \cdots + \mathfrak{B}_r + \cdots + \mathfrak{B}_m$$

as a direct sum of subspaces. The symbol m is used to denote the integral of f/2: $m = [f/2]$.

We now calculate the dimension of the subspace $\mathfrak{B}_r$. Let M(f, r) denot number of ways in which r bars may be placed on a row of f dots. The fir may have one end placed in any one of f positions and the other end placed i one of the remaining f - 1 positions. Since we do not distinguish between tl ends of a bar, the number of positions in which it may be placed is f(f - 1)/ For the second bar the same argument is valid with f replaced by f - 2, gi (f - 2)(f - 3)/2 possible positions. Finally, the r^{th} bar may be placed in (f - 2r + 2)(f - 2r + 1)/2 positions. However, the order in which the bars ai placed in position is immaterial; they may be permuted in r! ways without ing the bar structure. Hence the r bars may be placed in position in M(f, ı ways, where

$$M(f, r) = \frac{f(f-1)(f-2)\cdots(f-2r+1)}{2^r r!} = \frac{f!}{2^r (f-2r)!r!}.$$

The dimension N(f, r) of $\mathfrak{B}_r$ may now be derived. It is the number of diagrams. Since there are M(f, r) possible bar arrangements for each row since the remaining f - 2r dots in each row may be joined by lines in (f - 2 ways, we see that

$$N(f, r) = [M(f, r)]^2 (f-2r)! = \frac{(f!)^2}{(2^r r!)^2 (f-2r)!}$$

THEOREM 1.4A. *The algebra* ω_f^n *has the decomposition*

$$\omega_f^n = \mathfrak{B}_0 + \mathfrak{B}_1 + \cdots + \mathfrak{B}_r + \cdots + \mathfrak{B}_m \quad (m = [f/2])$$

as a direct sum of vector subspaces $\mathfrak{B}_r$. $\mathfrak{B}_r$ *is the subspace of* ω_f^n *spar by basis elements whose diagrams have exactly* r *bars. The dimension of is* N(f, r).

1.5. A CHAIN OF IDEALS IN ω_f^n

Properties P_2 and P_2' of the composition of diagrams (§1.2) show that i basis element of $\mathfrak{B}_r$ is multiplied by any other basis element of the algebr product is an element of $\mathfrak{B}_s$ with $s \geq r$. It follows that the subspace

$$\mathfrak{A}_r^* = \mathfrak{B}_r + \mathfrak{B}_{r+1} + \cdots + \mathfrak{B}_m \quad \text{(direct)}$$

is a two-sided ideal of ω_f^n. It may be described as the ideal spanned by all elements whose diagrams have r or more bars. In particular, $\mathfrak{B}_m = \mathfrak{A}_m^*$

two-sided ideal. By the definition of $\mathfrak{A}^*_r$ and the decomposition given in Theorem 1.4A, we see that ω_f^n possesses a chain of ideals

$$\omega_f^n = \mathfrak{A}^*_0 \supset \mathfrak{A}^*_1 \supset \cdots \supset \mathfrak{A}^*_r \supset \cdots \supset \mathfrak{A}^*_m . \tag{1}$$

1.6. A NOTATION FOR BASIS ELEMENTS

We now consider the construction of a diagram upon a skeleton consisting of two rows of f dots. In §1.4 we have seen that r bars may be arranged on f dots in $M = M(f, r)$ ways. We may therefore assign indices 1, 2, ⋯, M, one to each possible arrangement. Such a scheme of indices will be referred to as an "r-bar scheme". The bar structure of a diagram corresponding to a basis element of $\mathfrak{B}_r$ is therefore specified by a pair of indices $\binom{i}{j}$ which indicate that the bar structure of the upper row has the index i and that of the lower row has index j. We may also say that the corresponding basis element has bar structure $\binom{i}{j}$.

Now consider the construction of lines. Suppose that in each row of an incomplete diagram there is a set of s unoccupied dots. In each row we number these dots 1, 2, ⋯, s from left to right. A diagram for a permutation σ of $\mathfrak{S}_s$ may then be constructed upon these dots by joining dot k of the lower set to dot σk of the upper set, for $1 \leq k \leq s$. We then say that the diagram possesses the permutation σ of $\mathfrak{S}_s$ on the two sets of s dots and, when there is no fear of confusion, that the diagram, or the corresponding basis element, possesses the permutation σ of $\mathfrak{S}_s$.

The element whose bar structure is $\binom{i}{j}$ and which possesses the permutation σ of $\mathfrak{S}_{f-2r}$ is an element of $\mathfrak{B}_r$. A complete basis for $\mathfrak{B}_r$ is obtained by allowing i and j to range through the r-bar scheme and σ to range through $\mathfrak{S}_{f-2r}$. Elements possessing the identity permutation will be called special elements.

We will denote by U_j^i the basis element of $\mathfrak{B}_r$ whose bar structure is $\binom{i}{j}$ and which possesses the identity permutation of $\mathfrak{S}_{f-2r}$. To help the description we consider examples of 1-bar elements in the case $f = 5$. The bar arrangements 1, 2, ⋯, M ($M = M(5, 1) = 10$) are shown in Figure 6. The basis element U_5^3 is shown in Figure 7.

Fig. 6. One-bar scheme: f = 5.

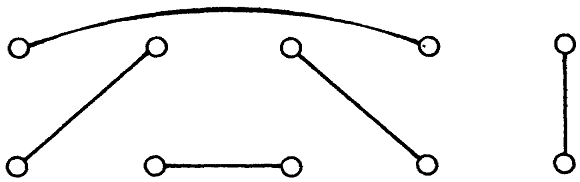

Figure 7.

The elements U_j^i of $\mathfrak{B}_r$ do not provide a complete basis when i and j through the r-bar scheme. We now develop a notation for the remaining el€

If i is an index of the r-bar scheme and if $\sigma \in \mathfrak{S}_{f-2r}$, then we define c that element of $\mathfrak{B}_0$ whose diagram is constructed as follows.

(i) The identity permutation of $\mathfrak{S}_{2r}$ is constructed upon those dots whi the end points of bars in the diagram for U_i^i.

(ii) The permutation σ of $\mathfrak{S}_{f-2r}$ is constructed upon those dots which end points of lines in the diagram for U_i^i.

It is now seen that $\sigma_i U_j^i$ is the basis element of $\mathfrak{B}_r$ whose bar structu and whose permutation is σ of $\mathfrak{S}_{f-2r}$. Since U_j^i possesses the identity pe of $\mathfrak{S}_{f-2r}$ and since the identity commutes with every other permutation, it that $\sigma_i U_j^i = U_j^i \sigma_j$. The set of elements $\sigma_i U_j^i$ forms a complete basis for $\mathfrak{B}$ σ ranges through $\mathfrak{S}_{f-2r}$ and i, j take all values in 1, 2, ⋯, M = M(f,r).

As an example we consider the element $(1\,3\,2)_3 U_5^3$ of $\mathfrak{B}_1$ in ω_5^n. The di

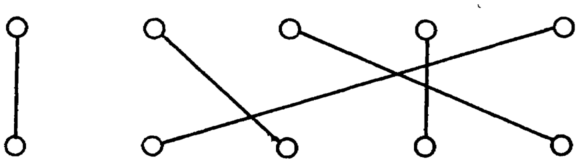

Figure 8.

$(1\,3\,2)_3$ is represented in Figure 8. We form the composite of this with the U_5^3 (Figure 7) and obtain the product diagram (Figure 9).

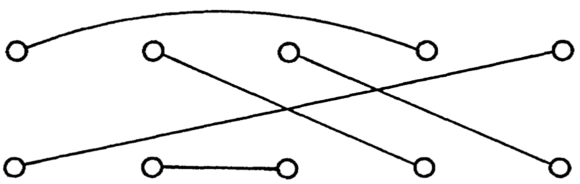

Figure 9.

This possesses the bar structure $\binom{3}{5}$ and the permutation (1 3 2) of $\mathfrak{S}_3$.

1.7. MULTIPLICATION OF BASIS ELEMENT OF $\mathfrak{B}_r$

The elements σ_i, τ_i and so forth correspond to diagrams, and therefor€ products in which they occur are associative. This helps in obtaining a rul€ multiplication of basis elements:

$$\left(\sigma_i U_j^i\right)\left(\tau_p U_q^p\right) = \left(\sigma_i U_j^i\right)\left(U_q^p \tau_q\right)$$
$$= \sigma_i\left(U_j^i U_q^p\right) \tau_q.$$

We consider now the product $U_j^i U_q^p$ of special elements.

Let U_j^i and U_q^p be special elements of $\mathfrak{B}_r$. We assume first that their product $U_j^i U_q^p$ is again an element of $\mathfrak{B}_r$. Then it possesses bar structure $\binom{i}{q}$ (by P_2 and P_2', § 1.2). It may also possess a permutation of $\mathfrak{S}_{f-2r}$ which need not be the identity, and a power of n may be introduced. It is not difficult to prove that the permutation and the power of n that occur depend only upon the bar structure of the lower row of U_j^i and of the upper row of U_q^p; that is, they depend upon the indices (j p), but not upon the indices (i q). We denote the power of n by ν(j p) and the permutation by σ(j p). Their product is an element of the group algebra Π_{f-2r} of $\mathfrak{S}_{f-2r}$ over K, and we denote it by $\sigma^*(j\,p) = \nu(j\,p)\sigma(j\,p)$. We may now write the product as

$$U_j^i U_q^p = \sigma_i^*(j\,p)\, U_q^i \qquad \left(\sigma^*(j\,p) \in \Pi_{f-2r}\right).$$

The multiplication rule becomes

$$\left(\sigma_i U_j^i\right)\left(\tau_p U_q^p\right) = \sigma_i \left(\sigma_i^*(j\,p) U_q^i\right) \tau_q$$
$$= \sigma_i\, \sigma_i^*(j\,p)\, \tau_i U_q^i$$
$$= (\sigma\sigma^*(j\,p)\,\tau)_i\, U_q^i.$$

For any element τ of $\mathfrak{S}_{f-2r}$ and a basis element $\sigma_i\, U_j^i$ of $\mathfrak{B}_r$, we define

$$\tau\left(\sigma_i U_j^i\right) = \tau_i\left(\sigma_i U_j^i\right).$$

If the definition is extended to the whole of the space $\mathfrak{B}_r$ by linearity, we see that we have a representation of $\mathfrak{S}_{f-2r}$ on $\mathfrak{B}_r$, i.e. for any ρ and τ of $\mathfrak{S}_{f-2r}$ and U of $\mathfrak{B}_r$, $\rho(\tau U) = (\rho\tau)U$. It is sufficient to check this for basis elements $\sigma_i\, U_j^i$:

$$\rho\left(\tau\left(\sigma_i U_j^i\right)\right) = \rho\left(\tau_i \sigma_i\, U_j^i\right) = \rho_i\, \tau_i\, \sigma_i\, U_j^i$$
$$= (\rho\,\tau)_i \sigma_i\, U_j^i = \rho\tau\left(\sigma_i U_j^i\right).$$

Since by our definition $\sigma U_j^i = \sigma_i\, U_j^i$, we may now drop the suffixes on permutations. We now have

R_1''. $\qquad \sigma(\tau U) = (\sigma\tau)U \quad (\sigma,\ \tau \in \mathfrak{S}_{f-2r};\ U \in \mathfrak{B}_r).$

The multiplication rule may be rewritten as

R_2''. $\qquad \left(\sigma U_j^i\right)\left(\tau\, U_q^p\right) = \sigma\sigma^*(j\,p)\ \tau\, U_q^i,$

and we observe that this rule is still only a partial rule, valid when $U_j^i U_q^p \in \mathfrak{B}_r$.

1.8. THE OPERATORS $\sigma^*(j\ p)$

For multiplication in $\mathfrak{B}_r$ the operators $\sigma^*(j\ p)$ have only been defined ducts $U^i_j U^p_q$ that are again in $\mathfrak{B}_r$. It is convenient for our later purposes t $\sigma^*(j\ p)$ to be a zero operator for products that are not in $\mathfrak{B}_r$. In this case P_2 and P'_2 (§1.2) ensure that the products will be in $\mathfrak{B}_s$ with $s > r$.

The $\sigma^*(j\ p)$ may be computed by forming composites of special diagran shows the values of $\sigma^*(j\ p)$ in the case of $\mathfrak{B}_1$ in ω^n_5. The identity element represented by 1.

TABLE 1

p \ q	1	2	3	4	5	6	7	8	9	10
1	n.1	1	(12)	(132)	1	(12)	(132)	0	0	0
2	1	n.1	1	(23)	1	0	0	(12)	(132)	0
3	(12)	1	n.1	1	0	1	0	(12)	0	(132)
4	(123)	(23)	1	n.1	0	0	1	0	(12)	(132)
5	1	1	0	0	n.1	1	(23)	1	(23)	0
6	(12)	0	1	0	1	n.1	1	1	0	(23)
7	(123)	0	0	1	(23)	1	n.1	0	1	(23)
8	0	(12)	(12)	0	1	1	0	n.1	1	1
9	0	(123)	0	(12)	(23)	0	1	1	n.1	1
10	0	0	(123)	(123)	0	(23)	(23)	1	1	n.1

We use the table to calculate the product $((1\,2)U^{10}_4)\,((1\,2\,3)U^2_7)$. By rule I equals $(1\,2)(2\,3)(1\,2\,3)U^{10}_7 = (1\,3\,2)U^{10}_7$. The multiplication is illustrated by Fig and 11, which correspond to the composite $((1\,2)U^{10}_4,\ (1\,2\,3)U^2_7)$ and the produc $(1\,3\,2)U^{10}_7$ respectively.

The following simple properties of $\sigma^*(j\ p)$ are useful; as before, we wri $\sigma^*(j\ p) = \nu(j\ p)\sigma(j\ p)$.

S_1. $\nu(j\ p) = \nu(p\ j)$.

S_2. $\sigma^*(j\ p) = 0 \Longrightarrow \sigma^*(p\ j) = 0$.

S_3. $\sigma^*(j\ p) \neq 0 \Longrightarrow \sigma(j\ p) = \sigma(p\ j)^{-1}$.

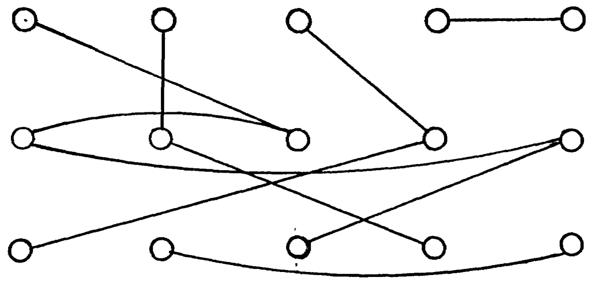

Figure 10

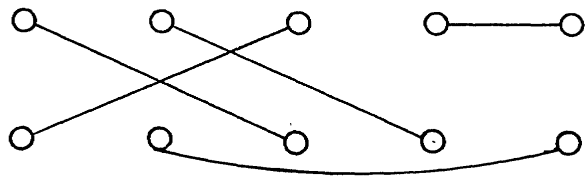

Figure 11

S_4. $\sigma^*(j\,j) = \nu(j\,j)\varepsilon = n^r\varepsilon$,

where ε denotes the identity permutation.

S_5. $\nu(j\,p) = n^k$ or zero, and $0 \le k < r$ for $j \ne p$.

Properties S_1, S_2 and S_3 are consequences of the following three observations:

(i) Whereas $\sigma^*(j\,p)$ arises from the composite $\left(U^i_j, U^p_q\right)$, $\sigma^*(p\,j)$ arises from the composite $\left(U^q_p, U^j_i\right)$.

(ii) The two composites in (i) are obtained from one another by an inversion, that is, by the operation "$\wedge$" described in P_4 (§1.2).

(iii) If σ is a permutation diagram, then $\hat{\sigma}$ is its inverse.

Properties S_4 and S_5 are consequences of the following two observations:

(iv) $\sigma^*(j\,j)$ arises from the composite $\left(U^i_j, U^j_q\right)$, where the bar structure of the lower row of U^i_j and the upper row of U^j_q are identical. There are then r cycles, and the permutation which arises is the identity.

(v) The numerical parts of S_4 and S_5 are consequences of P_3 (§1.2).

CHAPTER II

FACTOR ALGEBRAS OF ω^n_f

The factors of the chain

$$\omega^n_f = \mathfrak{A}^*_0 \supset \mathfrak{A}^*_1 \supset \cdots \supset \mathfrak{A}^*_r \supset \cdots \supset \mathfrak{A}^*_m \qquad \left(m = [f/2]\right)$$

will be denoted by $\mathbf{A}_r = \mathfrak{A}^*_r / \mathfrak{A}^*_{r+1}$ $(0 \le r \le m)$.

2.1. MULTIPLICATION IN THE FACTOR ALGEBRAS A_r

Since $\mathfrak{A}^*_r = \mathfrak{B}_r + \mathfrak{A}^*_{r+1}$ (direct sum) by definition, a basis for $\mathfrak{B}_r$, tak $\mathfrak{A}^*_{r+1}$, serves as a basis for the residue class algebra A_r. For the momen note the residue class modulo $\mathfrak{A}^*_{r+1}$ of an element U of $\mathfrak{A}^*_r$ by $\lfloor U \rfloor$. The r multiplication

$$\lfloor \sigma U^i_j \rfloor \lfloor \tau U^p_q \rfloor = \lfloor \sigma \sigma^*(j\ p)\tau U^i_q \rfloor$$

is now a complete rule in A_r, for when $U^i_j U^p_q$ does not lie in $\mathfrak{B}_r$, it lies in $\mathfrak{B}_s \subset \mathfrak{A}^*_{r+1}$ since $s > r$. It follows that its residue class is zero. This is with the special definition $\sigma^*(j\ p) = 0$ given at the beginning of §1.8.

Using the direct decomposition $\mathfrak{A}^*_r = \mathfrak{B}_r + \mathfrak{A}^*_{r+1}$, we can extend the dor the operators σ of $\mathfrak{S}_{f-2r}$ to the whole of $\mathfrak{A}^*_r$ by defining them to operate a tity transformations on $\mathfrak{A}^*_{r+1}$. Then we may use the usual definition of an o on a residue class, namely

$$\sigma \lfloor U \rfloor = \lfloor \sigma U \rfloor .$$

Writing $\lfloor U^i_j \rfloor = V^i_j$, we obtain $\sigma V^i_j = \lfloor \sigma U^i_j \rfloor$. We now have three rules for th these operators in A_r:

R'_1. $\quad \sigma(\tau V) = (\sigma\tau)V \quad (\sigma, \tau \in \mathfrak{S}_{f-2r};\ V \in A_r)$.

R'_2. $\quad \left(\sigma V^i_j\right)\left(\tau V^p_q\right) = \sigma\sigma^*(j\ p)\tau V^i_q$ for basis elements.

R'_3. $\quad (\rho V_1)V_2 = \rho(V_1 V_2) \quad (\rho \in \mathfrak{S}_{f-2r};\ V_1, V_2 \in A_r)$.

R'_1 and R'_2 are the forms which R''_1 and R''_2 take in A_r. R'_3 is an immediate quence of R'_1 and R'_2. Indeed, for basis elements σV^i_j and τV^p_q we have

$$\left(\rho(\sigma V^i_j)\right)(\tau V^p_q) = \left((\rho\sigma)V^i_j\right)\left((\tau V^p_q)\right) \quad (\text{by } R'_1)$$
$$= \rho\sigma\sigma^*(j\ p)\tau V^i_q \quad (\text{by } R'_2)$$

and

$$\rho\left((\sigma V^i_j)(\tau V^p_q)\right) = \rho(\sigma\sigma^*(j\ p)\tau V^i_q) \quad (\text{by } R'_2)$$
$$= \rho\sigma\sigma^*(j\ p)\tau V^i_q \quad (\text{by } R'_1).$$

All these rules may be extended to elements of the group algebra. For α, α α_2 in Π_{f-2r} and V, V_1 and V_2 in A_r, they become

R_1. $\quad \alpha_1(\alpha_2 V) = (\alpha_1\alpha_2)V$.

R_2. $\quad \left(\alpha_1 V^i_j\right)\left(\alpha_2 V^p_q\right) = \alpha_1\sigma^*(j\ p)\alpha_2 V^i_q$.

R_3. $\quad (\alpha V_1)V_2 = \alpha(V_1 V_2)$.

2.2. THE DECOMPOSITION OF A_r

Young's theory of the decomposition of the group algebra of the symmet into simple ideals is well known [4]. A corresponding decomposition of A_r tained in this section. We first outline the relevant parts of Young's theory.

There is a one-to-one correspondence between the partitions (λ) of f ar simple ideals $\Pi^{(\lambda)}$ of the group algebra Π_f of $\mathfrak{S}_f$ over a field K of chara tic zero. Π_f is a direct sum of these simple ideals:

$$\Pi_f = \sum_{(\lambda)} \Pi^{(\lambda)} \quad \text{(direct)},$$

where the summation extends over all partitions (λ) of f. The simple ideal $\Pi^{(\lambda)}$ is a total matrix algebra over the ground field K. Its dimension f_λ^2 can be stated in closed form. A basis $e_{\alpha\beta}^{(\lambda)}$ may be chosen for $\Pi^{(\lambda)}$, where $\alpha, \beta = 1, 2 \cdots, f_\lambda$. If this is done for each partition (λ) of f, then the whole set forms a basis for the algebra Π_f. The basis elements have the multiplication properties

$$e_{\alpha\beta}^{(\lambda)} e_{\gamma\delta}^{(\mu)} = 0 \quad \text{for all } \alpha, \beta, \gamma, \delta, \text{ if } (\lambda) \neq (\mu),$$

$$e_{\alpha\beta}^{(\lambda)} e_{\gamma\delta}^{(\lambda)} = \delta_{\beta\gamma} e_{\alpha\delta}^{(\lambda)}, \quad \text{where } \delta_{\beta\gamma} \text{ is the Kronecker delta.}$$

Each element $\xi \in \Pi_f$ has an expression of the form

$$\xi = \sum_{(\lambda)} \sum_{\alpha\beta} \xi_{\alpha\beta}^{(\lambda)} e_{\alpha\beta}^{(\lambda)} \qquad \left(\xi_{\alpha\beta}^{(\lambda)} \in K\right).$$

It is a consequence of this that

$$e_{\alpha\beta}^{(\lambda)} \xi e_{\gamma\delta}^{(\lambda)} = \xi_{\beta\gamma}^{(\lambda)} e_{\alpha\delta}^{(\lambda)}.$$

$\xi_{\alpha\beta}^{(\lambda)}$ will be called the $\alpha\beta$ coefficient of ξ in $\Pi^{(\lambda)}$. This theory is valid in an arbitrary field K of characteristic zero. In particular, the full reduction is valid over the field of rational numbers. In what follows, f is replaced by f-2r.

THEOREM 2.2A. *The linear subset of* A_r *spanned by the elements*

$$e_{\alpha\beta}^{(\lambda)} V_j^i \quad (1 \leq \alpha, \beta \leq f_\lambda;\ 1 \leq i, j \leq M(f, r))$$

is a two-sided ideal $A_r^{(\lambda)}$ *of* A_r, *and* A_r *is a direct sum of these ideals:*

(1) $$A_r = \sum_{(\lambda)} A_r^{(\lambda)} \quad \textit{(direct)}.$$

Proof: Rule R_2 (§2.1) states that $(\alpha_1 V_j^i)(\alpha_2 V_q^p) = \alpha_1 \sigma^*(j\ p) \alpha_2 V_q^i$. If the first factor on the left side lies in $A_r^{(\lambda)}$, then $\alpha_1 \in \Pi^{(\lambda)}$, and therefore the expression on the right side lies in $A_r^{(\lambda)}$. The same is true if the second factor of the left side lies in $A_r^{(\lambda)}$.

The second result (1) follows from the fact that the set of elements $e_{\alpha\beta}^{(\lambda)} V_j^i$ forms a basis for A_r when (λ) ranges through all partitions of f - 2r, $\alpha, \beta = 1, \cdots, f_\lambda$, and $i, j = 1, \cdots, M(f, r)$.

We now take a particular ideal $A_r^{(\lambda)}$; where there is no fear of confusion, we drop the index (λ), that is, we write $e_{\alpha\beta}^{(\lambda)} = e_{\alpha\beta}$, and so forth. We denote the $\beta\gamma$ coefficient of $\sigma*(j\ p)$ in $\Pi^{(\lambda)}$ by $\sigma*^{jp}_{\beta\gamma}$. Let $e_{\alpha\beta}V^i_j = E^{ij}_{\alpha\beta}$. These elements form a ba $A^{(\lambda)}_r$. With this new notation, the multiplication rule

$$(e_{\alpha\beta}V^i_j)(e_{\gamma\delta}V^p_q) = e_{\alpha\beta}\sigma*(j\ p)e_{\gamma\delta}V^i_q$$

becomes

$$E^{ij}_{\alpha\beta}E^{pq}_{\gamma\delta} = \sigma*^{jp}_{\beta\gamma}E^{iq}_{\alpha\delta}.$$

We now simplify the system of indexing. We replace the double indices $\binom{i}{\alpha}$ $\binom{p}{\gamma}$ and $\binom{q}{\delta}$ by single indices i, j, p and q. When the capitals E are repla small letters e, the rule becomes

$$R_0. \qquad e_{ij}\,e_{pq} = \sigma^*_{jp}\,e_{iq} \quad (\sigma^*_{jp}\in K).$$

Algebras which possess a basis $\{e_{ij}\}$ with a multiplication property of R_0 are the subject of an earlier paper [3]. They have been called "generaliz trix algebras." Let A be such an algebra, and let the range of i and j be f to n, so that the dimension of A is n^2. We denote the matrix (σ^*_{ij}) by Φ. T lowing properties of such algebras will be used.

1) Either (i) A is simple, or (ii) A possesses a radical $N \neq 0$ and the r class algebra A/N is simple.

2) A is simple if and only if it possesses an identity element.

3) If Φ has rank r, then the dimension of the radical of A is $n^2 - r^2$. T of nilpotence of the radical is at most 3.

4) A is simple if and only if Φ is non-singular.

2.4. A SUFFICIENT CONDITION FOR THE SEMISIMPLICITY OF A

The algebras $A^{(\lambda)}_r$ are generalized matrix algebras. The multiplication of $A^{(\lambda)}_r$ is

$$\Phi = \Phi^{(\lambda)}_r = \left(\sigma*^{ij}_{\alpha\beta}\right),$$

that is, the element of the $\binom{i}{\alpha}$ row and the $\binom{j}{\beta}$ column is

$$\sigma*^{ij}_{\alpha\beta} = \nu(ij)\sigma^{ij}_{\alpha\beta}.$$

This quantity is an element of K and is the $\alpha\beta$ coefficient of $\sigma*(i\ j)$ in the i $\Pi^{(\lambda)}$ of Π_{f-2r}. Since $A^{(\lambda)}_r$ is simple if and only if the matrix $\Phi^{(\lambda)}_r$ is non- we consider the determinant of this matrix.

The diagonal submatrices of Φ are obtained by fixing $i = j$ and letting α range through $1 \leq \alpha, \beta \leq f_\lambda$. S_4 of §1.8 shows that $\sigma(i\ i)$ is the identity per tion. It follows that $\sigma^{ii}_{\alpha\beta} = \delta_{\alpha\beta}$ (the Kronecker delta). The numerical part i $\nu(i\ i) = n^r$ (S_4) so that each diagonal submatrix of Φ is $n^r \cdot I$, where I is t matrix of order $f_\lambda \times f_\lambda$. The off-diagonal submatrices with $i \neq j$ have the nu part $\nu(i\ j)$ equal to zero or to n^k with $k < r$. Hence the main diagonal term provides us with a power of n in the determinant that is strictly higher than the power of any other term of the expansion. Hence det Φ is a non-zero polynomial in n with coefficients in K. The integral roots of the polynomial are then the values of n for which the algebra is non-semisimple. For all other integers the algebra is simple. Since det Φ has only a finite number of roots, the algebra is simple for all sufficiently large n.

The algebras $A_r^{(\lambda)}$ are two-sided ideals of A_r. Since A_r is a direct sum of a finite number of them, the result extends to the semisimplicity of A_r.

THEOREM 2.4A. *For a given integer* f, *the algebra* A_r *is semisimple for all sufficiently large* n.

2.5. THE EMBEDDING OF Π_{f-2r} IN A_r

When σ ranges through $\mathfrak{S}_{f-2r}$, the elements $\frac{1}{n^r}\sigma V_i^i$ of A_r form a group isomorphic to $\mathfrak{S}_{f-2r}$. However if we denote such elements by $\underline{\sigma}$, we do not have agreement between the operation of σ as an operator and $\underline{\sigma}$ as an element of the algebra; that is, in general $\underline{\sigma} V \neq \sigma V$. In this section we show that when A_r is semisimple, a satisfactory type of embedding may be obtained.

Let A_r be semisimple; then in particular there exists an identity element e_r in A_r. We define $\underline{\sigma} = \sigma e_r$. The mapping $\sigma \to \underline{\sigma}$ is a homomorphism of $\mathfrak{S}_{f-2r}$ onto a subgroup of A_r. Indeed,

$$\begin{aligned} \underline{\sigma}\,\underline{\tau} &= (\sigma e_r)(\tau e_r) \\ &= \sigma(e_r(\tau e_r)) && \text{(by } R_3\text{)} \\ &= \sigma(\tau e_r) = (\sigma\tau)e_r && \text{(by } R_1\text{)} \\ &= \underline{\sigma\tau} \end{aligned}$$

The mapping is an isomorphism, since the further mapping

$$\underline{\sigma} \to \frac{1}{n^r}\underline{\sigma} V_i^i = \frac{1}{n^r}\sigma V_i^i$$

is onto a group isomorphic to $\mathfrak{S}_{f-2r}$.

The embedding is now extended to elements α of the group algebra, by defining $\underline{\alpha} = \alpha e_r$.

2.6. THE STRUCTURE OF A_r IN THE SEMISIMPLE CASE

In Theorem 2.2A we have obtained an expression of A_r as a direct sum of two-sided ideals:

$$A_r = \sum_{(\lambda)} A_r^{(\lambda)} \quad \text{(direct).} \tag{1}$$

The ideals $A_r^{(\lambda)}$ are generalized matrix algebras, so that in the semisimple case this is the full decomposition of A_r into simple ideals. The $A_r^{(\lambda)}$ are indeed total matrix algebras over the ground field K. $A_r^{(\lambda)}$ has degree $f_\lambda \cdot M(f, r)$ over K.

Since $\underline{\sigma} V_j^i = \sigma V_j^i$, we may regard the products $\underline{\sigma} V_j^i$ as a basis for A_r. any basis $\{\alpha\}$ of Π_{f-2r} provides a basis $\{\underline{\alpha} V_j^i\}$ for A_r, where $1 \leq i, j \leq$ The ideal $A_r^{(\lambda)}$ is then spanned by the products $\underline{\alpha} V_j^i$, where $\underline{\alpha} = \alpha e_r$, $\alpha \epsilon$ $1 \leq i, j \leq M(f, r)$. The set of elements $\underline{\alpha}$ with α in $\Pi^{(\lambda)}$ form a subalgeb of A_r which is isomorphic to $\Pi^{(\lambda)}$. Hence $\underline{\Pi}^{(\lambda)}$ is a total matrix algebra f_λ that is a subalgebra of the total matrix algebra $A_r^{(\lambda)}$ of degree $f_\lambda \cdot M(f,$ follows that $A_r^{(\lambda)}$ contains a total matrix subalgebra $C_r^{(\lambda)}$ of degree $M(f, r)$ such that $A_r^{(\lambda)}$ may be written as a direct product:

(2)
$$A_r^{(\lambda)} = \underline{\Pi}^{(\lambda)} \times C_r^{(\lambda)}.$$

(1) may now be rewritten as

(3)
$$A_r = \sum_{(\lambda)} \underline{\Pi}^{(\lambda)} \times C_r^{(\lambda)} \quad \text{(direct)}.$$

While the $C_r^{(\lambda)}$ are not unique, we may select one in each $A_r^{(\lambda)}$. Since each matrix algebra of degree $M(f, r)$ over K, they are all isomorphic. Let C_r algebra which is isomorphic to them. Since Π_{f-2r} has the decomposition

$$\Pi_{f-2r} = \sum_{(\lambda)} \Pi^{(\lambda)} \quad \text{(direct)}.$$

a corresponding decomposition can be given for the direct product of Π_{f-2r} So we have

$$\Pi_{f-2r} \times C_r = \Big(\sum_{(\lambda)} \Pi^{(\lambda)}\Big) \times C_r$$
$$\cong \sum_{(\lambda)} \Big(\Pi^{(\lambda)} \times C_r\Big),$$

where both sums are direct.

Let $\alpha \; \epsilon \; \Pi_{f-2r}$ and $\alpha = \sum_{(\lambda)} \alpha^{(\lambda)}$, where $\alpha^{(\lambda)} \; \epsilon \; \Pi^{(\lambda)}$ for each (λ). F (λ) we will denote by γ and $\gamma^{(\lambda)}$ elements of C_r and $C_r^{(\lambda)}$ which correspon the isomorphism. Then the mapping given by

$$\alpha\gamma = \Big(\sum \alpha^{(\lambda)}\Big)\gamma \rightarrow \sum \alpha^{(\lambda)}\gamma \rightarrow \sum \underline{\alpha}^{(\lambda)}\gamma^{(\lambda)}$$

is an isomorphism between $\Pi_{f-2r} \times C_r$ and A_r. Under this mapping the set $\Pi_{f-2r} \times I_r$, where I_r is the identity element of C_r, corresponds to the subal $\underline{\Pi}_{f-2r}$ of A_r. If ε_r denotes the identity element of Π_{f-2r}, then we denote corresponding to $\varepsilon_r \times C_r$ by $\underline{C}_r$, and we have

$$A_r = \underline{\Pi}_{f-2r} \times \underline{C}_r.$$

THEOREM 2.6A. *In the semisimple case the algebra* A_r *contains an is copy of the group algebra of the symmetric group* $\underline{\Pi}_{f-2r}$, *and a total matrix* $\underline{C}_r$ *of degree* $M(f, r)$ *over* K; *and it is the direct product of them.*

CHAPTER III

THE STRUCTURE OF ω_f^n

3.1. THE SEMISIMPLICITY OF ω_f^n

LEMMA 3.1A. *If an algebra* $\mathfrak{A}$ *possesses an ideal* $\mathfrak{A}_0$ *that is semisimple, and if* $\mathfrak{A}/\mathfrak{A}_0$ *is semisimple, then the algebra* $\mathfrak{A}$ *is semisimple.*

Proof. Let $\mathfrak{N}$ be the radical of $\mathfrak{A}$; then $\mathfrak{A}_0 \cap \mathfrak{N}$ is contained in the radical of $\mathfrak{A}_0$. Since $\mathfrak{A}_0$ is semisimple, it follows that $\mathfrak{A}_0 \cap \mathfrak{N} = 0$.

In the canonical mapping of $\mathfrak{A}$ onto the residue class algebra $\mathfrak{A}/\mathfrak{A}_0$, the radical is mapped onto $(\mathfrak{A}_0 + \mathfrak{N})/\mathfrak{A}_0$. By the third isomorphism theorem, this is isomorphic to $\mathfrak{N}/(\mathfrak{A}_0 \cap \mathfrak{A}) = \mathfrak{N}$, since $\mathfrak{A}_0 \cap \mathfrak{N} = 0$. Hence $\mathfrak{N}$ is mapped isomorphically onto an ideal of $\mathfrak{A}/\mathfrak{A}_0$. The semisimplicity of $\mathfrak{A}/\mathfrak{A}_0$ shows that $\mathfrak{N}$ must be zero.

In §1.5 we obtained a chain of ideals of ω_f^n, namely:

$$\omega_f^n = \mathfrak{A}_0^* \supset \mathfrak{A}_1^* \supset \cdots \supset \mathfrak{A}_r^* \supset \cdots \supset \mathfrak{A}_m^* \qquad (m = [f/2]).$$

The factors of this chain are the residue class algebras $A_r = \mathfrak{A}_r^*/\mathfrak{A}_{r+1}^*$. The result of Lemma 3.1A may be applied inductively to these factors. ω_f^n is therefore semisimple if and only if each factor of the chain is semisimple. Since there are only a finite number of factors, the result of Theorem 2.4A applies to ω_f^n and we have

THEOREM 3.1B. *For a given integer* f, *the algebra* ω_f^n *is semisimple for all sufficiently large* n.

3.2. THE STRUCTURE OF ω_f^n IN THE SEMISIMPLE CASE

Theorem 3.1B shows that, for suitable values of the integers n and f, the algebra ω_f^n is semisimple. In such cases the structure theory given in Chapter II for the algebras A_r provides a complete structure theory for ω_f^n. The structure theory in the non-semisimple cases appears to be much more difficult and is not considered in this paper.

We suppose then that ω_f^n is semisimple. Let ε_r^* be the identity element of the ideal $\mathfrak{A}_r^*$ of ω_f^n (§1.5). We define $\varepsilon_r = \varepsilon_r^* - \varepsilon_{r+1}^*$. ε_r is then the identity element for an ideal $\mathfrak{A}_r$ of ω_f^n, and $\mathfrak{A}_r$ is the complementary ideal of $\mathfrak{A}_{r+1}^*$ in $\mathfrak{A}_r^*$, that is,

$$\mathfrak{A}_r^* = \mathfrak{A}_r + \mathfrak{A}_{r+1}^* \qquad \text{(direct sum)}.$$

This amounts to taking the Peirce decomposition of $\mathfrak{A}_r^*$ (see [1, Theorem 4.48]). Then we have that $\mathfrak{A}_r \cong A_r = \mathfrak{A}_r^*/\mathfrak{A}_{r+1}^*$. This decomposition leads inductively to an initial decomposition of ω_f^n:

$$\omega_f^n = \mathfrak{A}_0 + \mathfrak{A}_1 + \cdots + \mathfrak{A}_r + \cdots + \mathfrak{A}_m \qquad \text{(direct sum)}.$$

Since the summands in this decomposition are isomorphic to the corresponding factors in the chain (§1.5), that is, since $\mathfrak{A}_r \cong A_r$, we may apply the structure theory of Chapter II to them. We denote by $\mathfrak{A}_r^{(\lambda)}$ that ideal of $\mathfrak{A}_r$ which is isomorphic to $A_r^{(\lambda)}$. The results of Chapter II which may now be applied to ω_f^n are summarized in the following theorem.

MAIN THEOREM 3.2A. *In the semisimple case the algebra* ω_f^n *may b pressed as*

$$\omega_f^n = \sum_{r=0}^{m} \mathfrak{A}_r \quad (direct) = \sum_{r=0}^{m} \left(\Pi_{f-2r} \times \mathfrak{C}_r\right) \quad (direct),$$

where $m = [f/2]$. Π_{f-2r} *is a subalgebra of* $\mathfrak{A}_r$ *that is isomorphic to the g algebra of the symmetric group* $\mathfrak{S}_{f-2r}$. $\mathfrak{C}_r$ *is a total matrix algebra of d* $M(f, r)$ *over the ground field. Each summand is a two-sided ideal of* ω_f^n, *has the further decomposition*

$$\mathfrak{A}_r = \Pi_{f-2r} \times \mathfrak{C}_r = \sum_{(\lambda)} \Pi^{(\lambda)} \times \mathfrak{C}_r$$

$$= \sum_{(\lambda)} \mathfrak{A}_r^{(\lambda)} \quad (direct).$$

Each $\mathfrak{A}_r^{(\lambda)} = \Pi^{(\lambda)} \times \mathfrak{C}_r$ *is a simple two-sided ideal of* $\mathfrak{A}_r$. $\Pi^{(\lambda)}$ *is that sir ideal of* Π_{f-2r} *which corresponds to the partition* (λ) *of* $f - 2r$. *The sumn over all partitions* (λ).

The dimension of the simple ideal $\mathfrak{A}_r^{(\lambda)}$ *is* $f_\lambda^2 \cdot [M(f, r)]^2$, *where* f_λ *is t gree of* $\Pi^{(\lambda)}$ *over the ground field, and* $M(f, r)$ *is the degree of* $\mathfrak{C}_r$:

$$M(f, r) = \frac{f!}{2^r r! (f - 2r)!}.$$

CHAPTER IV

REPRESENTATIONS OF ω_f^n AND O(n) ON TENSOR SPACE

4.1. INTRODUCTION

Let P be a vector space of dimension n over K. At the outset, a basis $e_1, e_2, \cdots, e_n$ is chosen in P. A scalar product (x, y) is defined for element and y of P: if

$$x = \sum_{i=1}^{n} x_i e_i \quad \text{and} \quad y = \sum_{i=1}^{n} y_i e_i,$$

then $(x, y) = \sum_i x_i y_i$. The group O(n) of orthogonal transformations on P fined, relative to this scalar product, as the set of those nonsingular linear t mations A of P, $x \to Ax$, for which $(Ax, Ay) = (x, y)$ for all x and y in P.

Let P_f be the space of tensors of rank f over P. We take as basis for t the "products" $e_{i_1} e_{i_2} \cdots e_{i_f}$, where each index i has the range 1 to n. We d these products by $\varepsilon(i_1 i_2 \cdots i_f)$ or simply by $\varepsilon[i]$, $[i]$ denoting the f-tuple $(i_1 i_2 \cdots i_f)$. A tensor F of P_f may then be written in terms of its components as $F = \sum_{[i]} F[i]\,\varepsilon[i]$. A scalar product of tensors F and G in P_f is defined as $\sum_{[i]} F[i]\,G[i]$.

If a linear transformation A of O(n) is performed in P, the components of a general tensor of rank f undergo a linear transformation A^f. The set of transformations $\{A^f\}$ corresponding to elements A of O(n) forms a representation of O(n). For our present purposes, since bases for P and P_f have been chosen, an element A of O(n) may be regarded as an orthogonal matrix, and the corresponding element A^f as its Kronecker f^{th} power $A\times A\times\cdots\times A$ (f times).

In using f-tuples such as $[i]$ as suffixes for the components of tensors or matrices, we will need some method of ordering them. If the range of the i's is 1 to n, then there are n^f distinct f-tuples $[i] = (i_1 i_2 \cdots i_f)$. We assume that lexicographic ordering has been selected. The entry in the $[i]^{th}$ row and $[j]^{th}$ column of the matrix of any linear transformation of P_f may be written

$$a([i], [j]) = a(i_1 i_2 \cdots i_f;\ j_1 j_2 \cdots j_f),$$

and the matrix itself may be written as $[a([i], [j])]$. If $A^f = [a([i], [j])]$, then

$$a([i], [j]) = a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_f j_f},$$

where a_{ij} is the element of the i^{th} row and j^{th} column of A. Symbols such as F will be used to represent both a tensor of P_f and the column vector whose $[j]^{th}$ component is $F[j]$. Then, for any linear transformation U of P_f, UF will denote both the transform of the tensor F by U and the product of the $n^f \times n^f$ matrix U by the $n^f \times 1$ matrix F. So if $U = [u([i], [j])]$, we have

$$(UF)[i] = \sum_{[j]} u([i], [j])\,F[j].$$

For ω_f^n, the representation on P_f which we consider has been described by Brauer [2, Section 5]. The correspondence between a diagram U and the matrix U by which it is represented is as follows: f indices $i_1, i_2, \cdots, i_f$ are associated with the dots in the upper row of the diagram, and f indices $j_1, j_2, \cdots, j_f$ with the dots of the lower row; and

$$U([i], [j]) = \delta(k_1 k_2)\,\delta(k_3 k_4) \cdots \delta(k_{2f-1} k_{2f}),$$

where $k_1, k_2, \cdots, k_{2f}$ are the indices $i_1, i_2, \cdots, i_f, j_1, \cdots, j_f$, arranged in such a way that the dots corresponding to k_{2s-1} and k_{2s} are joined in U.

The importance of ω_f^n arises from the fact that, for a given n, its representation on P_f consists of all matrices which commute with every matrix of the representation of O(n) on P_f. It follows that if U denotes an element of ω_f^n and the transformation by which it is represented, then $UP_f = \{UF \mid F \in P_f\}$ is an invariant subspace in P_f for the representation of O(n). Indeed, if $A \in O(n)$, we have

$$A^f(UF) = (A^f U)F = (UA^f)F = U(A^f F) \in UP_f.$$

4.2. WEYL'S DECOMPOSITION OF TENSOR SPACE

A subspace of tensor space which is invariant under the representation will in the future be called simply an invariant subspace. The concluding p of §4.1 shows that, for each U in ω_f^n, UP_f is an invariant subspace.

In Chapter V, §6 of [5], Weyl obtains a decomposition of tensor space I invariant subspaces P_f^r. The subspaces are obtained by "trace operations' present chapter we will identify P_f^r with the invariant subspace $\varepsilon_r P_f$, whe is the identity element of the ideal $\mathfrak{A}_r$. Consequently we assume the existe $\mathfrak{A}_r$ and of its identity element. This assumption is not valid in general, so section of our theory is incomplete. Nevertheless, Weyl's decomposition o it is made without reference to ω_f^n, is valid as a direct decomposition of P variant subspaces, for any values of n and f. The remainder of this sectic description of Weyl's decomposition.

The 12-trace of a tensor F of P_f is a tensor F_{12} of P_{f-2} whose comp are obtained from those of F by contraction with respect to the first pair c In terms of components:

$$F_{12}(i_3\, i_4 \cdots i_f) = \sum_{i=1}^{n} F(i\, i\, i_3 \cdots i_f).$$

More generally, the $\alpha\beta$-trace of F is defined for $1 \leq \alpha, \beta \leq f$ as the tensor with components

$$F_{\alpha\beta}(i_1 \cdots \underset{\alpha}{|} \cdots \underset{\beta}{|} \cdots i_f) = \sum_{i_\alpha,\, i_\beta} \delta(i_\alpha\, i_\beta) F(i_1 i_2 \cdots i_f)$$

(a stroke like $\underset{\alpha}{|}$ means that the argument i_α is missing). The set of tenso whose traces are all zero is an invariant subspace P_f^0 of P_f.

The subspace P_f^{1*} (which Weyl denotes by $P_f^{\dagger}$) is defined as the space tensors Φ whose components are of the form

$$\Phi[i] = \delta(i_1\, i_2) F^{12}(i_3\, i_4 \cdots i_f) + \cdots \tag{1}$$

$$\cdots + \delta(i_\alpha i_\beta) F^{\alpha\beta}(i_1 \cdots \underset{\alpha}{|} \cdots \underset{\beta}{|} \cdots i_f) + \cdots .$$

There are f(f-1)/2 summands corresponding to all choices of α, β in $1 \leq$ Each $F^{\alpha\beta}$ is an element of P_{f-2}. Weyl shows that the expression

$$P_f = P_f^0 + P_f^{1*} \tag{2}$$

gives a direct decomposition of P_f into invariant subspaces. The subspace then treated in a similar way. A subspace P_f^1 is taken, of tensors of type but with the additional restriction that each tensor $F^{\alpha\beta}$ of P_{f-2} which occu have its traces zero, that is, $F^{\alpha\beta} \in P_{f-2}^0$.

The definition proceeds. P_f^{r*} is defined as the subspace of P_f spanned by tensors whose components are of the type

$$(3) \qquad \delta(i_{\alpha_1} i_{\alpha'_1}) \cdots \delta(i_{\alpha_r} i_{\alpha'_r}) \phi(i_{\beta_1} i_{\beta_2} \cdots i_{\beta_v}).$$

Here ϕ is any tensor of rank $v = f - 2r$, that is, $\phi \in P_v$.

$$|\alpha_1 \alpha'_1 \mid \alpha_2 \alpha'_2 \mid \cdots \mid \alpha_r \alpha'_r \mid \beta_1 \beta_2 \cdots \beta_v|$$

is any dissection of the row of indices 1, 2, $\cdots$ f into r portions of length 2 and one of length v; the arrangement of the r portions of length 2 and the order of the individual members within each portion are immaterial.

P_f^r is defined as the subspace of P_f^{r*} for which the tensors ϕ in (3) are subjected to the restriction that all their traces vanish, that is, that ϕ belongs to P_v^0 rather than to P_v. Weyl gives the name "tensors of valence r" to the elements of P_f^r. Each P_f^r and P_f^{r*} is an invariant subspace of P_f, and the expressions

$$(4) \qquad P_f = P_f^0 + P_f^1 + \cdots + P_f^r + \cdots + P_f^m \qquad (m = [f/2])$$

and

$$(5) \qquad P_f^{r*} = P_f^r + P_f^{r+1} + \cdots + P_f^m$$

are direct decompositions of P_f and P_f^{r*} into invariant subspaces.

If $F \in P_f$ and $F_{\alpha\beta}$ in P_{f-2} is the $\alpha\beta$-trace of F, we may call $F_{\alpha\beta}$ "a first trace" of F. A trace of $F_{\alpha\beta}$ may then be called "a second trace" of F, and by repeated contraction we may define an r^{th} trace. Indeed, if $|\alpha_1 \alpha'_1 \mid \alpha_2 \alpha'_2 \mid \cdots \mid \alpha_r \alpha'_r|$ are 2r distinct integers of the set 1, 2, $\cdots$, f, then the tensor of P_{f-2r} whose components are given as

$$(6) \qquad F_{(\alpha)}(i_{\beta_1} \cdots i_{\beta_v}) = \sum_{(\alpha)} \delta(i_{\alpha_1} i_{\alpha'_1}) \cdots \delta(i_{\alpha_r} i_{\alpha'_r}) F(i_1 i_2 \cdots i_f)$$

is an r^{th} trace of F.

By the definition of the space P_f^r, any of its elements may be written as a sum of elements of type (3), with ϕ in P_{f-2r}^0. It is easy to show that, for $s > r$, every s^{th} trace of an element of P_f^r is zero.

4.3. THE SUBSPACES $\varepsilon_r P_f$

We will assign specific bar arrangements to the indices 1 and M = M(f, r) of the r-bar scheme. These are most easily represented diagrammatically (see Figure 12).

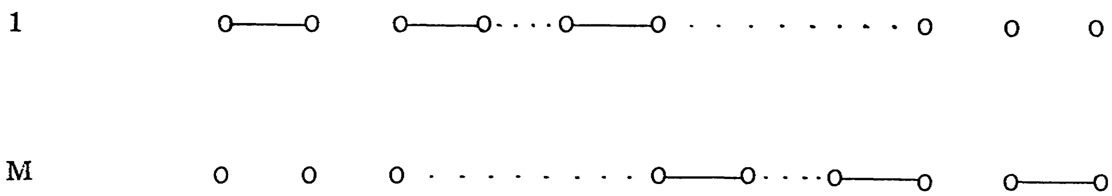

Figure 12.

LEMMA 4.3 A. *If* $U \in \mathfrak{A}_r^*$, *then* $UP \subset P_f^{r*}$.

Proof. The subspace $\mathfrak{B}_r$ of ω_f^n has a basis of diagram elements σU_j^i. show that, for any U in $\mathfrak{A}_r^*$ and each F in P_f, the tensor UF can be writte form

$$U_1^1 F_1 + U_2^2 F_2 + \cdots + U_M^M F_M,$$

with U_α^α in $\mathfrak{B}_r$ and $U_\alpha^\alpha F_\alpha$ in P_f^{r*} for each α in [1, M].

An element U of $\mathfrak{A}_r^*$ is a linear combination of basis elements whose dia have r or more bars. If $\sigma U_q^p \in \mathfrak{B}_s$, where $s \geq r$, then the upper row of it gram has s bars, and to any subset consisting of r of these bars correspor index α of the r-bar scheme. Since $U_\alpha^\alpha \in \mathfrak{B}_r$ has this bar arrangement in its rows, and since it possesses the identity permutation of $\mathfrak{S}_{f-2r}$, we see $\sigma U_q^p = \frac{1}{n^r} U_\alpha^\alpha (\sigma U_q^p)$. Any element U of $\mathfrak{A}_r^*$ may therefore be written as

$$U = U_1^1 U_1 + U_2^2 U_2 + \cdots + U_M^M U_M,$$

where the U_α^α have been defined and $U_1, \cdots, U_M$ are elements of $\mathfrak{A}_r^*$. In ge the expression is not unique. We now have that, for any F in P_f,

$$UF = \left(\sum_{\alpha=1}^{M} U_\alpha^\alpha U_\alpha \right) F = \sum U_\alpha^\alpha F_\alpha,$$

where $F_\alpha = U_\alpha F$. We now show that $U_1^1 F \in P_f^{r*}$ for every F in P_f.

$$\begin{aligned} (U_1^1 F)[i] &= \sum_{[j]} \delta(i_1\ i_2) \cdots \delta(i_{2r-1}\ i_{2r}) \delta(j_1\ j_2) \cdots \delta(j_{2r-1}\ j_{2r}) \\ &\quad \cdot \delta(i_{2r+1}\ j_{2r+1}) \cdots \delta(i_f\ j_f) F(j_1\ j_2 \cdots i_f) \\ &= \delta(i_1\ i_2) \cdots \delta(i_{2r-1}\ i_{2r}) \phi(i_{2r+1} \cdots i_f), \end{aligned}$$

where $\phi(i_{2r+1} \cdots i_f) = \sum F(i_1\ i_1\ i_2\ i_2 \cdots i_r\ i_r\ i_{2r+1} \cdots i_f)$. The summation i repeated indices. ϕ is seen to lie in P_{f-2r}, so that $U_1^1 F$ lies in P_f^{r*}. The showing that, for each F, $U_\alpha^\alpha F$ lies in P_f^{r*} is not essentially different. He UF $\in P_f^{r*}$, and the lemma is proved.

LEMMA 4.3 B. *Each tensor* Φ *in* P_f^r *may be written in the form*

$$\Phi = U_1^1 F_1 + U_2^2 F_2 + \cdots + U_M^M F_M,$$

where $U_\alpha^\alpha \in \mathfrak{B}_r$ *and* $F_\alpha \in P_f$ *for* $1 \leq \alpha \leq M$.

Proof. Φ is a linear combination of tensors of the type occurring in for (3) of §4.2. If α is the index of the r-bar scheme which is associated with t arrangement in which bars join dot α_i and dot α_i' for $1 \leq i \leq r$, then

$$(U_\alpha^\alpha F)[i] = \delta(i_{\alpha_1}\ i_{\alpha_1'}) \cdots \delta(i_{\alpha_r}\ i_{\alpha_r'}) F_{(\alpha)}(i_{\beta_1}\ i_{\beta_2} \cdots i_{\beta_v}),$$

where $F_{(\alpha)}$ is given by components in formula (6) of §4.2. It is easy to show that the mapping $F \to F_{(\alpha)}$ is of P_f onto P_{f-2r}, so that F can be chosen in such a way that $F_{(\alpha)} = \phi$, for any ϕ in P_{f-2r}.

THEOREM 4.3 C. *If the identity element* ε_r^* *of the ideal* $\mathfrak{A}_r^*$ *exists, then* $P_f^{r*} = \varepsilon_r^* P_f$.

Proof. (i) Let $F \in \varepsilon_r^* P_f$; then $F = \varepsilon_r^* F \in P_f^{r*}$ by Lemma 4.3 A, since $\varepsilon_r^* \in \mathfrak{A}_r^*$. Hence $\varepsilon_r^* P_f \subseteq P_f^{r*}$.

(ii) Let $F \in P_f^{r*}$. By Lemma 4.3 B, $F = \sum_\alpha U_\alpha^\alpha F$, with U_α^α in $\mathfrak{B}_r \subset \mathfrak{A}_r^*$. Hence $\varepsilon_r^* F = F$, and therefore $F \in \varepsilon_r^* P_f$. Hence $P_f^{r*} \subseteq \varepsilon_r^* P$, which with (i) gives the required result.

4.4. THE SUBSPACES $\varepsilon_r P_f$

The mapping $\sigma U_j^i \to \sigma^{-1} U_i^j = \hat{\sigma} \hat{U}_j^i$ of ω_f^n onto itself has been mentioned in §1.2, where it was introduced by the inversion of diagrams. We saw in P_4 that for basis elements it is an involution. By linearity it may be extended, as an involution, to the whole algebra. If an element U of ω_f^n has the matrix $[u([i], [j])]$, then $\hat{U}$ has the matrix $[u([j], [i])]$, i.e. the matrix for $\hat{U}$ is the transpose U^T of the matrix for U. It follows that if F and G are elements of P_f, and if U is an element of ω_f^n, then

$$(UF, G) = (F, U^T G) = (F, \hat{U}G),$$

where (F, G) is the scalar product defined in §4.1. We will, in the remainder of this chapter, assume the semisimplicity of $\mathfrak{A}_r^*$. This will imply the semisimplicity of $\mathfrak{A}_r$ and $\mathfrak{A}_{r+1}^*$ and the existence of identity elements ε_r^*, ε_r and ε_{r+1}^*.

LEMMA 4.4 A. $\varepsilon_r^* = \hat{\varepsilon}_r^*$.

Proof. Since the involute of an s-bar diagram is again an s-bar diagram, $\hat{\mathfrak{A}}_r^* = \mathfrak{A}_r^*$. In particular, $\hat{\varepsilon}_r^* \in \mathfrak{A}_r^*$. Since $\hat{\varepsilon}_r^*$ is an identity element for $\hat{\mathfrak{A}}_r^*$, it is also an identity for $\mathfrak{A}_r^*$. The uniqueness of a two-sided identity element gives the result.

COROLLARY 4.4 B. $\hat{\varepsilon}_r = \varepsilon_r$.

This is an immediate consequence of the lemma, since $\varepsilon_r = \varepsilon_r^* - \varepsilon_{r+1}^*$. Now it follows that, for any F and G in P_f, $(\varepsilon_r F, G) = (F, \varepsilon_r G)$.

Weyl's method of obtaining the decomposition of tensor space [(§4.2, formula (4)] amounts to showing that

$$(1) \qquad P_f^{r*} = P_f^r + P_f^{r+1*}$$

is a direct decomposition of an invariant subspace P_f^{r*} into invariant subspaces P_f^r and P_f^{r+1*}. For this Weyl shows that, over a real field, P_f^r is the orthogonal complement of P_f^{r+1*} in P_f^{r*} relative to the scalar product (F, G). An irrelevance argument shows that the decomposition is direct for an arbitrary field of characteristic zero.

THEOREM 4.4 C. $\varepsilon_r P_f = P_f^r$.

We first assume that the field K is real. We also assume Weyl's result that P_f^r is the orthogonal complement of P_f^{r+1*} in P_f^{r*}.

By Theorem 4.3 C, and by the fact that $\varepsilon_r^* = \varepsilon_r + \varepsilon_{r+1}^*$, we have

$$P_f^{r*} = \varepsilon_r P_f + \varepsilon_{r+1}^* P_f. \tag{2}$$

In addition, we have Weyl's decomposition

$$P_f^{r*} = P_f^r + P_f^{r+1*}. \tag{1}$$

Since $P_f^{r+1*} = \varepsilon_{r+1}^* P_f$, the subspace P_f^{r+1*} is common to both decomposit
Over a real field, the orthogonal complement is unique. It is therefore suff
to show that every element of $\varepsilon_r P_f$ is orthogonal to every element of

$$P_f^{r*} = \varepsilon_{r+1}^* P_f.$$

This is so since $(\varepsilon_r F, \varepsilon_{r+1}^* G) = (F, \varepsilon_r \varepsilon_{r+1}^* G) = 0$ for all F and G in P
when the field is real, $\varepsilon_r P_f = P_f^r$.

The whole theory up to this point is valid for any field K of characteris
In particular, the construction of identity elements ε_r goes through in the
rationals. The validity of our theorem over the rationals ensures that it is
over any extension.

Weyl has shown [5] that the algebra is semisimple for $n \geq 2f$. For this
makes use of certain properties of the representation of the algebra. A mo
application of his method can be used to show that the algebra is semisimpl
only if $n \geq f - 1$. It is expected that a paper exhibiting this result will be pu
shortly in the Annals of Mathematics.

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