The Interaction of Dirac Particles
with Non-Abelian Gauge Fields
and Gravity—Black Holes

FELIX FINSTER, JOEL SMOLLER,
& SHING-TUNG YAU

1. Introduction

Recently, the Einstein–Dirac–Yang/Mills (EDYM) equations were derived for a static, spherically symmetric system of a Dirac particle that interacts with both a gravitational field and the magnetic component of an SU(2) Yang/Mills field [5]. This system was constructed by choosing a representation of the rotation group that acts nontrivially on the YM index and by making an *ansatz* involving two real functions for the Dirac wave function, which is invariant under this group representation. The resulting equations were shown to admit stable particle-like solutions for physically relevant values of the coupling constants. In this paper, we study black-hole solutions of these EDYM equations.

We prove that the only black-hole solutions of our EDYM equations are the Bartnik–McKinnon (BM) black holes; that is, the spinors must vanish identically. In other words, the EDYM equations do not admit normalizable black-hole solutions. Thus, in the presence of quantum mechanical Dirac particles, static and spherically symmetric black-hole solutions do not exist. Another interpretation of our result is that Dirac particles can only either disappear into the black hole or escape to infinity. These results are proved under very weak regularity assumptions on the form of the event horizon; see assumptions (I)–(III) in the next section.

Our work here is a continuation of [3], where we showed that the Einstein–Dirac–Maxwell (EDM) equations do not admit normalizable black-hole solutions. However, in contrast to the EDM system [3], where an electric field is present, here we consider the influence of a magnetic field—more precisely, the influence of the magnetic component of a non-abelian gauge field. Furthermore, we point out that the Dirac particle considered here has zero total angular momentum. In analogy to [3], one could also form a spherically symmetric system out of $(2j+1)$ Dirac particles each having angular momentum j, $j = 1, 2, \ldots$. However, the EDYM equations for such a system would involve four real spinor functions. This is a considerably more difficult problem, which we are presently investigating.

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2. The Coupled EDYM Equations

In [5] we derived the spherically symmetric, static EDYM system of a Dirac parti-
cle and an SU(2) YM field with vanishing electric component. We shall not repeat
this derivation here, but merely write down the EDYM equations. We consider a
Lorentzian metric in polar coordinates $(t, r, \vartheta, \varphi)$ of the form

$$ds^2 = \frac{1}{T(r)^2} \, dt^2 - \frac{1}{A(r)} \, dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2)$$

with positive functions A and T. The Dirac wave function is described by two real
functions $(\alpha(r), \beta(r))$, and the potential $w(r)$ corresponds to the magnetic com-
ponent of an SU(2) YM field. Then the EDYM equations are

$$\sqrt{A}\alpha' = \frac{w}{r}\alpha - (m + \omega T)\beta, \tag{2.1}$$

$$\sqrt{A}\beta' = (-m + \omega T)\alpha - \frac{w}{r}\beta, \tag{2.2}$$

$$rA' = 1 - A - \frac{1}{e^2}\frac{(1 - w^2)^2}{r^2} - 2\omega T^2(\alpha^2 + \beta^2) - \frac{2}{e^2}Aw'^2, \tag{2.3}$$

$$2rA\frac{T'}{T} = -1 + A + \frac{1}{e^2}\frac{(1 - w^2)^2}{r^2} + 2mT(\alpha^2 - \beta^2) - 2\omega T^2(\alpha^2 + \beta^2)$$

$$+ 4\frac{T}{r}w\alpha\beta - \frac{2}{e^2}Aw'^2, \tag{2.4}$$

$$r^2 Aw'' = -(1 - w^2)w + e^2 rT\alpha\beta - r^2\frac{A'T - 2AT'}{2T}w'. \tag{2.5}$$

Equations (2.1) and (2.2) are the Dirac equations, (2.3) and (2.4) are the Einstein
equations, and (2.5) is the Yang/Mills equation. Here the constants m, ω, and e
are (respectively) the rest mass of the Dirac particle, its energy, and the YM coup-
ling constant.

We shall here consider black-hole solutions of this EDYM system. Thus we as-
sume (as in [3]) that the surface $r = \rho > 0$ is a black-hole event horizon:

$$A(\rho) = 0, \qquad A(r) > 0 \quad \text{if } r > \rho. \tag{2.6}$$

In addition, we assume the following conditions:

$$\int_{r_0}^{\infty} (\alpha^2 + \beta^2)\frac{\sqrt{T}}{A} \, dr < \infty \quad \text{for every } r_0 > \rho \tag{2.7}$$

(the spinors are normalizable),

$$\lim_{r \to \infty} \frac{r}{2}(1 - A(r)) < \infty \tag{2.8}$$

(finite ADM mass),

$$\lim_{r \to \infty} T(r) = 1$$

(the gravitational field is asymptotically flat Minkowskian), and

$$\lim_{r \to \infty}(w(r), w'(r)) \text{ is finite} \tag{2.9}$$

(well-behavedness of the Yang/Mills field).

Concerning the event horizon $r = \rho$, we make the following three regularity assumptions (cf. [3]).

(I) The volume element $\sqrt{|\det g_{ij}|} = |\sin \vartheta| r^2 A^{-1} T^{-2}$ is smooth and nonzero on the horizon; that is,

$$T^{-2}A^{-1}, T^2 A \in C^1([\rho, \infty)).$$

(II) The strength of the Yang/Mills field F_{ij} is given by

$$\text{Tr}(F_{ij}F^{ij}) = \frac{2Aw'^2}{r^2} + \frac{(1 - w^2)^2}{r^4}$$

(cf. [2]). We assume that this scalar is bounded near the horizon; that is, we assume that outside the event horizon and near $r = \rho$,

$$w \text{ and } Aw'^2 \text{ are bounded.} \tag{2.10}$$

(III) The function $A(r)$ is monotone increasing outside of and near the event horizon.

As discussed in [3], if assumption (I) or (II) were violated then an observer freely falling into the black hole would feel strong forces when crossing the horizon. Assumption (III) is considerably weaker than the corresponding assumption in [3]; indeed, in [3] we assumed that the function $A(r)$ obeyed a power law $A(r) = c(r - \rho)^s + \mathcal{O}((r - \rho)^{s+1})$, with positive constants c and s, for $r > \rho$.

3. Nonexistence of Black-Hole Solutions

Our main result is the following theorem.

THEOREM 3.1. *Every black-hole solution of the EDYM equations* (2.1)–(2.5) *satisfying the regularity conditions* (I), (II), *and* (III) *cannot be normalized and coincides with a BM black hole of the corresponding Einstein–Yang/Mills (EYM) equations; that is, the spinors α and β must vanish identically outside the event horizon.*

REMARKS. (A) In [6], it was proved that any black-hole solution of the EYM equations that has finite (ADM) mass (i.e., that satisfies (2.8)) must be one of the BM black-hole solutions whose existence was first demonstrated in [7]. Thus, amending the EYM equations by taking quantum mechanical effects into account—in the sense that we allow both the gravitational and Yang/Mills fields to interact with Dirac particles—does not yield any new types of black-hole solutions.

(B) In [3] we studied black-hole solutions of the Einstein–Dirac–Maxwell equations, and we proved a corresponding result for this EDM system. As mentioned before, in [3] we assumed that near the event horizon,

$$A(r) = c(r - \rho)^s + \mathcal{O}((r - \rho)^{s+1}),$$

where $c > 0$ and $s > 0$. We were able to prove that if $s \neq 2$ then the spinors must vanish outside of the event horizon; if $s = 2$, we could only obtain this result numerically. Theorem 3.1 thus gives an even stronger result in the case of a YM field.

In the remainder of this paper, we shall give the proof of Theorem 3.1. We assume that we have a black-hole solution of the EDYM equations (2.1)–(2.9), with event horizon at $r = \rho$, satisfying the regularity assumptions (I)–(III), and where the spinors $(\alpha(r), \beta(r))$ do not vanish identically in the region $r > \rho$. We will show that this leads to a contradiction. We consider two cases: either $A^{-1/2}$ is integrable near the horizon or $A^{-1/2}$ fails to be integrable near the horizon.

Case 1: $A^{-1/2}$ Is Integrable Near $r = \rho$

In order to obtain the desired contradiction, we shall need a few preliminary results as follows.

LEMMA 3.2.   *If $A^{-1/2}$ is integrable near $r = \rho$, then there exist positive constants c, ε such that*

$$c \leq \alpha^2(r) + \beta^2(r) \leq \frac{1}{c} \quad \text{if } \rho < r < \rho + \varepsilon. \tag{3.1}$$

Proof. If we multiply (2.1) by α and (2.2) by β and then add, we obtain

$$\sqrt{A}\,\frac{d}{dr}(\alpha^2 + \beta^2) = 2\begin{pmatrix} \alpha \\ \beta \end{pmatrix}^T \begin{pmatrix} w/r & -m \\ -m & -w/r \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.2}$$

$$\leq 2\sqrt{m^2 + \frac{w^2}{r^2}}(\alpha^2 + \beta^2),$$

where in the last estimate we computed the eigenvalues of the preceding 2×2 matrix. Since (α, β) is a nontrivial solution, the uniqueness theorem for ODEs implies that $(\alpha^2 + \beta^2)(r) \neq 0$ on all intervals of the form $(\rho, \rho + \varepsilon)$. Thus, dividing (3.2) by $\sqrt{A}(\alpha^2 + \beta^2)$ and integrating from $r > \rho$ to $\rho + \varepsilon$ gives

$$|\log(\alpha^2 + \beta^2)(\rho + \varepsilon) - \log(\alpha^2 + \beta^2)(r)|$$

$$\leq 2\int_r^{\rho+\varepsilon} A^{-1/2}(s)\sqrt{m^2 + \frac{w^2(s)}{s^2}}\,ds. \tag{3.3}$$

Since $w(r)$ is bounded near the event horizon (by assumption (II)) and $A^{-1/2}$ is integrable near the event horizon, we can take the limit $r \searrow \rho$ in (3.3) to achieve the desired result.   □

COROLLARY 3.3.   *If $A^{-1/2}$ is integrable near $r = \rho$, then $\omega = 0$.*

Proof. Combining (2.3) and (2.4), we have

$$r(AT^2)'$$
$$= -4\omega T^4(\alpha^2 + \beta^2) + T^3\left[2m(\alpha^2 - \beta^2) + \frac{4w}{r}\alpha\beta\right] - \frac{4}{e^2}(Aw'^2)T^2. \quad (3.4)$$

From assumption (II) and Lemma 3.2, we see that the coefficients of T^4, T^3, and T^2 on the right-hand side of (3.4) are bounded. From assumption (I) we see that the left-hand side of (3.4) is bounded near the event horizon. Since $T(r) \to \infty$ as $r \searrow \rho$ (again by (I)), we conclude from (3.4) and Lemma 3.2 that $\omega = 0$.   □

In view of this last result, we can write the Dirac equations (2.1) and (2.2) as

$$\sqrt{A}\alpha' = \frac{w}{r}\alpha - m\beta, \quad (3.5)$$

$$\sqrt{A}\beta' = -m\alpha - \frac{w}{r}\beta. \quad (3.6)$$

If we multiply (3.5) by β and (3.6) by α and then add, we obtain

$$\sqrt{A}(\alpha\beta)' = -m(\alpha^2 + \beta^2) < 0, \quad (3.7)$$

so that $\alpha\beta$ is monotone decreasing. Thus $(\alpha\beta)(r)$ has a limit for $r \to \infty$ (which might be $-\infty$). Since $\alpha^2 + \beta^2 \geq 2|\alpha\beta|$, we see that the normalization condition (2.7) will be satisfied only if this limit is zero. We thus have proved the following lemma.

LEMMA 3.4.   *If $\omega = 0$, then $(\alpha\beta)(r)$ is a positive decreasing function tending to 0 as $r \to \infty$.*

REMARK.   We did not use the fact that $A^{-1/2}$ is integrable to prove this lemma.

The YM equation (2.5) can be written equivalently as an equation for Aw',

$$r^2(Aw')' = -w(1 - w^2) + e^2\frac{r(\sqrt{AT})\alpha\beta}{\sqrt{A}} + r^2\frac{(AT^2)'}{2AT^2}(Aw'). \quad (3.8)$$

Since Aw'^2 is bounded, we see that $A^2w'^2 \to 0$ as $r \searrow \rho$ and thus $Aw' \to 0$ as $r \searrow \rho$. In view of Lemma 3.4 together with assumption (I), we see that for r near ρ we can write (3.8) in the form

$$(Aw')' = d(r) + \frac{c(r)}{\sqrt{A(r)}}, \quad (3.9)$$

where $d(r)$ is bounded and $c(r)$ is a positive function bounded away from zero near the event horizon. It thus follows from (3.9) that we have

$$(Aw')'(r) \geq d + \frac{c}{\sqrt{A(r)}} \quad (3.10)$$

for $r > \rho$ (r near ρ), where d and c are constants and $c > 0$. In order to analyze this inequality, we shall need the following lemma.

LEMMA 3.5. *If $A(r)^{-1/2}$ is integrable near $r = \rho$, then there exists a function $g(r)$ with $0 < \lim_{r \searrow \rho} g(r) < \infty$ and such that, for r near ρ,*

$$A(r) = (r - \rho)g(r). \tag{3.11}$$

Proof. First note that A cannot have a zero of infinite order at $r = \rho$. Indeed, were this not true then we could write $(r - \rho)^2 > A(r)$ for r near ρ, so that $A^{-1/2}(r) > (r - \rho)^{-1}$, thereby violating the integrability of $A^{-1/2}$. Hence there exists an $s > 0$ such that

$$g(r) \equiv \frac{A(r)}{(r - \rho)^s} \nrightarrow 0 \quad \text{as } r \searrow \rho \tag{3.12}$$

while for any ε, $s > \varepsilon > 0$,

$$(r - \rho)^\varepsilon g(r) = \frac{A(r)}{(r - \rho)^{s - \varepsilon}} \rightarrow 0 \quad \text{as } r \searrow \rho. \tag{3.13}$$

According to (3.12), there is a positive constant η such that, for all r ($\rho < r < \rho + \eta$),

$$\frac{1}{\sqrt{A(r)}} = (r - \rho)^{-s/2} \frac{1}{\sqrt{g(r)}}; \tag{3.14}$$

since $A > 0$, we see that $g^{-1/2} > 0$. Pick any $\bar{r}$ in the interval $(\rho, \rho + \eta)$. Setting $\delta = \bar{r} - \rho$, (3.14) yields that

$$\frac{1}{\sqrt{A(\bar{r})}} = \delta^{-s/2} \frac{1}{\sqrt{g(\bar{r})}}. \tag{3.15}$$

Since $A \searrow 0$ monotonically as $r \searrow \rho$, we have $A^{-1/2}$ tending monotonically to infinity and so

$$\frac{1}{\sqrt{A(r)}} > \delta^{-s/2} \frac{1}{\sqrt{g(\bar{r})}}, \quad \rho < r < \bar{r}. \tag{3.16}$$

If we now integrate (3.10) from ρ to $\bar{r}$ and note that $(Aw')(\rho) = 0$, we obtain

$$(Aw')(\bar{r}) \geq d(\bar{r} - \rho) + c\delta^{-s/2} \frac{1}{\sqrt{g(\bar{r})}}(\bar{r} - \rho)$$

$$= d\delta + \frac{c}{\sqrt{g(\bar{r})}} \delta^{1-s/2}, \tag{3.17}$$

where we have used (3.16). On the other hand, writing $B(\bar{r}) := (\sqrt{A}w')(\bar{r})$, we see that B is uniformly bounded near the horizon because of (2.10). Then (3.15) gives

$$(Aw')(\bar{r}) = \sqrt{A(\bar{r})}(\sqrt{A}w')(\bar{r}) = B\sqrt{g(\bar{r})}\delta^{s/2}.$$

Using this together with (3.17), for sufficiently small δ and for $\varepsilon > 0$ we have

$$B \geq \frac{d\delta^{1-s/2+\varepsilon/2}}{\sqrt{\delta^\varepsilon g(\bar{r})}} + \frac{c\delta^{1-s+\varepsilon}}{\delta^\varepsilon g(\bar{r})} = \frac{\delta^{1-s+\varepsilon}\left[d\sqrt{\delta^\varepsilon g(\bar{r})}\delta^{(s-\varepsilon)/2} + c\right]}{\delta^\varepsilon g(\bar{r})}$$

$$\geq \frac{\delta^{1-s+\varepsilon}\left[-\frac{c}{2} + c\right]}{\delta^\varepsilon g(\bar{r})} \quad \text{(if } \delta \text{ is small)}$$

$$\geq \delta^{1-s+\varepsilon}\frac{c}{2}, \tag{3.18}$$

so that $1 - s + \varepsilon \geq 0$ and thus

$$1 \geq s.$$

To obtain the reverse inequality, let $0 < \varepsilon < 1$ and note that

$$\frac{A(r)}{(r-\rho)^\varepsilon} = \frac{A(r)}{(r-\rho)}(r-\rho)^{1-\varepsilon} \to 0$$

as $r \searrow \rho$, because (2.3) (with $\omega = 0$) shows that A' is bounded at ρ; thus, $s \geq 1$.

We know $A(r) = (r-\rho)g(r)$ and that $\lim_{r\searrow\rho}(r-\rho)^{-1}A(r)$ exists. Hence $\lim_{r\searrow\rho} g(r)$ exists and is finite. Since (3.12) shows that $\limsup_{r\searrow\rho} g(r) > 0$, we see that $\lim_{r\searrow\rho} g(r) > 0$. □

From (3.11) we have

$$A(\rho) = 0, \qquad A'(\rho) > 0,$$

and $A'(\rho)$ is finite. Our Einstein metric thus has the same qualitative features as the Schwarzschild metric. Hence, the metric singularity can be removed via a Kruskal transformation [1]. In these Kruskal coordinates, the Yang/Mills potential is continuous and bounded (as is easily verified). As a consequence, the arguments from [4] go through and show that the spinors must vanish identically outside the horizon. For this, one must notice that continuous zero-order terms in the Dirac operator are irrelevant for the derivation of the matching conditions in [4, Sec. 2.4]. Thus, the matching conditions [4, (2.31), (2.34)] are valid without changes in the presence of our YM field. Using conservation of the (electromagnetic) Dirac current and its positivity in time-like directions, the arguments of [4, Sec. 4] all carry over. Thus, the proof of Theorem 3.1 in the case that $A^{-1/2}$ is integrable near the event horizon is considered complete.

Case 2: $A(r)^{-1/2}$ Is Not Integrable Near $r = \rho$

We break the proof up into two subcases:

(i) $\omega \neq 0$,
(ii) $\omega = 0$.

Suppose first that we are in case (i), $\omega \neq 0$. We begin with the following lemma.

LEMMA 3.6.   *The function* $[|(m/\omega T)'| + |(w/r\omega T)'|]$ *is integrable near the horizon* $r = \rho$.

Proof. We first consider $|(1/T)'|$. Toward this end, we write $1/T = \sqrt{A}/\sqrt{A}T$ and compute

$$\left|\left(\frac{1}{T}\right)'\right| = \frac{\left|(\sqrt{A}T)(\sqrt{A})' - \sqrt{A}(\sqrt{A}T)'\right|}{AT^2} \leq \left|\frac{(\sqrt{A}T)(\sqrt{A})'}{AT^2}\right| + \left|\frac{\sqrt{A}(\sqrt{A}T)'}{AT^2}\right|.$$

According to assumption (I), near the event horizon $r = \rho$, AT^2 is bounded away from zero and $\left|(\sqrt{A}T)'\right|$ is bounded and thus integrable. Moreover, since A is increasing near $r = \rho$, we see that $\left|\sqrt{A}T(\sqrt{A})'\right| = (\sqrt{A}T)(\sqrt{A})'$, and since $A' > 0$, this term too is integrable near $r = \rho$. Thus $|(1/T)'|$ is integrable near $r = \rho$.

We handle $|(w/rT)'|$ similarly; namely, write

$$\frac{w}{rT} = \frac{\sqrt{A}w}{r(\sqrt{A}T)}.$$

Then

$$\left|\left(\frac{w}{rT}\right)'\right| = \frac{\left|(r\sqrt{A}T)(\sqrt{A}w)' - (\sqrt{A}w)(r\sqrt{A}T)'\right|}{r^2AT^2}$$

$$= \frac{\left|r(\sqrt{A}T)\left[\sqrt{A}w' + (\sqrt{A})'w\right] - (\sqrt{A}w)\left[r(\sqrt{A}T)' + \sqrt{A}T\right]\right|}{r^2AT^2}.$$

Since AT^2, w, and Aw'^2 are bounded and $A' > 0$ near $r = \rho$, we see as before that $|(w/rT)'|$ is integrable near $r = \rho$; this proves the lemma. $\square$

PROPOSITION 3.7. *Assume that $\omega \neq 0$. Then there exist constants $c_1 > 0$ and $\varepsilon_1 > 0$ such that*

$$c_1 \leq (\alpha^2 + \beta^2)(r) \leq \frac{1}{c_1} \quad \text{for } \rho < r < \rho + \varepsilon_1. \tag{3.19}$$

Proof. We rewrite the Dirac equations (2.1) and (2.2) in matrix form:

$$\sqrt{A}\Phi' = \begin{pmatrix} w/r & -(m + \omega T) \\ -m + \omega T & -w/r \end{pmatrix} \Phi, \tag{3.20}$$

where $\Phi = (\alpha, \beta)^T$. Furthermore, let

$$a = \omega T, \quad b = \frac{w}{r}, \quad c = -m,$$

and define the matrix $B(r)$ by

$$B(r) = \begin{pmatrix} 1 + \dfrac{c}{a} & -\dfrac{b}{a} \\ -\dfrac{b}{a} & 1 - \dfrac{c}{a} \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{m}{\omega T} & -\dfrac{w}{r\omega T} \\ -\dfrac{w}{r\omega T} & 1 + \dfrac{m}{\omega T} \end{pmatrix}.$$

Our proof is based on an extension of [4, Lemma 5.1]. In this earlier paper we used the fact that b/a and c/a are monotone near the horizon. In the case considered here, we do not have these hypotheses and must therefore work harder.

Since $T(r) \to \infty$ as $r \searrow \rho$ and $w(r)$ is bounded (by assumption (II)), we see that both $m/\omega T$ and $w/r\omega T$ tend to zero as $r \searrow \rho$. Thus, $B(r)$ is close to the identity matrix when r is near ρ. If we define $F(r)$ by

$$F(r) = \langle \Phi(r), B(r)\Phi(r) \rangle$$

(where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product), then we can find constants $c > 0$ and $\varepsilon_1 > 0$ such that

$$\frac{1}{c}|\Phi(r)|^2 \le F(r) \le c|\Phi(r)|^2, \quad \rho < r < \rho + \varepsilon_1. \tag{3.21}$$

Furthermore, an easy calculation using the Dirac equation (3.20) yields that

$$F'(r) = \langle \Phi(r), B'(r)\Phi(r) \rangle;$$

thus, if $\rho < r < \rho + \varepsilon_1$,

$$-\sqrt{2}\left[\left|\left(\frac{m}{\omega T}\right)'\right| + \left|\left(\frac{w}{r\omega T}\right)'\right|\right]|\Phi|^2 \le F'(r)$$

$$\le \sqrt{2}\left[\left|\left(\frac{m}{\omega T}\right)'\right| + \left|\left(\frac{w}{r\omega T}\right)'\right|\right]|\Phi|^2,$$

because the sup-norm of the matrix B' is bounded by

$$|B'| \le \sqrt{2}\left[\left|\left(\frac{m}{\omega T}\right)'\right| + \left|\left(\frac{w}{r\omega T}\right)'\right|\right].$$

From (3.21) we therefore obtain

$$-c\left[\left|\left(\frac{m}{\omega T}\right)'\right| + \left|\left(\frac{w}{r\omega T}\right)'\right|\right]F(r) \le \frac{F'(r)}{\sqrt{2}}$$

$$\le c\left[\left|\left(\frac{m}{\omega T}\right)'\right| + \left|\left(\frac{w}{r\omega T}\right)'\right|\right]F(r) \tag{3.22}$$

on $\rho < r < \rho + \varepsilon_1$. Dividing this inequality by $F(r)$ and applying Lemma 3.6, we obtain

$$-a(r) \le \frac{F'(r)}{F(r)} \le a(r), \tag{3.23}$$

where $a(r)$ is integrable near the event horizon $r = \rho$. If we integrate (3.23) from ρ to $\rho + \varepsilon_1$, we see that $\log F(r)$ is bounded from above and below near $r = \rho$. Upon exponentiating and substituting into (3.21), we see (as in the proof of Lemma 3.2) that (3.19) holds. $\qquad \square$

We return now to the proof of Theorem 3.1 for $A^{-1/2}$ not integrable near $r = \rho$ and $\omega \ne 0$ (subcase (i)). Toward this end, we consider the differential equation for (AT^2), (3.4); we observe that, as $r \searrow \rho$, the right-hand side tends to $-\infty$ while the left-hand side is bounded. This contradiction implies that $\omega \ne 0$ cannot hold.

For subcase (ii) of $A^{-1/2}$ not integrable near the event horizon, $\omega = 0$, we note that Lemma 3.4 holds (cf. the remark following the statement of that lemma). As in Case 1, we find that (3.10) holds near $r = \rho$. Integrating this inequality from

$r > \rho$ to $\rho + \varepsilon$, we see that for r near ρ, the left-hand side is bounded while the right-hand side can be made arbitrarily large. This contradiction completes the proof of Theorem 3.1.

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F. Finster
Max Planck Institute MIS
Inselstr. 22-26
04103 Leipzig
Germany

Felix.Finster@mis.mpg.de

J. Smoller
Mathematics Department
University of Michigan
Ann Arbor, MI 48109

smoller@umich.edu

S.-T. Yau
Mathematics Department
Harvard University
Cambridge, MA 02138

yau@math.harvard.edu