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Davis's Inequality for Orthogonal Martingales under Differential Subordination

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1. Introduction

Consider two $\mathbb{H}$-valued semimartingales X and Y, where $\mathbb{H}$ is a separable Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. We denote by $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ their common filtration, which is a family of right-continuous sub-σ-fields in a probability space $\{\Omega, \mathcal{A}, P\}$. We also assume that $\mathcal{F}_0$ contains all the sets of probability zero. We use the notation $[X, Y] = \{[X, Y]_t\}_{t\geq 0}$ to denote the quadratic covariation process between X and Y (see e.g. [DM]). Unless otherwise stated, we assume that all semimartingales have right-continuous paths with left limits (r.c.l.l.). For notational simplicity, we use $[X] = \{[X]_t\}_{t\geq 0}$ to denote $[X, X]$.

Since all the results in the paper are invariant under Hilbert space isomorphisms, we can restrict to the spaces of square integrable sequences.

We say that Y is *differentially subordinate* to X if $[X]_t - [Y]_t$ is nondecreasing and nonnegative as a function of t. A slightly weaker notion of martingale differential subordination was first introduced by Burkholder for discrete-time martingales and certain stochastic integrals (see [Bu1; Bu2; Bu3; Bu4; Bu5; Bu6] for connections and applications to various settings in Banach spaces). For continuous parameter martingales with continuous paths, this definition was introduced by Bañuelos and Wang [BW1] and for continuous parameter martingales by Wang [W]. With this definition of subordination, Bañuelos and Wang [BW1] and Wang [W] extended various sharp martingale inequalities of Burkholder [Bu1–Bu5] from the discrete-time and certain stochastic integral settings to general continuous parameter martingales. In particular, the following theorem was proved in Wang [W] (see also [BW1]). We use the notation $\|X\|_p$ to denote $\sup_{t\geq 0}\|X_t\|_p$.

Theorem 1.1. *Let X and Y be two $\mathbb{H}$-valued continuous-time parameter martingales such that Y is differentially subordinate to X. Then, for $1 < p < \infty$,*

$$\|Y\|_p \leq (p^* - 1)\|X\|_p. \tag{1.1}$$

This inequality is sharp, and it is also strict if $p \neq 2$ and $0 < \|X\|_p < \infty$.

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This sharp martingale inequality has many important applications in analysis. For example, let $B\colon L^p(\mathbb{C}) \to L^p(\mathbb{C})$ be the Beurling–Ahlfors operator defined by

$$Bf(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(\xi)}{(\xi - z)^2}\,dm(z).$$

Using (1.1) and the representation of this operator as the conditional expectation of certain stochastic integrals, it was proved in [BW1] that

$$\|Bf\|_p \le 4(p^* - 1)\|f\|_p, \quad 1 < p < \infty,$$

where $p^* = \max\{p, q\}$ with q the conjugate of p. The interest in estimating $\|B\|_p$ comes from the now well-known conjecture of Iwaniec [I1] which asserts that $\|B\|_p = p^* - 1$. This conjecture has many applications in partial differential equations, quasi-conformal mapping, and complex analysis (see [A; IK; IM; IMNS]). The constant $4(p^* - 1)$ is the best-known upper bound for $\|B\|_p$. Furthermore, there is no analytic proof of this bound available. An extension of this estimate to the Beurling–Ahlfors operator in several dimensions is presented in Bañuelos and Lindeman [BL].

Another interesting application of martingale differential subordination in analysis is to the norms of the Riesz transforms $R_j\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $j = 1, \dots, n$, defined by

$$R_j = c_n \int_{R^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y)\,dy, \tag{1.2}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.$$

In order to calculate the norms $\|R_j\|_p$ $(1 < p < \infty)$, we need to introduce another definition. We say that two $\mathbb{H}$-valued martingales $X = (X_1, X_2, \dots)$ and $Y = (Y_1, Y_2, \dots)$ are orthogonal if, for each i, j, $[X_i, Y_j]_t = 0$ for all $t \ge 0$. This definition was introduced in [BW1], where the following theorem was proved when both X and Y have continuous paths. The result for general continuous parameter martingales is contained in [BW2]. For $1 < p < \infty$, set

$$C_p = \cot\left(\frac{\pi}{2p^*}\right) \quad \text{and} \quad E_p = \csc\left(\frac{\pi}{2p^*}\right).$$

THEOREM 1.2. *Let X and Y be two $\mathbb{R}$-valued continuous-time orthogonal martingales such that Y is differentially subordinate to X. Then, for $1 < p < \infty$,*

$$\|Y\|_p \le C_p\|X\|_p \quad \textit{and} \quad \left\|\sqrt{X^2 + Y^2}\right\|_p \le E_p\|X\|_p. \tag{1.3}$$

These inequalities are sharp, and they are strict unless $p = 2$ or $\|X\|_p = \infty$. Moreover, if $1 < p \le 2$ then X may be taken to be $\mathbb{H}$-valued, and if $2 \le p < \infty$ then Y may be taken to be $\mathbb{H}$-valued.

The best constants in these inequalities are not known when both X and Y are $\mathbb{H}$-valued. The inequalities (1.3) are martingale versions of the inequalities of

Pichorides [P] and Essén [E] for conjugate harmonic functions. Using (1.3) and the representation of the Riesz transforms as conditional expectation of martingale transforms [B1; B2], it was proved in [BW1] that $\|R_j\|_p = C_p$ and that $\left\|\sqrt{|R_j|^2 + |I|^2}\right\|_p = E_p$ for $1 < p < \infty$, where I denotes the identity operator. The first of these results had been proved earlier by Iwaniec and Martin [IM] using the Calderón–Zygmund method of rotations.

Several weak-type inequalities analogous to Theorem 1.1 are also well known. The following inequality was first proved by Burkholder [Bu2] for discrete-time martingales and certain stochastic integrals and by Wang [W] for general continuous parameter martingales.

THEOREM 1.3. *Let X and Y be two $\mathbb{H}$-valued continuous-time parameter martingales such that Y is differentially subordinate to X. Then, for any $\lambda \geq 0$,*

$$\lambda P(|X| + |Y| \geq \lambda) \leq 2\|X\|_1; \tag{1.4}$$

the inequality is sharp.

Inequality (1.4), and all the similar weak-type inequalities to follow, should be interpreted as

$$\lambda P\Big(\sup_{t\geq 0}(|X_t| + |Y_t|) \geq \lambda\Big) \leq 2\|X\|_1.$$

As in the L^p cases, this inequality has important applications to systems of conjugate harmonic functions. The following theorem is due to Burkholder [Bu3; Bu5].

THEOREM 1.4. *Let u and v be two $\mathbb{H}$-valued harmonic functions on a domain $D \subset \mathbb{R}^n$. Let D_0 be a bounded subdomain of D such that $\partial D_0 \subset D$, where ∂D_0 denotes the boundary of D_0. For $\xi \in D_0$, assume*

$$|v(\xi)| \leq |u(\xi)|, \tag{1.5}$$

$$|\nabla v| \leq |\nabla u| \ \text{ on } D. \tag{1.6}$$

Then, for $\lambda \geq 0$,

$$\lambda\mu_\xi(|u| + |v| \geq \lambda) \leq 2\int_{\partial D_0}|u|\,d\mu_\xi, \tag{1.7}$$

where μ_ξ is the harmonic measure on ∂D_0 with respect to ξ. Moreover, the inequality is sharp because it is already sharp in the inequality

$$\lambda\mu_\xi(|v| \geq \lambda) \leq 2\int_{\partial D_0}|u|\,d\mu_\xi. \tag{1.8}$$

If the harmonic functions u and v satisfy conditions (1.5) and (1.6), we say that v is differentially subordinate to u (as defined in [Bu3]). If the harmonic function v is the conjugate function of u and if $v(\xi) = u(\xi)$, then conditions (1.5) and (1.6) are satisfied. In addition, the Cauchy–Riemann equations imply that $\nabla u \cdot \nabla v = 0$.

The analog of inequality (1.8) is the Kolmogorov weak-type inequality for conjugate functions. The best constant for this inequality was obtained by Davis [Da], who showed that—if $\xi = 0$, $v(0) = 0$, and D_0 is the unit disk—then

$$\lambda\mu_0(|v| \geq \lambda) \leq K \int_{\partial D_0} |u|\, d\mu_0; \tag{1.9}$$

here (and for the rest of this paper)

$$K = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots}.$$

Furthermore, the inequality is sharp.

The inequality (1.9) was recently generalized by Choi [C] to an arbitrary pair of harmonic functions u and v which satisfy the differential subordination and the orthogonality condition but which are not necessarily conjugates of each other. Following the martingale definition, we say that two real-valued harmonic functions u and v defined on D are *orthogonal* if

$$\nabla u \cdot \nabla v = 0 \text{ on } D.$$

Choi's result is as follows.

THEOREM 1.5 [C]. *Let u and v be two $\mathbb{R}$-valued harmonic functions on a domain $D \subset \mathbb{R}^n$. Let D_0 be a bounded subdomain of D such that $\partial D_0 \subset D$. For $\xi \in D_0$, assume*

$$|v(\xi)| \leq |u(\xi)|,$$

$$|\nabla v| \leq |\nabla u| \text{ on } D,$$

$$\nabla u \cdot \nabla v = 0 \text{ on } D.$$

Then, for $\lambda \geq 0$,

$$\lambda\mu_\xi(|v| \geq \lambda) \leq K \int_{\partial D_0} |u|\, d\mu_\xi.$$

The inequality is sharp.

The interplay between martingales and harmonic functions is very rich and broad. In almost all situations, sharp inequalities for harmonic functions correspond to sharp martingale inequalities under the appropriate setting. In this paper we will establish this relationship for Theorem 1.5. Namely, we will prove the analog of Theorem 1.5 in the martingale setting, which can also be viewed as an analog of Theorem 1.3 with the extra condition of orthogonality. This result is motivated by the fact that the orthogonal martingales arising in the representations of the Riesz transforms are not simply harmonic functions composed with Brownian motion. Hence, knowing only the theorems for harmonic functions gives no information for the Riesz transforms. The main result of the paper is the following theorem.

THEOREM 1.6. *Let X and Y be two $\mathbb{R}$-valued continuous-time–parameter orthogonal martingales such that Y is differentially subordinate to X. Then, for any $\lambda \geq 0$,*

$$\lambda P(|Y| \geq \lambda) \leq K\|X\|_1,$$

where K is the constant in (1.9). *The inequality is sharp. Moreover, X can be $\mathbb{H}$-valued and Y can be a nonnegative supermartingale.*

REMARK 1.7. The numerator in the expression for K can be written as $\frac{3}{4}\zeta(2)$, where $\zeta(2)$ is the Riemann zeta function evaluated at 2. The value of this expression is $\pi^2/8$. The denominator, as pointed out to us by the referee, is the famous Catalan $\beta(2)$ constant (see [AbS]), which arises in several settings and whose value is 0.9159655 .... (It is interesting that the connection between Davis's constant and Catalan's constant had not been noticed before, as far as we know.)

REMARK 1.8. The extremal case is already covered by the classical case of conjugate harmonic function composed with Brownian motion. Note also that, under the condition of Theorem 1.6 (as shown by Lemma 2.1), the martingale Y necessarily has continuous paths.

The results just discussed, as well as the methods employed in proving them, have raised several interesting problems related to singular integrals and also to quasi-convexity and rank-1 convexity—notions that arise in the calculus of variations. We refer the reader to [BaM; BL; I2], where many of these connections and problems are discussed. Here, we just mention a few problems concerning L^p estimates for some basic and classical singular integrals.

The Beurling–Ahlfors operator and the composition of two Riesz transforms R_iR_j are examples of a singular integral of even kernel. It follows from Theorem 1.1 that, for $1 < p < \infty$, $\|R_iR_j\|_p \leq (p^* - 1)$ for any $i, j = 1, \dots, n$ (see [BW1]). This inequality is not sharp. However, it is interesting that, as in the upper bound for the Beurling–Ahlfors operator discussed previously, there is no analytic proof available for this bound. These are all examples of singular integrals of even kernels for which the analytic techniques available give no useful information about their L^p constants. For example, the best one can say with the classical Calderón–Zygmund method of rotations is that $\|R_iR_j\|_p \leq \cot^2(\pi/2p^*)$. This constant does not even have the correct behavior as $p \to \infty$ or 1.

Another interesting open problem is the identification of the best constant C in the weak-type inequality

$$m\{x : |R_j f(x)| \geq \lambda\} \leq \frac{C}{\lambda}\|f\|_1.$$

At this point, nothing seems to be known about C outside of what follows from the general theory of singular integrals. It is not even known if the constant C can be taken as independent of the dimension; this problem is raised in [S2]. The reason we cannot make any conclusions about C (from Theorem 1.6 and the representation of the Riesz transforms as conditional expectations of orthogonal martingales) is that—unlike the L^p inequalities—weak-type inequalities are not preserved by the conditional expectation operator. Nevertheless, we believe there is now considerable information to *conjecture* that the best constant for the Riesz transforms should not be larger than that for the Hilbert transform—that is, Davis's constant K.

2. Proof of Theorem 1.6

Let X and Y be two semimartingales that are r.c.l.l. We denote the "jump process" of X and Y by

$$\Delta X = \{\Delta X_t\}_{t\geq 0} = \{X_t - X_{t-}\}_{t\geq 0}$$

and

$$\Delta Y = \{\Delta Y_t\}_{t\geq 0} = \{Y_t - Y_{t-}\}_{t\geq 0},$$

respectively. We also use $X^c = \{X_t^c\}_{t\geq 0}$ and $Y^c = \{Y_t^c\}_{t\geq 0}$ to denote the continuous part of X and Y, respectively. Lemma 1 in [W] shows that, for every $t \geq 0$, $|\Delta Y_t| \leq |\Delta X_t|$ if Y is differentially subordinate to X. If X and Y are orthogonal, then Lemma 1 in [BW2] shows that $\langle \Delta X_t, \Delta Y_t\rangle = 0$ for every $t \geq 0$. If Y is one-dimensional then, for every $t \geq 0$, either $\Delta X_t = 0$ or $\Delta Y_t = 0$. Thus, $|\Delta Y_t| = 0$ for all $t \geq 0$ if Y is differentially subordinate to X. Therefore, combining the two lemmas yields the following observation.

LEMMA 2.1. *Let X be an $\mathbb{H}$-valued semimartingale, and let Y be a real-valued semimartingale in the same filtration. Then Y is differentially subordinate and orthogonal to X if and only if $[X^c]_t - [Y^c]_t$ is a nondecreasing and nonnegative function of t, X^c and Y^c are orthogonal, $|Y_0| \leq |X_0|$, and Y has continuous paths.*

We now assume that X is an $\mathbb{H}$-valued martingale and that Y is a real-valued martingale such that Y is differentially subordinate to X and that X and Y are orthogonal. In order to prove that

$$\lambda P\left\{\sup_{t\geq 0}|Y_t| \geq \lambda\right\} \leq K\|X\|_1,$$

we may assume that $\lambda = 1$. Also, by a "stopping time" argument (see [W]) it is enough to prove that

$$P\{|Y_t| \geq 1\} \leq K\|X_t\|_1 \quad \text{for all } t \geq 0. \tag{2.1}$$

Since

$$P\{|Y_t| \geq 1\} - K\|X_t\|_1 = E(I_{\{|Y_t|\geq 1\}} - K|X_t|),$$

proving (2.1) is equivalent to proving that

$$EV(X_t, Y_t) \leq 0, \tag{2.2}$$

where $V\colon \mathbb{H}\times\mathbb{R} \to \mathbb{R}$:

$$V(x,y) = \begin{cases} 1 - K|x| & \text{if } |y| \geq 1,\\ -K|x| & \text{if } |y| < 1.\end{cases}$$

To prove (2.2), we introduce a new function $\bar{W}\colon \mathbb{H}\times\mathbb{R} \to \mathbb{R}$ given by $\bar{W}(x,y) = W(|x|,|y|)$, where $W\colon \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$W(x,y) = \begin{cases} |x| & \text{if } |y| \geq 1,\\ \mathcal{W}(\varphi(x,y)) & \text{if } |y| < 1.\end{cases}$$

Here $\varphi\colon S \to H$ is a conformal mapping from the strip $S = \{z = (x, y) : |y| < 1\}$ into the upper half-plane $H = \mathbb{R}^2_+ = \{\zeta = (\alpha, \beta) : \beta > 0\}$, given explicitly by

$$\zeta = \varphi(z) = \varphi(x, y) = ie^{(\pi/2)z} = \left(e^{(\pi/2)x}\cos\left(\tfrac{\pi}{2}(y+1)\right), e^{(\pi/2)x}\sin\left(\tfrac{\pi}{2}(y+1)\right)\right);$$

$\mathcal{W}(\alpha, \beta)\colon H \to \mathbb{R}$ is the Poisson integral of the function $\alpha \to \frac{2}{\pi}|\ln|\alpha||$. That is,

$$\mathcal{W}(\alpha, \beta) = \frac{2}{\pi^2}\int_{-\infty}^{\infty} \frac{\beta|\ln|t||}{(\alpha - t)^2 + \beta^2}\, dt.$$

We first study the function W. The following properties have been established in [C].

LEMMA 2.2. *The function W satisfies the following properties.*

(a) *W is continuous on $\mathbb{R}^2$ and harmonic everywhere except on the set $\{(x, y) : x = 0, |y| > 1\}$ and possibly at the boundary of S; $|y| = 1$. Moreover,*

$$W_{xx}(x, y) \geq 0 \text{ and } W_{yy}(x, y) \leq 0$$

in S.

(b) *W is symmetric in x and in y. That is,*

$$W(x, y) = W(-x, y) = W(x, -y).$$

Therefore,

$$W_x(0, y) = W_y(x, 0) = W_{xy}(x, 0) = W_{xy}(0, y) = 0.$$

(c) *We have that*

$$\frac{1}{W(0,0)} = K = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots}.$$

(d) *$W(x, y) \geq |x|$ for all $(x, y) \in \mathbb{R}^2$.*

(e) *If $U(x, y) = 1 - KW(x, y)$, then $U(x, y) \leq 0$ if $|y| \leq |x|$. Moreover, $V(x, y) \leq U(x, y)$ for all $(x, y) \in \mathbb{R}^2$.*

By (e) of this lemma and the fact that $|Y_0| \leq |X_0|$ (which follows from Lemma 2.1), inequality (2.2) is implied by

$$EU(|X_t|, |Y_t|) \leq EU(|X_0|, |Y_0|). \tag{2.3}$$

Define the stopping time as

$$T = \inf\{t \geq 0 : |Y_t| \geq 1\}.$$

Let $Z = \{Z_t\}_{t\geq 0} = \{U(|X_{t\wedge T}|, |Y_{t\wedge T}|)\}_{t\geq 0}$. Our goal now is to show that

$$EU(|X_t|, |Y_t|) \leq EZ_t \tag{2.4}$$

and that

$$Z_t \text{ is a supermartingale.} \tag{2.5}$$

Clearly, if the two preceding inequalities hold then

$$EU(|X_t|, |Y_t|) \le EZ_t \le EZ_0 = EU(|X_0|, |Y_0|),$$

which proves (2.3).

To prove (2.4), note that

$$EU(|X_t|, |Y_t|) = E[U(|X_t|, |Y_t|)I_{\{t \ge T\}}] + E[U(|X_t|, |Y_t|)I_{\{t<T\}}].$$

By Lemma 2.2(d), $\bar{W}(X_{t,} Y_t) \ge |X_t|$. Since X is a martingale, $|X|$ is a submartingale. Thus,

$$E[|X_t|I_{\{t\ge T\}}] = E(E(|X_t|\|\mathcal{F}_T)I_{\{t\ge T\}}) \ge E[|X_T|I_{\{t\ge T\}}].$$

Consequently,

$$\begin{aligned} E[U(|X_t|, |Y_t|)I_{\{t\ge T\}}] &= E[(1 - K\bar{W}(X_t, Y_t))I_{\{t\ge T\}}] \\ &\le E[(1 - K|X_t|)I_{\{t\ge T\}}] \\ &\le E[(1 - K|X_T|)I_{\{t\ge T\}}] \\ &= E[(1 - K\bar{W}(X_T, Y_T))I_{\{t\ge T\}}] \\ &= E[U(|X_T|, |Y_T|)I_{\{t\ge T\}}]. \end{aligned}$$

Here we have used the fact that $|Y_T| = 1$ on $\{t \ge T\}$, since (by Lemma 2.1) Y has continuous paths.

REMARK 2.3. Note that the foregoing argument shows that

$$EU(|X_t|, |R_t|) \le EU(|X_{t\wedge S}|, |R_{t\wedge S}|) \tag{2.6}$$

for any real-valued right-continuous path-adapted process $R = \{R_t\}_{t\ge 0}$, where $S = \inf\{t \ge 0 : |R_t| \ge 1\}$. In particular, it holds for nonnegative supermartingales with right-continuous paths.

It remains to prove (2.5). This is equivalent to showing that

$$\{\bar{W}(X_{t\wedge T}, Y_{t\wedge T})\}_{t\ge 0} = \{W(|X_{t\wedge T}|, |Y_{t\wedge T}|)\}_{t\ge 0} \tag{2.7}$$

is a submartingale. For this we shall need a few more properties of W. These are given in the following sequence of lemmas. Most of these are extensions of the arguments given by Choi. First we observe that the argument given in [C, Lemma 3] shows the following result.

LEMMA 2.4. *We have*

$$\lim_{\substack{x\to\infty \\ (x,y)\in S}} W_x(x, y) = 1, \qquad \lim_{\substack{x\to-\infty \\ (x,y)\in S}} W_x(x, y) = -1,$$

$$\lim_{\substack{x\to\infty \\ (x,y)\in S}} W_y(x, y) = 0, \qquad \lim_{\substack{x\to-\infty \\ (x,y)\in S}} W_y(x, y) = 0,$$

$$\lim_{\substack{x\to\infty \\ (x,y)\in S}} W_{xxx}(x, y) = 0, \qquad \lim_{\substack{x\to-\infty \\ (x,y)\in S}} W_{xxx}(x, y) = 0.$$

LEMMA 2.5. *Define the function* $A(x, y)\colon H \to \mathbb{R}$ *by*

$$A(x, y) = \frac{1}{\pi} \int_{-1}^{1} \frac{y|t|}{(x-t)^2 + y^2}\, dt,$$

which is the Poisson extension of the function $t \to |t| I_{|t|\le 1}$. *Then*

$$0 \le \liminf_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x>0}} A_x(x, y) \le \limsup_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x>0}} A_x(x, y) \le 1,$$

$$-1 \le \liminf_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x<0}} A_x(x, y) \le \limsup_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x<0}} A_x(x, y) \le 0$$

Moreover,

$$\liminf_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x<0}} A_{xxx}(x, y) \ge 0 \quad \text{and} \quad \limsup_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x>0}} A_{xxx}(x, y) \le 0.$$

Proof. It is easy to see that

$$\pi A(x, y) = \frac{y}{2}\big(\ln((x-1)^2 + y^2) + \ln((x+1)^2 + y^2) - 2\ln(x^2 + y^2)\big) + x\left(2\arctan\frac{x}{y} - \arctan\frac{x-1}{y} - \arctan\frac{x+1}{y}\right).$$

Hence,

$$\begin{aligned} \pi A_x(x, y) &= \frac{y}{(x+1)^2 + y^2} - \frac{y}{(x-1)^2 + y^2} + 2\arctan\frac{x}{y} - \arctan\frac{x-1}{y} - \arctan\frac{x+1}{y} \\ &= \frac{y}{(x+1)^2 + y^2} - \frac{y}{(x-1)^2 + y^2} - \arctan\frac{2xy}{y^2 - x^2 + 1} + 2\arctan\frac{x}{y}. \end{aligned}$$

Since $y > 0$, we have

$$0 \le \liminf_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x>0}} A_x(x, y) \le \limsup_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x>0}} A_x(x, y) \le 1,$$

$$-1 \le \liminf_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x<0}} A_x(x, y) \le \limsup_{\substack{(x,y)\to(0,0)\\ (x,y)\in H\\ x<0}} A_x(x, y) \le 0.$$

If $(x, y) \in H$ then we also have

$$\pi A_{xx}(x, y) = -\frac{2(x+1)y}{((x+1)^2 + y^2)^2} + \frac{2(x-1)y}{((x-1)^2 + y^2)^2} - \frac{y}{(x-1)^2 + y^2} - \frac{y}{(x+1)^2 + y^2} + \frac{2y}{x^2 + y^2}$$

and

$$\pi A_{xxx}(x,y) = -\frac{4xy}{(x^2+y^2)^2} + \frac{2y((x+1)^3+3(x+1)^2+xy^2)}{((x+1)^2+y^2)^3} + \frac{2y((x-1)^3-3(x-1)^2+xy^2)}{((x-1)^2+y^2)^3}.$$

Thus,

$$\liminf_{\substack{(x,y)\to(0,0)\\(x,y)\in H\\x<0}} A_{xxx}(x,y) \ge 0 \quad \text{and} \quad \limsup_{\substack{(x,y)\to(0,0)\\(x,y)\in H\\x>0}} A_{xxx}(x,y) \le 0. \qquad \square$$

These give our next lemma.

LEMMA 2.6. *The following hold for W:*

$$0 \le \liminf_{\substack{(x,y)\to(0,\pm1)\\(x,y)\in S\\x>0}} W_x(x,y) \le \limsup_{\substack{(x,y)\to(0,\pm1)\\(x,y)\in S\\x>0}} W_x(x,y) \le 1,$$

$$-1 \le \liminf_{\substack{(x,y)\to(0,\pm1)\\(x,y)\in S\\x<0}} W_x(x,y) \le \limsup_{\substack{(x,y)\to(0,\pm1)\\(x,y)\in S\\x<0}} W_x(x,y) \le 0,$$

$$0 \le \liminf_{\substack{(x,y)\to(0,\pm1)\\(x,y)\in S\\x<0}} W_{xxx}(x,y), \quad \textit{and} \quad \limsup_{\substack{(x,y)\to(0,\pm1)\\(x,y)\in S\\x>0}} W_{xxx}(x,y) \le 0.$$

Proof. We will prove

$$0 \le \liminf_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x>0}} W_x(x,y) \le \limsup_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x>0}} W_x(x,y) \le 1,$$

$$-1 \le \liminf_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x<0}} W_x(x,y) \le \limsup_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x<0}} W_x(x,y) \le 0,$$

$$0 \le \liminf_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x<0}} W_{xxx}(x,y), \quad \text{and} \quad \limsup_{\substack{(x,y)\to(0,-1)\\(x,y\in S\\x>0}} W_{xxx}(x,y) \le 0.$$

The other limits are similar (simply replace $A(x, y+1)$ in the proof by $A(x, y-1)$ and perform a reflection about $y = 1$).

Following the proof of [C, Lemma 5], let $B(x,y)\colon S \to \mathbb{R}$ be given by

$$B(x,y) = W(x,y) - A(x,y+1).$$

Then B is harmonic on S and, for any $|x_0| < 1$,

$$\lim_{\substack{(x,y)\to(x_0,-1)\\(x,y)\in S}} B(x,y) = 0.$$

Thus, by the reflection principle, there is a harmonic extension $B^*(x, y)$ of $B(x, y)$ on $\{(x, y) : -1 < x < 1, -3 < y < -1\}$ such that

$$B^*(x, y) = -B(x, -2 - y) = -B^*(x, -2 - y), \quad -1 \le y < -3.$$

Since

$$B^*(x, -1) = 0,$$

we have

$$B^*_x(x, -1) = 0 \quad \text{and} \quad B^*_{xxx}(x, -1) = 0.$$

Therefore, the conclusions follow from Lemma 2.5. □

If in the proof we replace $A(x, y \pm 1)$ by $|x|$, we have the following.

LEMMA 2.7.

$$\lim_{\substack{(x,y)\to(x_0,\pm1)\\(x,y)\in S}} W_x(x, y) = 1$$

when $x_0 > 0$, *and*

$$\lim_{\substack{(x,y)\to(x_0,\pm1)\\(x,y)\in S}} W_x(x, y) = -1$$

when $x_0 < 0$. *Moreover,*

$$\lim_{\substack{(x,y)\to(x_0,\pm1)\\(x,y)\in S}} W_{xxx}(x, y) = 0.$$

We may combine these lemmas as follows.

LEMMA 2.8. *Let* $(x, y) \in S$. *Then* $-1 \le W_x(x, y) \le 0$, $W_{xxx}(x, y) \ge 0$ *if* $x < 0$ *and* $0 \le W_x(x, y) \le 1$, *and* $W_{xxx}(x, y) \le 0$ *if* $x > 0$.

Proof. We prove the first part; the rest is similar. Clearly, W_x is harmonic on $\{(x, y) : (x, y) \in S, x < 0\}$ and continuous on S. By Lemma 2.7,

$$\lim_{\substack{(x,y)\to(x_0,\pm1)\\(x,y)\in S}} W_x(x, y) = -1 \quad \text{and} \quad \lim_{\substack{(x,y)\to(x_0,\pm1)\\(x,y)\in S}} W_{xxx}(x, y) = 0$$

when $x_0 < 0$. By Lemma 2.4,

$$\lim_{\substack{x\to-\infty\\(x,y)\in S}} W_x(x, y) = -1 \quad \text{and} \quad \lim_{\substack{x\to\infty\\(x,y)\in S}} W_{xxx}(x, y) = 0.$$

Part (b) of Lemma 2.2 implies that

$$W_x(0, y) = 0 \quad \text{and} \quad W_{xxx}(0, y) = 0$$

if $-1 < y < 1$. Finally, Lemma 2.6 implies that

$$-1 \le \liminf_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x<0}} W_x(x, y) \le \limsup_{\substack{(x,y)\to(0,-1)\\(x,y)\in S\\x<0}} W_x(x, y) \le 0$$

and

$$\liminf_{\substack{(x,y)\to(0,-1)\\ (x,y)\in S\\ x<0}} W_{xxx}(x, y) \geq 0.$$

Thus, the maximum and minimum principle for harmonic functions implies that

$$-1 \leq W_x(x, y) \leq 0 \quad \text{and} \quad W_{xxx}(x, y) \geq 0$$

for all $(x, y) \in S$, $x < 0$. □

LEMMA 2.9. *Let* $(x, y) \in S$. *Then* $W_x(x, y) - xW_{xx}(x, y) \geq 0$ *when* $x > 0$.

Proof. Fix a $|y| < 1$. Define

$$G^y(x) = W_x(x, y) - xW_{xx}(x, y).$$

Then, by Lemma 2.2(b),

$$G^y(0) = W_x(0, y) = 0.$$

If $x > 0$ then

$$\frac{d}{dx} G^y(x) = -xW_{xxx}(x, y) \geq 0$$

by Lemma 2.8; hence, by the mean value theorem,

$$G^y(x) \geq 0.$$

This finishes the proof. □

We shall now prove (2.7). For $x \in \mathbb{H}$, let

$$\tilde{x} = \frac{x}{|x|} \quad \text{if } |x| \neq 0.$$

Then, for $|x||y| \neq 0$,

$$\frac{\partial \bar{W}(x, y)}{\partial x_i} = W_x(|x|, |y|)\frac{x_i}{|x|},$$

$$\frac{\partial \bar{W}(x, y)}{\partial y} = W_y(|x|, |y|)\frac{y}{|y|}.$$

Therefore, for $h \in \mathbb{H}$ and $k \in \mathbb{R}$,

$$\langle \nabla_x \bar{W}(x, y), h\rangle = W_x(|x|, |y|)\langle \tilde{x}, h\rangle,$$

$$\frac{\partial \bar{W}(x, y)}{\partial y}k = W_y(|x|, |y|)\frac{y}{|y|}k.$$

Similarly, if $|x||y| \neq 0$ then, since

$$\frac{\partial^2 \bar{W}(x, y)}{\partial x_i \partial x_j} = W_{xx}(|x|, |y|)\frac{x_i x_j}{|x|^2} - W_x(|x|, |y|)\frac{x_i x_j}{|x|^3}, \quad i \neq j$$

$$\frac{\partial^2 \bar{W}(x, y)}{\partial x_i^2} = W_{xx}(|x|, |y|)\frac{x_i^2}{|x|^2} + W_x(|x|, |y|)\frac{|x|^2 - x_i^2}{|x|^3}, \quad i = j$$

$$\frac{\partial^2 \bar{W}(x, y)}{\partial y^2} = W_{yy}(|x|, |y|),$$

we have

$$\langle \nabla_x^2 \bar{W}(x,y)h,h\rangle = W_{xx}(|x|,|y|)\langle \tilde{x},h\rangle^2 + W_x(|x|,|y|)\frac{|h|^2-\langle \tilde{x},h\rangle^2}{|x|},$$

$$\frac{\partial^2 \bar{W}(x,y)}{\partial y^2}k^2 = W_{yy}(|x|,|y|)k^2.$$

Hence, if $|y|<1$ and $|x||y|\neq 0$ then, for all $h\in\mathbb{H}$ and $k\in\mathbb{R}$, we have

$$\begin{aligned}
&\langle \nabla_x^2 \bar{W}(x,y)h,h\rangle + \frac{\partial^2 \bar{W}(x,y)}{\partial y^2}k^2 \\
&\quad = W_{xx}(|x|,|y|)\langle \tilde{x},h\rangle^2 + W_x(|x|,|y|)\frac{|h|^2-\langle \tilde{x},h\rangle^2}{|x|} + W_{yy}(|x|,|y|)k^2 \\
&\quad = (W_x(|x|,|y|) - |x|W_{xx}(|x|,|y|))\frac{|h|^2-\langle \tilde{x},h\rangle^2}{|x|} + W_{xx}(|x|,|y|)(|h|^2-k^2) \\
&\quad \geq W_{xx}(|x|,|y|)(|h|^2-k^2). \qquad (2.8)
\end{aligned}$$

Here we have used Lemma 2.9 and the fact that $W_{yy}(|x|,|y|) = -W_{xx}(|x|,|y|)$, since W is harmonic in S.

Next, let $x,h\in\mathbb{H}$, $|y|<1$, and $|x||y|\neq 0$. We want to show that

$$\bar{W}(x+h,y) - \bar{W}(x,y) - \langle \nabla_x \bar{W}(x,y)h,h\rangle \geq 0. \qquad (2.9)$$

Write

$$\begin{aligned}
&\bar{W}(x+h,y) - \bar{W}(x,y) - \langle \nabla_x \bar{W}(x,y)h,h\rangle \\
&\qquad\qquad = W(|x+h|,|y|) - W(|x|,|y|) - W_x(|x|,|y|)\langle \tilde{x},h\rangle,
\end{aligned}$$

and let

$$G(t) = W(|x+th|,|y|) - W(|x|,|y|) - tW_x(|x|,|y|)\langle \tilde{x},h\rangle.$$

First observe that $(|x|,|y|)\in S$ implies that $(|x+th|,|y|)\in S$ for all $0\leq t\leq 1$. Now assume that, for all $0\leq t\leq 1$, $|x+th|\neq 0$. Then

$$G'(t) = W_x(|x+th|,|y|)\langle \widetilde{x+th},h\rangle - W_x(|x|,|y|)\langle \tilde{x},h\rangle$$

and, by Lemma 2.2 and 2.8,

$$\begin{aligned}
G''(t) &= W_{xx}(|x+th|,|y|)\langle \widetilde{x+th},h\rangle^2 + W_x(|x+th|,|y|)\frac{|h|^2-\langle \widetilde{x+th},h\rangle^2}{|x+th|} \\
&\geq 0.
\end{aligned}$$

Hence, by the mean value theorem and the fact that $G(0)=G'(0)=0$, we have $G(1)\geq 0$; this is equivalent to

$$\bar{W}(x+h,y) - \bar{W}(x,y) - \langle \nabla_x \bar{W}(x,y)h,h\rangle \geq 0.$$

To complete the proof of (2.9), assume there is a $0<t_0\leq 1$ such that $|x+t_0h| = 0$. The preceding argument shows that

$$G(t_0)\geq 0 \quad \text{and} \quad G'(t_0)\geq 0.$$

Since $|x+t_0h| = 0$, it follows that $h = -(1/t_0)x$. Therefore, $|x+th| = |t-t_0||h|$ and $|h| \neq 0$. When $t_0 \leq t \leq 1$,

$$\begin{aligned} G(t) &= W(|t-t_0||h|, |y|) - tW_x(|t_0||h|, |y|)|h| - W(|t_0||h|, |y|) \\ &= W((t-t_0)|h|, |y|) - tW_x(|t_0||h|, |y|)|h| - W(|t_0||h|, |y|). \end{aligned}$$

Thus,

$$G'(t) = W_x((t-t_0)|h|, |y|)|h| - W_x(|t_0||h|, |y|)|h|$$

and, by Lemma 2.2,

$$G''(t) = W_{xx}((t-t_0)|h|, |y|)|h|^2 \geq 0.$$

Once again, the mean value theorem and the fact that $G(t_0) \geq 0$ and $G'(t_0) \geq 0$ imply that

$$G(1) \geq G(t_0).$$

Combining these, we have

$$G(1) \geq 0,$$

which proves (2.9).

To finish the proof, note that the range of Z is contained in the closure of S, where the function $\bar{W}$ is C^2 except possibly at the boundary of S or on $|x||y| = 0$. Applying [BW2, Prop. 1] to $f = -W$, we have shown (2.7).

If Y is a nonnegative supermartingale then (2.7) also follows by Itô's lemma and the convolution argument used in [BW2, Prop. 1], together with the fact that W_y is nonpositive in $\{(x, y) \in S : x \geq 0,\ y \geq 0\}$. To see this last fact, simply use the mean value theorem, $W_{yy} \leq 0$ in S, and $W_y(x, 0) = 0$ for all $x \in \mathbb{R}$. Therefore, we have completed the proof of the Theorem 1.6. □

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