Michigan Math. J. 47 (2000)

Fixed-Point Indices, Homoclinic Contacts, and Dynamics of Injective Planar Maps

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1. Introduction

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a C^1 diffeomorphism. Let $p \in \mathrm{Fix}(f)$ be a fixed point that is a *direct saddle*; that is, the derivative of f at p has eigenvalues λ, μ such that $0 < \lambda < 1 < \mu$. The Hartman–Grobman theorem implies that there is a topological coordinate system centered at p representing f as a linear map with these eigenvalues (see [13; 20]).

Our main results assert that, when the closures of the stable and unstable curves at p intersect in certain ways, there exists a fixed-point free Jordan curve $J \subset S$ whose index under f (defined below) is 1. Such a curve surrounds a *block* B of fixed points (i.e., B is open and closed in $\mathrm{Fix}(f)$) that is disjoint from p and has fixed point index 1. In particular, *there exists a fixed point different from* p. When fixed points are isolated with negative indices, the dynamics is shown to be rather simple.

For nonplanar surfaces, even a homoclinic point does not guarantee a second fixed point, as shown by the diffeomorphism of the torus $\mathbf{R}^2/\mathbf{Z}^2$ induced by the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Section 3 contains background material and proofs of the main results. Section 4 shows that many results on planar homeomorphisms generalize to homologically nilpotent injective maps in planar surfaces. In particular, the theorems of Brouwer and Brown, as well as the new results, hold for such maps.

Terminology. All maps are assumed to be continuous; diffeomorphisms are C^1 (continuously differentiable). A set X is *forward invariant* for a map g if $gX \subset X$, *overflowing* if $gX \supset X$, and *invariant* if $gX = X$. Homeomorphism is indicated by $\approx$. The set of fixed points of g is denoted by $\mathrm{Fix}(g)$. The *omega limit set* $\omega(x)$ is the limit set of the sequence $\{g^n x\}_{n \in \mathbf{N}}$. The set of natural numbers is $\mathbf{N} = \{0, 1, \dots\}$ and the set of integers is $\mathbf{Z}$. The Euclidean norm of $x \in \mathbf{R}^2$ is denoted $\|x\|$.

2. Background

Brouwer's 1912 plane translation theorem [1] can be stated as follows.

Received March 16, 1999. Revision received July 26, 1999.
This research was partially supported by National Science Foundation grant DMS-9404261.

THEOREM 2.1 (Plane Translation Theorem). *Let f be an orientation preserving homeomorphism of the plane $\mathbf{R}^2$ having no fixed point. Then every point has an open neighborhood W with the following properties: The boundary of W consists of two disjoint closed sets A, B, each homeomorphic to the line, such that $f(A) = B$ and $f^n W \cap W = \varnothing$ for all $n \neq 0$.*

For an elegant recent proof, see Franks [10].

This theorem is usually used contrapositively to obtain fixed points, as follows. Recall that a point p is *wandering* for a map g if it has a neighborhood disjoint from its images under iterates of g; otherwise, p is *nonwandering.* The closed invariant set $\mathrm{NW}(g)$ of nonwandering points contains all periodic points and omega limit points. An immediate consequence of the plane translation theorem is the following.

THEOREM 2.2 (Wandering Theorem). *If an orientation preserving homeomorphism of the plane has no fixed point, then every point is wandering.*

We will use the following two theorems, which embody further developments of Brouwer's ideas.

A surface map f is *free* (see [3]) provided that, for every closed 2-cell D, if $D \cap f(D) = \varnothing$ then $f^m(D) \cap f^n(D) = \varnothing$ for all $m \neq n$. This implies that every nonwandering point is fixed but is a strictly stronger condition. Brown [3] proves the following result, crediting it to Brouwer [1].

THEOREM 2.3 (Index Theorem). *If an orientation preserving homeomorphism of $\mathbf{R}^2$ is not free, then there exists a Jordan curve with index* 1.

A *translation line* for a map f is a set having the form $\bigcup_{-\infty<k<\infty} f^k C$, where C is a closed 1-cell meeting $f(C)$ only at a common, nonfixed endpoint. If β is a branch of a stable or unstable curve at a direct saddle p (see below), then $\beta \setminus \{p\}$ is a translation line. Here we restate [4, Thm. 4.7].

THEOREM 2.4 (Translation Line Theorem). *For any free, orientation preserving homeomorphism of $\mathbf{R}^2$, every translation line is homeomorphic to $\mathbf{R}$.*

The main results for plane diffeomorphisms are developed in the next section. The chief dynamical application is Theorem 3.3. In the last section, results are extended to injective, homologically nilpotent diffeomorphisms in open subsets of the plane.

All the results are valid for homeomorphisms that are local diffeomorphisms at saddle fixed points. And the full strength of saddles is not needed; the proofs can be adapted to negative index fixed points having finitely many stable and unstable branches and having a suitable canonical form. It is sufficient that there be a local branched covering space for which the fixed point is covered by a saddle.

FIXED-POINT INDICES. Fixed-point indices are briefly reviewed here; for the general theory see [5; 9].

Let $V \subset \mathbf{R}^2$ be an open set, $J \subset V$ a Jordan curve, and $f\colon V \to \mathbf{R}^2$ a map. We say that f has *index* $d \in \mathbf{Z}$ on J, denoted by $\mathrm{Ind}(f, J) = d$, provided f has no fixed point on J and d is the degree of the map

$$J \to S^1, \quad x \mapsto \frac{x - f(x)}{\|x - f(x)\|},$$

where $S^1 \subset \mathbf{R}^2$ denotes the unit circle and the orientations of S and J are induced from that of $\mathbf{R}^2$.

When the boundary ∂D of a closed 2-cell $D \subset V$ contains no fixed points, we calculate $\mathrm{Ind}(f, \partial D)$ as follows. When D contains a unique fixed point p, then $\mathrm{Ind}(f, \partial D)$ depends only on p; it is denoted by $\mathrm{Ind}(f, p)$ and can be computed from any convenient small disk around p. In general, let $g\colon V \to \mathbf{R}^2$ be an ε-approximation to f with only finitely many fixed points in D. If $\varepsilon > 0$ is small enough then

$$\mathrm{Ind}(f, \partial D) = \mathrm{Ind}(g, \partial D) = \sum_{p \in \mathrm{Fix}(g) \cap D} \mathrm{Ind}(g, p).$$

It is known that $\mathrm{Fix}(f) \cap D \neq \varnothing$ if $\mathrm{Ind}(f, \partial D) \neq 0$.

It is easy to see that a direct saddle has fixed point index -1. This yields the useful fact:

> *If D is a closed 2-cell such that $\mathrm{Ind}(f, \partial D) > 0$, then D contains a fixed point that is not a direct saddle.*

3. Fixed-Point Indices and Homoclinic Contacts

HYPOTHESIS. In this section, $S \subset \mathbf{R}^2$ is an open set and $p \in S$ is a direct saddle for an orientation preserving diffeomorphism $f\colon S \to S$.

The *stable curve* at p is

$$W_s = W_s(p) = W_s(p, f) = \left\{ x \in \mathbf{R}^2 : \lim_{n \to \infty} f^n(x) = p \right\}.$$

This curve is the image of a C^1 immersion $\gamma\colon \mathbf{R} \to \mathbf{R}^2$ [16]. Assuming $\gamma(0) = p$, we define the images of $(-\infty, 0]$ and $[0, -\infty)$ to be the two *stable branches* at p. The *unstable curve* is defined as $W_u = W_s(p, f^{-1})$; unstable branches are defined analogously.

The *limit set* of a branch β is the closed invariant set

$$\mathcal{L}(\beta) = \bigcap_{t \geq 0} \mathrm{clos}(\zeta([t, \infty))),$$

where $\zeta\colon [0, \infty) \to \beta$ is any bijective map; it is a closed invariant set independent of ζ. The limit set $\mathcal{L}(W_u)$ is defined to be the union of the limit sets of the two branches of W_u, and similarly for $\mathcal{L}(W_s)$.

Points in $W_u \cap W_s \setminus \{p\}$ are *homoclinic points* for p. We introduce the more general notion of a *homoclinic contact*, meaning a point of the closed invariant set

$$(W_u \cap W_s \setminus \{p\}) \cup \mathcal{L}(W_u) \cap \bar{W}_s \cup \mathcal{L}(W_s) \cap \bar{W}_u.$$

The horseshoe theorem of Smale [22] shows that when W_u and W_s cross transversely at a homoclinic point, $\mathcal{L}(W_u) \cap \mathcal{L}(W_s)$ contains a compact invariant set in which there are periodic orbits of arbitrarily high cardinality. I do not know whether this holds for homoclinic contacts. For related results see [6; 7; 8; 12; 18; 19; 21].

LEMMA 3.1. *Every homoclinic contact z for p is nonwandering.*

Proof. We must show that an arbitrary neighborhood U of z meets some $f^k U$ $(k \neq 0)$. We may assume $z \neq f(z)$. Then there is a smooth open arc $A \subset U$ through z such that A meets W_u and W_s transversely at points a and b, respectively. By a basic result known as the "λ-lemma" [20] or the "inclination lemma" [17], there is a sequence of diffeomorphisms $j_n \colon [0,1] \approx J_n \subset A$ onto nested neighborhoods $J_n \supset J_{n+1} \supset \cdots$ $(n \geq 1)$ of a in A such that the C^1 embeddings $f^n \circ j_n \colon [0,1] \to \mathbf{R}^2$ $(n > 0)$ converge to a C^1 embedding whose image arc J' is a neighborhood of p in W_u. Likewise, there is a sequence of diffeomorphisms $k_m \colon [0,1] \approx K_m \subset A$ onto nested neighborhoods $K_m \supset K_{m+1} \supset \cdots$ $(m \geq 1)$ of b in A such that the C^1 embeddings $f^{-m} \circ j_m \colon [0,1] \to \mathbf{R}^2$ $(n > 0)$ converge to a C^1 embedding whose image arc K' is a neighborhood of p in W_s. Since K' and J' cross at p, their respective approximations $f^n J$ and $f^{-m} K$ meet for sufficiently large $n, m > 0$; therefore, $U \cap f^{n+m} U \neq \varnothing$. □

THEOREM 3.2. *There exists a Jordan curve J such that* $\operatorname{Ind}(f, J) = 1$ *in each of the following cases*:

(a) $\mathcal{L}(W_s \cup W_u) \cap (W_s \cup W_u) \neq \varnothing$;
(b) *p admits a homoclinic contact z such that either $z = p$ or $z \neq f(z)$*;
(c) *there is a branch γ at p with $\bar{\gamma}$ compact and* $\bar{\gamma} \cap \operatorname{Fix}(f) = \{p\}$;
(d) *there exists a branch β at p that is not homeomorphic to* $[0, \infty)$.

Proof. We will prove in each case that f is not free; the index theorem will then complete the proof.

If $p \in \mathcal{L}(W_u)$ then a λ-lemma argument (see the proof of Lemma 3.1) shows $\operatorname{NW}(f)$ contains a stable branch at p, and analogously if $p \in \mathcal{L}(W_s)$; whence f is not free. Therefore we assume $p \notin \mathcal{L}(W_s \cup W_u)$. This precludes (a) by invariance of $W_u \cup W_s$, and with (c) it implies $\mathcal{L}(\gamma) \cap \operatorname{Fix}(f) = \varnothing$. So under (c), $\operatorname{NW}(f) \setminus \operatorname{Fix}(f)$ is not empty, as it contains $\omega(x)$ for every $x \in \gamma \setminus \{p\}$. Lemma 3.1 implies the theorem when (b) holds. Under assumption (d), $\beta \setminus \{p\}$ is a translation line that is not homeomorphic to $\mathbf{R}$, whence f is not free by the translation line theorem. □

THEOREM 3.3. *Assume that every fixed point is isolated and has index* ≤ 0. *Then the following statements hold*.

(i) *For every x, as n goes to $\pm\infty$, either $f^n x$ goes to a fixed point or $\|f^n x\| \to \infty$.*
(ii) *For each direct saddle p, every homoclinic contact is a fixed point $\neq p$ and each branch at p is homeomorphic to* $[0, \infty)$.

(iii) *If the only fixed point is a direct saddle p, then there are no homoclinic contacts and every branch is unbounded.*

Proof. As there cannot be a Jordan curve of positive index, the wandering theorem implies (i). Conclusion (ii) follows from Theorem 3.2(d). For (iii), suppose there is a homoclinic contact z. Then z is nonwandering (Lemma 3.1). If either $z = p$, or $z \neq p$ (whence $z \neq f(z)$), there is an index-1 curve by Theorem 3.2(b), contradicting the hypothesis. Let β be a bounded branch, assumed unstable to fix ideas. Let $x \in \mathcal{L}(\beta) \setminus \{p\}$. If $p \in \omega(x)$ then x is a homoclinic point, contradicting (ii), but any point of $\omega(x) \setminus \{p\}$ is a nonfixed nonwandering point, yielding an index-1 curve by the translation line theorem. Thus, every branch is bounded. □

Notice that Theorem 3.3 does not say every branch is closed and unbounded. Little is known about limit sets of branches.

The following result is an analog of Theorem 3.2 for a heteroclinic cycle.

THEOREM 3.4. *For some $m \geq 1$, let $p_0, \ldots, p_m = p_0$ be direct saddles for an orientation preserving diffeomorphism f of $\mathbf{R}^2$. Suppose that an unstable branch α_{i-1} at p_{i-1} meets a stable branch β_i at p_i at a point $z_i \notin \{p_{i-1}, p_i\}$ for $i = 1, \ldots, m$. Then there exists a Jordan curve of index 1 and hence there is a fixed point different from all the p_i.*

Proof. Each z_i is nonwandering, by a well-known argument similar to the proof of Lemma 3.1; now apply the index theorem. □

4. Homologically Nilpotent Maps

Let M denote a surface. The first rational singular homology group of M, denoted by $H = H_1(M; \mathfrak{q})$, is a vector space over the rational field $\mathfrak{q}$. A map $g: M \to M$ induces a linear map $g_*: H \to H$.

Call $g: M \to M$ *homologically nilpotent* if each $c \in H$ is annihilated by some iterate g_*^k, $k = k(c)$. This holds when $g_*^n = 0$ for some $n \geq 1$, for example, when gM lies in a simply connected subset of M. For a more interesting example, take M to be the complement in $\mathbf{R}^2$ of the positive integer points on the horizontal axis; then $H_1(M; \mathfrak{q})$ has the countably infinite basis $\{e_k\}_{k \in \mathbf{N}_+}$, where e_k is represented by a small oriented circle centered at $(k, 0)$. Let $g: M \to M$ be the restriction of a map $\mathbf{R}^2 \to \mathbf{R}^2$ taking $(k, 0)$ to $(k-1, 0)$. Then g is homologically nilpotent because $g_*^k e_k = 0$ for all k.

LEMMA 4.1. *If M admits an injective homologically nilpotent map g, then M embeds in the sphere.*

Proof. We argue by contradiction. If M is not embeddable in the sphere, then there exist Jordan curves C, C' crossing each other at $q \in M$ and otherwise disjoint, with mod 2 intersection number $C \# C' \neq 0$. By injectivity, $g^n C \cap g^n C' = \{g^n q\}$ for all $n \geq 1$; and $g^n C$ crosses $g^n C'$ at $g^n q$ because g^n, being injective, maps M

homeomorphically onto an open set by Brouwer's invariance of domain. Therefore, $g^n C \# g^n C' \neq 0$. This implies $g_*^n c \neq 0$, where $c \in H_1(M; \mathfrak{q})$ is the homology class defined by an orientation of C. Thus g is not homologically nilpotent. $\square$

In contrast, every orientable, connected noncompact surface M admits a homologically nilpotent map $M \to M$ that is *locally* injective. For there exists an immersion of M in the plane (see e.g. [14, Thm. 4.7]) and hence there is a locally injective map $f \colon M \to M$ whose image lies in an open 2-cell.

HYPOTHESIS. From now on, $S \subset \mathbf{R}^2$ is an open set and $f \colon S \to S$ is injective (thus a homeomorphism onto an open set), homologically nilpotent, and orientation preserving.

The construction in the following proof gives a canonical embedding of S in a surface $E \approx \mathbf{R}^2$ together with a canonical extension of f to a bijective homeomorphism $h \colon E \to E$.

PROPOSITION 4.2. *There exists a surface $E \approx \mathbf{R}^2$, a homeomorphism $j : S \approx jS \subset E$ onto an open set, and an orientation preserving homeomorphism $h : E \approx E$ such that the following hold.*

(a) $h \circ j = j \circ f$.
(b) *For every $x \in E$, there exists a natural number $n = n(x)$ such that $h^n(x) \in j(S)$.*
(c) *If $f \colon S \to fS$ is a diffeomorphism, then E has a natural differential structure diffeomorphic to $\mathbf{R}^2$, and j and h are diffeomorphisms.*

Proof. We obtain $h \colon E \to E$ as the direct limit of the infinite sequence

$$S \xrightarrow{f} S \xrightarrow{f} \cdots .$$

An element of E is an equivalence class $[x, k]$ (where $(x, k) \in S \times \mathbf{N}$) under the equivalence relation $[x, k] = [y, l]$ if there exists $n \geq k, l$ with $f^{n-k}x = f^{n-l}y$. Give E the largest topology making continuous the natural projection $S \times \mathbf{N} \to E$, $(x, k) \mapsto [x, k]$. Define $j \colon S \to E$ by $j(x) = [x, 0]$, and set

$$h \colon E \to E, \qquad [x, k] \mapsto [x, k-1] = [fx, k].$$

It is easy to see that, for each $n \in \mathbf{N}$, the map $j_n \colon S \to E$, $x \mapsto [x, n]$ maps S homeomorphically onto an open subset $E_n \subset E$, and $E = \bigcup E_n$. Hence E is a two-dimensional manifold, and it is not hard to show that E is metrizable, connected, and noncompact. We have $H_1(E; \mathfrak{q}) = 0$ because f is homologically nilpotent and the singular homology functor commutes with direct limits. The classification of surfaces therefore implies that E is homeomorphic to the plane. The homeomorphism h is surjective because, for any $[x, k]$,

$$h([x, k+1]) = [fx, k+1] = [x, k];$$

h is injective because if $h([x, j]) = h([y, k])$ then $[f(x), j] = [fy, k]$ and there exists $n \geq j, k$ such that $f^{n-j+1}x = f^{n-k+1}y$, implying $[x, j] = [y, k]$. Conclusions (a) and (b) follow from the construction. When f is a diffeomorphism, the maps j_n define a differential structure on E; the resulting manifold is diffeomorphic to $\mathbf{R}^2$ by the classification of smooth surfaces, and the rest of (c) is easy to verify. □

COROLLARY 4.3. *Let f, S, h, E, j be as in Proposition 4.2. Then*:

(a) *each compact subset of E has a neighborhood U such that $h^n U \subset jS$ for some $n \geq 0$;*
(b) *$y \in E$ is nonwandering for h if and only if $y = j(x)$, where $x \in S$ is nonwandering for f;*
(c) *$h^n y = y \in E$ if and only if $y = j(x)$ and $f^n x = x$;*
(d) *if a disk $D \subset E$ has no fixed points on its boundary, there exists $n \geq 0$ such that $h^n D \subset S$ and $\mathrm{Ind}(f, j(h^n J)) = \mathrm{Ind}(h, J)$.*

These results enable us to extend the earlier theorems on planar homeomorphisms to surface maps $f\colon S \to S$ that are injective and homologically nilpotent. For example, the plane translation theorem extends as follows.

THEOREM 4.4. *If f has no fixed point, then every point is wandering. More precisely, every point lies in an open set V with the following properties*:

(a) *$f^n V \cap V = \varnothing$ for all $n \neq 0$;*
(b) *$\bar{V}$ is surface whose boundary ∂V is the union of disjoint nonempty sets A, B that are closed in ∂V, and each component of A and B is homeomorphic to* $\mathbf{R}$;
(c) *$f(\bar{V}) \cap \bar{V} = f(A) \subset B$;*
(d) *$f^k(\bar{V}) \cap \bar{V} = \varnothing$ for all $k \geq 2$.*

Proof. By Theorem 4.2 and Corollary 4.3(c), we assume $jS \subset \mathbf{R}^2$ is the inclusion of an open subset, $h\colon \mathbf{R}^2 \approx \mathbf{R}^2$ is orientation preserving and fixed-point free, $hS \subset S$, and $f = h|S$. Given $x \in S$, there is open set $W \subset E$ containing x and satisfying the plane translation theorem applied to h. Now define $V = W \cap S$. □

The index theorem may be generalized as follows.

THEOREM 4.5. *If f is not free, then f has index* 1 *on the boundary of some disk D.*

Proof. Make the same identifications as in the preceding proof. By the index theorem, h has index 1 on the boundary of a disk $D' \subset \mathbf{R}^2$. For some $n > 0$, the disk $D = h^n D'$ lies in S and satisfies the theorem, by Corollary 4.3(a) and (d). □

The other results can be similarly extended. Thus, the theorems in Section 3 extend to orientation preserving diffeomorphisms $f : \mathbf{R}^2 \approx X \subset \mathbf{R}^2$.

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