Michigan Math. J. 47 (2000)

A New Characterization of Hyperellipticity

Bernard Maskit

1. Introduction

A (geodesic) *necklace* on a closed Riemann surface of genus $p \geq 2$ is a cyclically ordered set of $2p + 2$ simple nondividing closed geodesics (in the hyperbolic metric) $L_1, \dots, L_{2p+2}$, where each L_i intersects L_{i-1} exactly once, intersects L_{i+1} exactly once, and is otherwise disjoint from every other geodesic in the necklace. In this note we give a new characterization of hyperellipticity in terms of geodesic necklaces; this characterization is distinct from that given by Schmutz-Schaller [11]. We also give a geometric proof of Jørgensen's theorem [5], which states that, on a hyperbolic orbifold of dimension 2, there are infinitely many closed geodesics passing through every point of intersection of closed geodesics.

We denote the hyperbolic plane by $\mathbb{H}^2$; we will usually regard this as the upper half-plane. The group of all orientation preserving isometries of $\mathbb{H}^2$ can be canonically identified with $\mathrm{PSL}(2,\mathbb{R})$, the group of real 2×2 matrices with unit determinant.

A discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is *elementary* if it is a finite extension of a cyclic group. For our purposes, a Fuchsian group is a finitely generated nonelementary discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

We will use the following notation throughout. Matrices in $\mathrm{PSL}(2,\mathbb{R})$ are denoted by $\tilde{a}, \tilde{b}, \dots$; the corresponding hyperbolic isometries are denoted by $a, b, \dots$. If the transformation a is hyperbolic, its axis is denoted by A_a; further, if a is a hyperbolic element of the discrete group G, then we denote by L_a the projection of A_a, which is a geodesic on $\mathbb{H}^2/G$.

Elliptic elements of order 2 are called *half-turns.* The fixed point of a half-turn in $\mathbb{H}^2$ is its *center* (or *vertex*). In general, for any group H and for any set A, the *stabilizer* of A in H is given by

$$\mathrm{Stab}(A) = \{h \in H \mid h(A) = A\}.$$

The author wishes to thank William Abikoff for several useful suggestions.

Received September 23, 1997. Revision received November 4, 1999.
Research supported in part by NSF grant DMS 9500557.

2. The One-Holed Torus

Let a and b be hyperbolic transformations whose axes cross, where $H = \langle a, b\rangle$ is discrete. (We use $\langle a, \dots\rangle$ to denote the group generated by $a, \dots$.) It is well known that $\mathbb{H}^2/H$ is a torus with one hole if and only if the trace of the commutator $[\tilde{a}, \tilde{b}]$ is less than -2. If $S = \mathbb{H}^2/H$ is not a torus with a hole, then S has finite area, and S has one of the signatures $(0, 3; \nu_1, \nu_2, \nu_3)$, $2 \le \nu_i \le \infty$; $(0, 4; 2, 2, 2, \nu)$, $2 < \nu \le \infty$; or $(1, 1; \nu)$, $2 \le \nu \le \infty$ (see [10]). It is also well known that if $S = \mathbb{H}^2/H$ is not a torus with a hole and if H is a subgroup of the discrete group G, then H is of finite index in G, which also has one of the above signatures.

It follows from the preceding remarks that, if the hyperbolic transformations a and b (with crossing axes) generate a purely hyperbolic Fuchsian group H, then H is free on these two generators and $T = \mathbb{H}^2/H$ is a torus with a hole. Then, using the fact that all pairs of generators of H can be obtained from a and b by using Nielsen transformations, it follows that, if a' and b' are any generators of H, then $A_{a'}$ and $A_{b'}$ also cross.

2.1. THE CANONICAL EXTENSION. In any case, let z be the point of intersection of A_a and A_b, and let j be the half-turn with center at z. Then $j(A_a) = A_a$ but with reversed orientation, and $j(A_b) = A_b$ but with reversed orientation. It follows that $jaj^{-1} = a^{-1}$, and $jbj^{-1} = b^{-1}$. (Even though $j^{-1} = j$, we write jaj^{-1} to emphasize the fact that this is a conjugation.) Since $j^2 = 1$, this implies that $j_a = ja^{-1}$ and $j_b = jb^{-1}$ are also half-turns. The center of j_a lies on A_a halfway between z and $a(z)$; call this point z_a. Likewise, the center of j_b is at $z_b \in A_b$, halfway between z and $b(z)$.

In the case that $S = \mathbb{H}^2/H$ is a torus with a hole, then the three points z, z_a, z_b project to three distinct points on S.

2.2. BOUNDARY AXES AND HALF-PLANES. In general, a hyperbolic element $a \in G$ is a *boundary element* if its axis A_a bounds a half-plane that is precisely invariant under a hyperbolic cyclic subgroup of G. In this case, the axis A_a is called a *boundary axis.* In the case that $\mathbb{H}^2/G$ is a torus with a hole, the commutator $[a, b]$ is a primitive boundary element, and every boundary element of G is conjugate to some power of it. The projections of the boundary axes are the *boundary geodesics.*

Let G be any Fuchsian group. The convex hull $C(G)$ is the smallest hyperbolically closed and convex set whose Euclidean closure contains the limit set, $\Lambda(G)$. If G is of the first kind, then $C(G) = \mathbb{H}^2$. If G is of the second kind, then $C(G)$ is bounded by a disjoint collection of boundary axes and hyperbolic lines that (i) have limit points of G at both ends and (ii) bound half-planes that are precisely invariant under the identity. If G is finitely generated, then $C(G)$ is bounded only by boundary axes, and (modulo G) there are only finitely many of these. The *convex core* of G is $c(G) = C(G)/G$.

PROPOSITION 2.1. *Let T be a torus with a hole. Then there is a unique conformal involution $\eta\colon T \to T$. Furthermore, η has exactly three fixed points on T.*

Proof. Let a and b generate the universal covering group H of T; then a and b are hyperbolic with crossing axes. Let z be the point where A_a crosses A_b, and let j be the half-turn about z. Then j normalizes H; hence j projects to a conformal involution $\eta\colon T \to T$.

Let x, x_a, x_b be the projections of z, z_a, z_b, respectively. Then x, x_a, and x_b are three distinct fixed points of η on T.

Now let η be any conformal involution on T. It is well known that T can be conformally embedded in a closed torus $\hat{T}$, so that η has a conformal extension to $\hat{T}$ (see e.g. [7]). We call this conformal extension by the same name; then $\eta\colon \hat{T} \to \hat{T}$ is also a conformal involution. Since η preserves T, it also preserves its complement on $\hat{T}$, which is topologically a disc. Hence η has at least one fixed point on $\hat{T}$. Every conformal homeomorphism of a closed torus has either four fixed points or none, we conclude that η has four fixed points on $\hat{T}$. Moreover, because η is a conformal involution on the complement of T in $\hat{T}$ (which is a closed disc), η has exactly one fixed point there. We conclude that η has exactly three fixed points in T.

Since η is a hyperbolic isometry, it preserves the boundary geodesic on T; hence, it preserves $c(T)$, whose interior $c^0(T)$ is also a torus with a hole. We next reproduce the foregoing argument, starting with $c^0(T)$ rather than T. We embed $c^0(T)$ in $\hat{T}'$, so that η extends to a conformal involution on $\hat{T}'$, and conclude (as before) that η has three fixed points on $c^0(T)$.

Let S be the double of $c^0(T)$, so that S is a symmetric closed Riemann surface of genus 2. The symmetry $r\colon S \to S$ has one dividing geodesic of fixed points—this is the boundary geodesic of $c^0(T)$, and $S/r = c(T)$. Since η is continuous up to the boundary of $c(T)$, it has a unique extension to a conformal homeomorphism (which we call by the same name) acting on S. Now, $\eta\colon S \to S$ is a conformal involution with six fixed points; hence it is the hyperelliptic involution. Since the conformal structure on S is uniquely determined by the conformal structure on T and the reflected conformal structure on its double, and since the hyperelliptic involution on any surface is unique, it follows that η is uniquely determined by the conformal structure on T. □

Remark 2.1. One can prove Proposition 2.1 using the fact that a torus with a hole has a unique representation as a closed torus with a Euclidean disc removed. It was also remarked by Abikoff (personal communication) that one can prove this fact using existence and uniqueness of the infinite Nielsen extension. However, there does not appear to be a simple direct proof of the uniqueness of the involution.

It follows from the preceding that, if a and b are any pair of generators of a purely hyperbolic Fuchsian group H representing a torus with a hole, and if j is the half-turn with fixed point at the point of intersection of A_a and A_b, then j projects to the unique conformal involution η on $T = \mathbb{H}^2/H$. Also, L_a and L_b each pass through two of the three fixed points of η.

Proposition 2.2. *Let L be a simple closed nondividing geodesic on the torus with a hole T. Then L passes through two of the three fixed points of the unique conformal involution $\eta\colon T \to T$.*

Proof. It is well known that, given L, there is another simple closed geodesic M crossing L exactly once. We choose directions on each of these geodesics, so that they determine elements a and b in the fundamental group (i.e., $L = L_a$ and $M = L_b$). It is also well known that a and b generate $\pi_1(T)$.

We saw before that L_a and L_b each necessarily pass through two of the three fixed points of η. □

PROPOSITION 2.3. *Let T be a torus with a hole, and let x be a fixed point of the unique conformal involution $\eta\colon T \to T$. There are infinitely many simple closed geodesics passing through x.*

Proof. Let a and b be generators of the Fuchsian group representing T, and let j be the half-turn with center at the crossing point z of A_a and A_b. Then L_a and L_b are simple geodesics on T crossing exactly once. Call the point of intersection x_1, and let $x_2 \in L_a$ and $x_3 \in L_b$ be the other two fixed points of η.

Let c be either ab or $a^{-1}b$. Then L_c is a simple geodesic on T, crossing both L_a and L_b exactly once. Since L_c contains exactly two of the three fixed points of η and since L_c crosses each of L_a and L_b exactly once, L_c passes through x_2 and x_3 but not x_1.

We have shown that, if a and b are generators of $\pi_1(T)$ (where the corresponding geodesics L_a and L_b cross at x_1) and if L_a also passes through x_2 and L_b also passes through x_3, then the elementary Nielsen transformations $(a, b) \mapsto (a, ab)$ and $(a, b) \mapsto (a, a^{-1}b)$ both yield two new generators whose (simple) geodesics cross at x_2; also, the elementary Nielsen transformations $(a, b) \mapsto (ab, b)$ and $(a, b) \mapsto (a^{-1}b, b)$ both yield two new generators whose (simple) geodesics cross at x_3. Since the (outer) automorphism group of a free group is an infinite group generated by Nielsen transformations, we can now easily construct a sequence of distinct pairs of generators crossing at any one of the three points x_1, x_2, x_3. □

3. Jørgensen's Theorem

We next prove Jørgensen's theorem [5] (see also [6]) on crossing points of geodesics.

PROPOSITION 3.1. *Let L and M be closed geodesics crossing at the point x on the hyperbolic orbifold S. Then there are infinitely many distinct closed geodesics passing through x.*

Proof. Let G be the Fuchsian group representing S, let A_a and A_b be the axes of the elements a and b (respectively), where A_a projects to $L = L_a$ and A_b projects to $M = L_b$. We can assume without loss of generality that a and b have been chosen so that A_a and A_b cross at z, which projects to x.

The four fixed points of the hyperbolic elements a and b are distinct, so there is a sufficiently high-power n such that $H = \langle a^n, b^n\rangle$ is the uniformizing group of a torus with a hole T. We know from Proposition 2.3 that there are infinitely many closed geodesics on T passing through the projection of z. Hence, there are infinitely many axes of hyperbolic transformations of G passing through z. Since the stabilizer of z in G is finite and since each closed geodesic on S can pass through x only finitely often, these infinitely many axes project to infinitely many distinct closed geodesics on S. $\square$

4. Precisely Embedded Subgroups

From here on, we assume that all Fuchsian groups are purely hyperbolic.

Let H be a subgroup of the Fuchsian group G. If H is non-elementary, then its convex hull $C(H)$ has non-empty interior; if H is a hyperbolic cyclic group, then $C(H)$ consists of a single hyperbolic line, the axis of H; if H is trivial, then $C(H) = \emptyset$. In any case, $C(H)$ is H-invariant.

Assume that H is non-elementary. We define $C^0(H)$ to be the interior of $C(H)$. We say that H is *precisely embedded* in G if $g(C^0(H)) \cap C^0(H) = \emptyset$ for all $g \in G - H$; that is, $C^0(H)$ is precisely invariant under H in G.

Since $H \subset G$, there is a natural covering map $p_H : \mathbb{H}^2/H \to \mathbb{H}^2/G$. One easily sees that the statement that H is precisely embedded in G is equivalent to the statement that p_H is injective on $c^0(H)$. Also, since G is torsion-free, p_H is a local homeomorphism. It follows that if H is precisely embedded in G then we can regard $c^0(H)$ as a subsurface of $c^0(G)$, which in turn is a subsurface of $S = \mathbb{H}^2/G$.

4.1. The Elementary Case. If H is hyperbolic cyclic, then $C(H)$ is a single hyperbolic line L. In this case, we say that H is *precisely embedded* if L projects to a simple closed geodesic on S and if $H = \mathrm{Stab}(L)$.

If H is a hyperbolic cyclic precisely embedded subgroup of G, then we choose some sufficiently small $\varepsilon > 0$ and define $C^0(H)$ to be the set of all points at distance less than ε from L. The collar lemma (see e.g. [1] or [2, p. 94]) asserts that there is an $\varepsilon > 0$ depending only on the length of L/H such that $C^0(H)$ is precisely invariant under H in G; that is, $C^0(H)$ projects to an embedded topological annulus (collar) on S.

The trivial subgroup is precisely embedded in every group.

Proposition 4.1. *If H is precisely embedded in the finitely generated Fuchsian group G, then H is finitely generated.*

Proof. There is nothing to prove if H is elementary, so we assume that H is non-elementary.

It is well known that a non-elementary Fuchsian group G is finitely generated if and only if $c^0(G)$ has finite area. Since G is finitely generated, $c^0(G)$ has finite area. Since $H \subset G$, $C^0(H) \subset C^0(G)$. Then, since p_H is injective on $c^0(H)$ and $p_H(c^0(H)) \subset c^0(G)$, it follows that $c^0(H)$ has finite area. $\square$

Proposition 4.2. *Let G be a Fuchsian group, and let S_0 be a subsurface of $S = \mathbb{H}^2/G$. Let $\tilde{S}_0 \subset \mathbb{H}^2$ be a connected component of the preimage of S_0, and let $H = \mathrm{Stab}(\tilde{S}_0)$. Then H is precisely embedded in G.*

Proof. There is nothing to prove if H is trivial, so we assume that H is nontrivial.

Since S_0 is a subsurface of S, $\tilde{S}_0$ is precisely invariant under its stabilizer H in G. Also, since $\tilde{S}_0$ is connected, for every primitive hyperbolic element $a \in H$ there is an a-invariant path $V_a \subset \tilde{S}_0$; since it is a-invariant, the endpoints of V_a lie at the endpoints of A_a.

Let a and b be not necessarily distinct (hyperbolic) elements of H, and let g be some element of G, $g \notin H$. Since $\tilde{S}_0$ is precisely invariant under H, we have that $g(V_a) \cap V_b = \emptyset$. Looking at the endpoints of these curves on the circle at infinity, we see that $g(A_a) \cap A_b = \emptyset$.

It then follows that, if $H = \langle a \rangle$ is cyclic, then no translate of A_a crosses A_a. Hence, in this case, H is precisely embedded. From here on, we assume that H is non-elementary.

The complement of $C(H)$ consists of an H-invariant set of boundary half-planes, where each boundary half-plane is either a boundary axis or is bounded by a line whose stabilizer is the identity, where both of its endpoints are limit points of H; this latter possibility occurs only if H is infinitely generated.

We first take up the case that H is finitely generated, in which case $C(H)$ is bounded only by boundary axes. Let b be a boundary element of H. Then, since no axis of $\Lambda(gHg^{-1})$ crosses A_b, the entire limit set of gHg^{-1} lies on one side of A_b. Either there is a particular boundary axis $A_{b'}$ separating $\Lambda(H)$ from $\Lambda(gHg^{-1})$ or not. If not, then $C(gHg^{-1}) \subset C(H)$, from which it follows that there is an $a \in H$, so that $A_{gag^{-1}} \subset C(H)$. Then, since the fixed points of hyperbolic elements are dense in $\Lambda(H) \times \Lambda(H)$, there is some $b \in H$ with A_b crossing $A_{gag^{-1}} = g(A_a)$, which cannot be.

For the infinitely generated case, we note that the hyperbolic lines on the boundary of $C(H)$ are either boundary axes (in which case the previous argument applies) or they are lines that are precisely invariant under the identity in H, where both endpoints are limit points of H. In this latter case, we can approximate the line on the boundary of $C(H)$ with axes of H. As before, $\Lambda(gHg^{-1})$ lies entirely on one side of each of these axes, so either there is a line on the boundary of $C(H)$ separating $C^0(gHg^{-1})$ from $C^0(H)$ or $g(C(H)) = C(gHg^{-1}) \subset C(H)$. Again, since the axes of hyperbolic elements are dense in $\Lambda(H) \times \Lambda(H)$, this latter possibility cannot occur. $\square$

5. Conjugation Invariant Subgroups

Let j be a half-turn. A subgroup $H \subset G$ is *conjugation invariant under* j (or simply *conjugation invariant*, if there is no danger of confusion) if $jHj^{-1} = H$. One sees at once that, if H is conjugation invariant, then $j(C(H)) = C(H)$.

PROPOSITION 5.1. *Let G be a finitely generated Fuchsian group; let j be a half-turn, and let H_1 and H_2 be nontrivial, conjugation invariant, precisely embedded subgroups of G. Then there is a conjugation invariant precisely embedded subgroup $H \subset G$ with $H \supset (H_1 \cup H_2)$.*

Proof. Let z be the fixed point of j, and let x be the projection of z to $S = \mathbb{H}^2/G$. For $i = 1, 2$, since H_i is precisely embedded in G, $S_i = c^0(H_i)$ is embedded in

$c^0(G) \subset S$; hence we can consider S_1 and S_2 as subsurfaces of S. Also, since $C^0(H_i)$ is connected, S_i is connected.

If H_i is cyclic, then S_i consists of a metric tubular neighborhood of a simple closed geodesic; in this case, the fixed point x necessarily lies on that geodesic and so is an interior point of S_i. If H_i is non-elementary then, by Proposition 4.1, H_i is finitely generated. Then S_i is bounded by a finite number of simple disjoint closed geodesics; here again, one easily sees that x necessarily lies in the interior of S_i.

Since H_i is conjugation invariant, j projects to a conformal involution η_i of S_i with fixed point x. Since η_1 and η_2 are both projections of j, there is a neighborhood $U \subset S_1 \cap S_2$ of x for which $\eta_1 \big| U = \eta_2 \big| U$. By the identity theorem, there is a single involution η acting on $S_1 \cup S_2$, where $\eta \big| S_i = \eta_i$.

Let $\tilde{S} \subset \mathbb{H}^2$ be the connected component of the preimage of $S_1 \cup S_2$ containing z, and let $H = \mathrm{Stab}(\tilde{S})$. Since $S_1 \cup S_2$ is η-invariant and x is an interior point of $S_1 \cap S_2$, it follows that $\tilde{S}$ is j-invariant. We conclude that $H = \mathrm{Stab}(\tilde{S})$ is conjugation invariant. Also, since S_1 and S_2 are both connected and $S_1 \cap S_2$ has non-empty interior, $S_1 \cup S_2$ is connected.

Since $S_1 \cup S_2$ is connected, $\tilde{S}$ contains liftings of both S_1 and S_2; that is, $C^0(H_i) \subset \tilde{S}$ $(i = 1, 2)$. It follows that $H = \mathrm{Stab}(\tilde{S})$ contains the stabilizers of both $C^0(H_1)$ and $C^0(H_2)$. Hence H contains both H_1 and H_2.

Since $S_1 \cup S_2$ is a subsurface of S, $\tilde{S}$ is precisely invariant under its stabilizer H. It then follows from Proposition 4.2 that H is precisely embedded in G. □

PROPOSITION 5.2. *Let G be a finitely generated Fuchsian group, and let j be a half-turn. Then there is a unique maximal, conjugation invariant, precisely embedded subgroup $H \subset G$.*

Proof. For any (finitely generated) Fuchsian group K, let $\mathrm{Area}(K)$ denote the hyperbolic area of $c(K)$. We have already observed that if H is a precisely embedded subgroup of G then $c^0(H)$ is embedded in $c^0(G)$, from which it follows that $\mathrm{Area}(H) \leq \mathrm{Area}(G)$.

Further, if H is precisely embedded in G and $H \neq G$ then, since we are dealing only with purely hyperbolic groups, there is a hyperbolic element $a \in G$ whose axis lies outside $C(G)$. It follows that, in this case, $c^0(H)$ is properly contained in $c^0(G)$, from which we conclude that $\mathrm{Area}(H) < \mathrm{Area}(G)$.

Next observe that, if H is precisely embedded in G, then it is precisely embedded in every intermediate subgroup $H \subset H' \subset G$. Hence, if $H' \supset H$, $H' \neq H$, and H' is also precisely embedded and conjugation invariant, then $\mathrm{Area}(H') > \mathrm{Area}(H)$. Our result now follows from Proposition 5.1, together with the fact that the set of areas of (unramified) hyperbolic surfaces is discrete in $\mathbb{R}$. □

6. The Index of a Half-Turn

Let j be a half-turn; let G be a finitely generated Fuchsian group; and let H be the maximal, precisely embedded, conjugation invariant subgroup of G.

We first take up the case that H is of the first kind; that is, its limit set is the entire circle at infinity. Then $C^0(H) = \mathbb{H}^2$. Since H is precisely embedded in G,

$H = G$. Then $S = \mathbb{H}^2/G$ is a closed surface of some genus p; we set the rank of H (relative to j) to be $2p$.

If H is of the second kind then, since G is finitely generated and purely hyperbolic, H is either trivial or free on a finite number n of generators. If H is trivial, then we define the rank of H (relative to j) to be 0. If H is nontrivial, then the rank of H (relative to j) is the rank of the free group H. In any case, if $H = G$ then $jGj^{-1} = G$, so j projects to a conformal involution η acting on $S = \mathbb{H}^2/G$.

The involution j has a unique fixed point $z \in \mathbb{H}^2$; we also think of the rank of H as being the rank of z or of its projection $x \in S = \mathbb{H}^2/G$.

7. Geodesic Necklaces

Let $\mathcal{L} = L_1, \ldots, L_{2p+2}$ be a *geodesic necklace* on the closed surface S of genus $p \geq 2$. That is, $L_1, \ldots, L_{2p+2}$ are distinct, unoriented, simple nondividing geodesics, called the *links* of $\mathcal{L}$; regarding these links as being circularly ordered, each L_i crosses L_{i-1} at exactly one point, crosses L_{i+1} at exactly one point, and crosses no other link of $\mathcal{L}$. The $2p + 2$ crossing points of the links are called the *ties* of $\mathcal{L}$. A link in $\mathcal{L}$ is *evenly spaced* if the two arcs between the two ties lying on it have equal length; $\mathcal{L}$ itself is *evenly spaced* if every link is evenly spaced.

A closed Riemann surface of genus p is hyperelliptic if it admits an orientation preserving conformal involution, called the hyperelliptic involution, with exactly $2p + 2$ fixed points. Equivalently, S is hyperelliptic if it is conformally equivalent to a two-sheeted covering of the Riemann sphere (necessarily branched over $2p + 2$ points). It is well known that there is at most one hyperelliptic involution on any Riemann surface. It is also well known that, on a hyperelliptic surface, the Weierstrass points are exactly the fixed points of the hyperelliptic involution. Proofs of these facts can be found for example in [3].

REMARK 7.1. The construction of a geodesic necklace on a hyperelliptic surface, where every link is invariant under the hyperelliptic involution and the ties occur exactly at the Weierstrass points, appears in [9]; for the convenience of the reader, we outline the construction here.

PROPOSITION 7.1. *If S is hyperelliptic then there is an evenly spaced geodesic necklace $\mathcal{L}$ on S, where the individual links of $\mathcal{L}$ are invariant under the hyperelliptic involution η and the ties of $\mathcal{L}$ are at the fixed points of η.*

Proof. Regard S as a two-sheeted covering of the Riemann sphere branched over the points $z_1, \ldots, z_{2p+2}$, where these points have been ordered in some fashion. Construct paths $V_1', \ldots, V_{2p+2}'$, where V_i' connects z_i to z_{i+1} (V_{2p+2}' connects z_{2p+2} to z_1) and where these $2p + 2$ paths are nonintersecting, except that z_i is a common point of V_{i-1}' and V_i'. Let V_i be the corresponding geodesic on the sphere, where the hyperbolic metric is defined by making these $2p+2$ points special points of order 2. Let L_i be the preimage on S of V_i. Using the standard conformal structure near a special point of order 2 (that is, two directions at an angle of α at z_i

lift to two directions at an angle of $\alpha/2$ at the preimage of z_i on S), it is easy to observe that each L_i is an η-invariant simple closed geodesic; that L_i and L_{i+1} intersect exactly at the preimage of z_i, which is a fixed point of η; and that L_i and L_j intersect only if $j = i \pm 1$. Hence $\mathcal{L} = L_1, \ldots, L_{2p+2}$ is a geodesic necklace. Since the conformal involution η has its fixed points at the ties of $\mathcal{L}$, every geodesic passing through a fixed point of η is necessarily η-invariant; hence, every link of $\mathcal{L}$ is η-invariant.

Near each fixed point, η interchanges the two arcs of each link passing through that fixed point; hence η interchanges the two arcs of each link cut out by the two fixed points lying on that link. Since η is a hyperbolic isometry, each link is evenly spaced. Therefore, $\mathcal{L}$ is evenly spaced. □

PROPOSITION 7.2. *If there is a geodesic necklace $\mathcal{L} = L_1, \ldots, L_{2p+2}$ on S with $2p - 2$ successive evenly spaced links, then*:

(i) *S is hyperelliptic*;
(ii) *every link of $\mathcal{L}$ is invariant under the hyperelliptic involution*;
(iii) *the ties of $\mathcal{L}$ lie at the Weierstrass points on S*; *and*
(iv) *$\mathcal{L}$ is evenly spaced*.

Proof. Suppose there is a geodesic necklace $\mathcal{L}$ on S with $2p-2$ successive evenly spaced links; we label the links of $\mathcal{L}$, in order, so that the evenly spaced links are $L_2, \ldots, L_{2p-1}$. For $i = 1, \ldots, 2p+1$, let x_i be the tie between L_i and L_{i+1}. We orient each link in some manner and reparameterize it (if necessary) using arc-length as the parameter, so that x_1 is the base point of both L_1 and L_2 and so that, for $i = 3, \ldots, 2p$, x_{i-1} is the base point of L_i. Let L_i^+ be the arc of L_i going in the positive direction from x_{i-1} to x_i. We choose x_1 as the base point for $\pi_1(S)$. For each $i = 3, \ldots, 2p$, we choose the path $v_i = L_2^+ \cdots L_{i-1}^+$ from x_1 to x_{i-1}. (We use $a \cdot b$ to denote the path: first a then b.) Set $L_1' = L_1$, $L_2' = L_2$, and, for $i = 3, \ldots, 2p$, $L_i' = v_i \cdot L_i \cdot v_i^{-1}$.

It is easy to see that, if one cuts S along the geodesics $L_1, \ldots, L_{2p}$, then the resulting compact 2-manifold is both connected and simply connected. Hence, any loop on S that is based at x_1 can be deformed so as to lie on the graph formed by $L_1, \ldots, L_{2p}$; further, any loop based at x_1 is homotopic to a product of the L_i' and their inverses. We conclude that $L_1', \ldots, L_{2p}'$ generate $\pi_1(S, x_1)$.

We choose some point $z_1 \in \mathbb{H}^2$ lying over x_1 and, for $i = 2, \ldots, 2p$, let z_i be the endpoint of the lifting of v_i starting at z_1.

Let A_1 (resp. A_2) be the hyperbolic line lying over L_1 (resp. L_2) and passing through z_1. For each $i = 3, \ldots, 2p$, let A_i be the hyperbolic line lying over L_i and passing through z_{i-1}. We can regard A_i as being oriented with the orientation lifted from L_i. There is a unique primitive element $a_i \in G$, where the (oriented) axis of a_i is A_i. We have constructed a_i so that, under the natural correspondence, it is the element of G corresponding to L_i'. Hence $G = \langle a_1, \ldots, a_{2p} \rangle$.

For $i = 2, \ldots, 2p-1$, L_i is evenly spaced. This means that, on A_i, z_i lies halfway between z_{i-1} and $a_i(z_{i-1})$. For $i = 2, \ldots, 2p-1$, let j_i be the half-turn with fixed point at z_i. Then the fact that z_i lies halfway between z_{i-1} and $a_i(z_{i-1})$

can be interpreted as saying that $j_{i-1}\circ a_i^{-1}=j_i$ for $i=2,\dots,2p-1$. Since j_{i-1} is an involution, we obtain

$$j_{i-1}=j_i\circ a_i=a_i^{-1}\circ j_i,\quad i=2,\dots,2p-1. \tag{1}$$

Also, since j_i is the half-turn with fixed point at z_i, which is the point of intersection of L_i with L_{i+1}, we have that, for all $i=1,\dots,2p-1$,

$$j_i\circ a_i\circ j_i^{-1}=a_i^{-1},\qquad j_i\circ a_{i+1}\circ j_i^{-1}=a_{i+1}^{-1}. \tag{2}$$

Let $j=j_1$ be the half-turn with fixed point at z_1. We have already observed in (2) that

$$j\circ a_1\circ j^{-1}=a_1^{-1}\quad\text{and}\quad j\circ a_2\circ j^{-1}=a_2^{-1}.$$

Furthermore, using equations (1) and (2) yields

$$\begin{aligned} j\circ a_3\circ j^{-1}&=j_2\circ a_2\circ a_3\circ a_2^{-1}\circ j_2^{-1}\\ &=a_2^{-1}\circ j_2\circ a_3\circ j_2^{-1}\circ a_2\\ &=a_2^{-1}\circ a_3^{-1}\circ a_2. \end{aligned}$$

Thus we have shown that $j\circ a_3\circ j^{-1}\in G$.

Continuing in this fashion, for every $i=4,\dots,2p$ we have that

$$\begin{aligned} j\circ a_i\circ j^{-1}&=j_1\circ a_i\circ j_1^{-1}\\ &=a_2^{-1}\circ j_2\circ a_i\circ j_2^{-1}\circ a_2\\ &\;\;\vdots\\ &=a_2^{-1}\circ a_3^{-1}\circ\cdots\circ a_{i-1}^{-1}\circ j_i\circ a_i\circ j_i^{-1}\circ a_{i-1}\circ\cdots\circ a_2\\ &=a_2^{-1}\circ\cdots\circ a_{i-1}^{-1}\circ a_i^{-1}\circ a_{i-1}\circ\cdots\circ a_2. \end{aligned}$$

We have shown that $j\circ a_i\circ j^{-1}\in G$ for all $i=1,\dots,2p$. Therefore, j has rank $2p$ in G; that is, G is conjugation invariant.

It follows from equation (1) that $j=j_1,\dots,j_{2p-1}$ all project to the same conformal homeomorphism η of S. Since each j_i is an involution, so is η. Of course, the points $z_1,\dots,z_{2p-1}$ project to the $2p-1$ distinct ties $x_1,\dots,x_{2p-1}$ of $\mathcal{L}$; hence η has at least $2p-1$ fixed points. It is an easy consequence of Hurwitz's theorem that the number of fixed points of a conformal involution on a closed surface of genus p is either 0 or of the form $2p+2-4k$, $0\le k\le[p/2]$. Since η has $2p-1$ fixed points, it has $2p+2$ fixed points. Thus, S is hyperelliptic and η is the unique hyperelliptic involution. This concludes the proof of (i).

We have shown that η has fixed points at those ties of $\mathcal{L}$ that lie between the given $2p-2$ evenly spaced links $L_2,\dots,L_{2p-1}$. It follows that η keeps each of these links invariant. Also, we have defined η as the projection of $j=j_1$, which keeps invariant the axis of a_1; hence $\eta(L_1)=L_1$. Similarly, j_{2p-1}, which also projects to η, keeps invariant the axes of a_{2p-1} and a_{2p}; it follows that $\eta(L_{2p})=L_{2p}$. We note that L_{2p+1} is the unique (unoriented) simple closed geodesic that is disjoint from $L_1, \ldots, L_{2p-1}$ and crosses L_{2p} exactly once. Since η keeps $L_1, \ldots, L_{2p}$ invariant, it also keeps L_{2p+1} invariant. Finally, L_{2p+2} is the unique simple closed geodesic that is disjoint from $L_2, \ldots, L_{2p}$ and crosses each of L_1 and L_{2p+1} exactly once. Since η preserves $L_1, \ldots, L_{2p+1}$, it also preserves L_{2p+2}. This concludes the proof of (ii).

We already know that the links $L_2, \ldots, L_{2p-1}$ are evenly spaced and that the ties between these links are at fixed points of η. We also know that the tie between L_1 and L_2 is a fixed point of η. Since $\eta(L_1) = L_1$, η has a second fixed point on L_1. Likewise, since $\eta(L_{2p-1}) = L_{2p-1}$, it follows that η has a second fixed point on L_{2p-1}. Then, since $\eta(L_{2p}) = L_{2p}$, the second fixed of η on L_{2p-1} must lie on the tie between L_{2p-1} and L_{2p}. Next, η keeps L_{2p} invariant and so has a second fixed point on L_{2p}; then, since η keeps L_{2p+1} invariant, this second fixed point must be the tie between L_{2p} and L_{2p+1}. Next, η has a fixed point on L_{2p+1} and so has a second fixed point on L_{2p+1}; then, since η keeps L_{2p+2} invariant, this second fixed point is at the tie between L_{2p+1} and L_{2p+2}. Finally, once we have one fixed point of η on L_{2p+2}, there must be a second fixed point at the tie between L_{2p+2} and L_1. This concludes the proof of (iii).

Each link of $\mathcal{L}$ is η-invariant, and the ties lie at the fixed points of η, which acts as an isometry in the hyperbolic metric. Conclusion (iv) now follows.

Combining the results of Propositions 7.1 and 7.2, we have proven the following.

THEOREM 7.1. *A closed Riemann surface S of genus $p \geq 2$ is hyperelliptic if and only if there is an evenly spaced geodesic necklace on it, which in turn occurs if and only if there is a necklace on it with $2p-2$ successive evenly spaced links.*

8. Special Case of Genus 2

PROPOSITION 8.1. *On a closed Riemann surface of genus* 2, *every geodesic necklace is evenly spaced, with its ties lying at the fixed points of the hyperelliptic involution.*

Proof. It was shown by Haas and Susskind [4] that, in genus 2, every simple closed geodesic is invariant under the hyperelliptic involution. Thus, if v and w are a pair of simple closed nondividing geodesics meeting at one point, then this point of intersection must be a Weierstrass point—that is, a fixed point of the hyperelliptic involution. Hence every tie of every geodesic necklace is a Weierstrass point. It follows that the hyperelliptic involution interchanges the two arcs of each link between the ties. Hence, as before, every geodesic necklace is evenly spaced. □

REMARK 8.1. Even in genus 2, the relationships among the six lengths and angles of a geodesic necklace can be quite complicated. A set of six real parameters for the Teichmüller space of surfaces of genus 2, using three of these lengths and two of these angles together with the hyperbolic length of one other geodesic arc, appears in [8].

References

[1] A. Basmajian, *Constructing pairs of pants,* Ann. Acad. Sci. Fenn. Ser. A I Math. 15 (1990), 65–74.

[2] P. Buser, *Geometry and spectra of compact Riemann surfaces,* Birkhäuser, Boston, 1992.

[3] H. M. Farkas and I. Kra, *Riemann surfaces,* Springer-Verlag, New York, 1980.

[4] A. Haas and P. Susskind, *The geometry of the hyperelliptic involution in genus two,* Proc. Amer. Math. Soc. 105 (1989), 159–165.

[5] T. Jørgensen, *Closed geodesics on Riemann surfaces,* Proc. Amer. Math. Soc. 72 (1978), 140–142.

[6] M. Lustig, *Weierstrass points and multiple intersection points of closed geodesics,* Geom. Dedicata 48 (1993), 1–14.

[7] B. Maskit, *The conformal group of a plane domain,* Amer. J. Math. 90 (1968), 718–722.

[8] ———, *New parameters for Fuchsian groups of genus 2,* Proc. Amer. Math. Soc. 127 (1999), 3643–3652.

[9] H. E. Rauch and A. Lebowitz, *Elliptic functions, theta functions, and Riemann surfaces,* Williams & Wilkins, Baltimore, 1973.

[10] G. Rosenberger, *All generating pairs of all two-generator Fuchsian groups,* Arch. Math. (Basel) 46 (1986), 198–204.

[11] P. Schmutz-Schaller, *Geometry of Riemann surfaces based on closed geodesics,* Bull. Amer. Math. Soc. (N.S.) 35 (1998), 193–214.

Mathematics Department
The University at Stony Brook
Stony Brook, NY 11794-3651

bernie@math.sunysb.edu