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A Complete Bounded Minimal Cylinder in $\mathbb{R}^3$

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1. Introduction

Calabi asked if it were possible to have a complete minimal surface in $\mathbb{R}^3$ entirely contained in a half-space. As a consequence of the strong half-space theorem [6], no such surfaces are properly immersed. The first examples of complete orientable nonflat minimal surfaces with a bounded coordinate function were obtained by Jorge and Xavier [7]. Their construction is based on an ingenious idea of using Runge's theorem. Later, Brito [1] discovered a new method to construct surfaces of this kind. Examples of complete minimal surfaces with nontrivial topology, contained in a slab of $\mathbb{R}^3$, were obtained by Rosenberg and Toubiana [12], López [8; 9], Costa and Simoes [3], and Brito [2], among others.

A few years ago, Nadirashvili [10] used Runge's theorem in a more elaborate way to produce a complete minimal disc inside a ball in $\mathbb{R}^3$ (see also [4]).

In this paper we generalize the techniques used by Nadirashvili to obtain new examples of complete minimal surfaces inside a ball in $\mathbb{R}^3$ that have the conformal structure of an annulus. To be more precise, we have proved the following.

Theorem 1. *There exist an open set A of $\mathbb{C}$ and a complete minimal immersion $X: A \to \mathbb{R}^3$ satisfying*:

(1) *$X(A)$ is a bounded set of $\mathbb{R}^3$*;
(2) *A has the conformal type of an annulus.*

This theorem is proved in Section 3.

We have obtained the immersion X as limit of a sequence of bounded minimal annuli with boundary. To construct the sequence we require a technical lemma whose proof is exhibited in Section 4. This lemma allows us to modify the intrinsic metric of a minimal annulus around the boundary without excessively increasing the diameter of the annulus in $\mathbb{R}^3$.

2. Background and Notation

The aims of this section are to establish the principal notation used in the paper and to summarize some results about minimal surfaces.

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We set $D_r = \{z \in \mathbb{C} : |z| < r\}$, $S_r = \{z \in \mathbb{C} : |z| = r\}$, and $D^* = D_1 \setminus \{0\}$. Let $X\colon D^* \to \mathbb{R}^3$ be a conformal minimal immersion. Then

$$\phi_j = \frac{\partial X_j}{\partial u} - i\frac{\partial X_j}{\partial v}, \quad j = 1, 2, 3, \;\; z = u + iv, \tag{1}$$

are holomorphic functions on D^*, with real residues at 0, verifying $\sum_{j=1}^{3} \phi_j^2 \equiv 0$ and $\sum_{j=1}^{3} |\phi_j|^2 \not\equiv 0$. If we define

$$f = \phi_1 - i\phi_2 \quad \text{and} \quad g = \frac{\phi_3}{\phi_1 - i\phi_2}, \tag{2}$$

then g is a meromorphic function on D^* that coincides with the stereographic projection of the Gauss map. The behavior of f is determined by the rule that f is holomorphic on D^*, with zeroes precisely at the poles of g, but with twice order.

Conversely, if f and g are (respectively) a holomorphic and meromorphic function on D^* such that

$$\phi_1 = \frac{f}{2}(1 - g^2), \quad \phi_2 = i\frac{f}{2}(1 + g^2), \quad \phi_3 = fg \tag{3}$$

are holomorphic functions on D^* and if ϕ_1, ϕ_2, ϕ_3 have no real periods in zero, then

$$X\colon D^* \to \mathbb{R}^3,$$
$$X(z) = \operatorname{Re} \int_{z_0}^{z} (\phi_1(w), \phi_2(w), \phi_3(w))\, dw + c, \quad z_0 \in D^*, \;\; c \in \mathbb{R}^3, \tag{4}$$

is a conformal minimal immersion. It is usual to label $\phi = (\phi_1, \phi_2, \phi_3)$ as the Weierstrass representation of the immersion X. We can write the conformal metric associated to the immersion X, $\lambda_X^2(z)\langle\cdot,\cdot\rangle$, in terms of the Weierstrass representation as follows:

$$\lambda_X(z) = \frac{1}{2}|f(z)|(1 + |g(z)|^2) = \frac{\|\phi(z)\|}{\sqrt{2}}. \tag{5}$$

For more details on minimal surfaces, see [11].

If $\phi\colon D^* \to \mathbb{C}^3$ is holomorphic then we say that ϕ is of z^2*-type* if $\phi_j(z) = \hat{\phi}_j(z^2)$ for $j = 1, 2, 3$, where $\hat{\phi}_j$ are holomorphic functions on D^*. When the Weierstrass representation ϕ is a z^2-type map, then $X(z) + X(-z)$ is constant on D^*. Hence, we define $S(X) = X(z) + X(-z)$ for any one particular $z \in D^*$.

Let α be a curve in D^*. By length(α, X) we mean the length of α with the metric associated to immersion X. For $T \subset D^*$ we define the following distance: If $a, b \in T$ let $\operatorname{dist}_{(X,T)}(a, b) = \inf\{\text{length}(\alpha, X) \mid \alpha\colon [0, 1] \to T,\ \alpha(0) = a,\ \alpha(1) = b\}$. If $A \subset T$, then $\operatorname{dist}_{(X,T)}(z, A)$ means the distance between point z and set A. Any other distance or length that we use without mentioning the metric will be associated to the Euclidean metric.

By a *polygonal pair* (P, Q) we mean a pair of closed simple curves in $\mathbb{R}^2$ formed by a finite number of straight segments verifying:

(a) $\overline{D_{1/3}} \subset \operatorname{Int}(Q) \subset \overline{\operatorname{Int}(Q)} \subset D_{2/3} \subset \overline{D_{2/3}} \subset \operatorname{Int}(P) \subset \overline{\operatorname{Int}(P)} \subset D_1$;
(b) $-z \in P$ for all $z \in P$ and $-z \in Q$ for all $z \in Q$,

where $\mathrm{Int}(\alpha)$ denotes the interior domain bounded by a Jordan curve α; the exterior domain is denoted by $\mathrm{Ext}(\alpha)$. For a pair (P, Q), we write $T = \mathrm{Int}(P) \setminus \overline{\mathrm{Int}(Q)}$. If $\xi > 0$ is small enough then (P^ξ, Q^ξ) represents a new polygonal pair, parallel to (P, Q), such that:

(i) the Euclidean distance in $\mathbb{R}^2$ from P to P^ξ is ξ;
(ii) the Euclidean distance in $\mathbb{R}^2$ from Q to Q^ξ is ξ;
(iii) the corresponding set T^ξ associated to (P^ξ, Q^ξ) is contained in T

(see Figure 1 on page 505).

3. Proof of the Theorem

In order to prove the main theorem, we need the following lemma.

LEMMA 1. *Let* $X\colon D^* \to \mathbb{R}^3$ *be a conformal minimal immersion. Consider the polygonal pair* (P, Q), $\rho, r > 0$ *and* $1 > k > 0$, *satisfying*:

(1) $(1-k)\rho < \mathrm{dist}_{(X,T)}(z, S_{2/3}) < \rho$ *for all* $z \in P \cup Q$;
(2) $X(T) \subset B_r = \{p \in \mathbb{R}^3 : \|p\| < r\}$;
(3) $X(z) = \mathrm{Re}\big(\int_{2/3}^{z} \phi(w)\,dw\big) + c$, *where* $c \in \mathbb{R}^3$ *and* $\phi\colon D^* \to \mathbb{C}^3$ *is of* z^2*-type*;
(4) $S(X) = 0$.

Then, for any $\varepsilon > 0$ *and for any* $s, \xi, k' > 0$ *verifying*

$$(1-k)\rho < \mathrm{dist}_{(X,T^\xi)}(z, S_{2/3}) < \rho \quad \forall z \in P^\xi \cup Q^\xi, \tag{6}$$

$$\rho < (1-k')(\rho + s), \tag{7}$$

$$\rho k < s, \tag{8}$$

there exist a polygonal pair $(\tilde{P}, \tilde{Q})$ *and a conformal minimal immersion* $Y\colon D^* \to \mathbb{R}^3$ *such that*:

(1) $(1-k')(\rho+s) < \mathrm{dist}_{(Y,\tilde{T})}(z, S_{2/3}) < \rho + s$ *for all* $z \in \tilde{P} \cup \tilde{Q}$;
(2) $Y(\tilde{T}) \subset B_R$ *where* $R = \sqrt{r^2 + (2s)^2} + \varepsilon$;
(3) $Y(z) = \mathrm{Re}\big(\int_{2/3}^{z} \psi(w)\,dw\big) + c'$, *where* $c' \in \mathbb{R}^3$ *and* $\psi\colon D^* \to \mathbb{C}^3$ *is of* z^2*-type*;
(4) $S(Y) = 0$;
(5) $\|Y - X\| < \varepsilon$ *in* T^ξ;
(6) $T^\xi \subset \mathrm{I}(\tilde{T})$ *and* $\tilde{T} \subset \mathrm{I}(T)$, *where* $\mathrm{I}(O)$ *denotes the topological interior of the set* O.

This lemma is similar in spirit to that used by Nadirashvili in [10]. However, we have worked with non–simply connected planar domains bounded by polygonal pairs, and so a period problem arises. To solve this problem we have made our Weierstrass data ϕ a z^2-type map. Furthermore, when we take the limit in the conformal structure of our minimal annuli, this structure must not degenerate. This is why we have dealt with pairs of parallel annuli T and T^ξ.

Lemma 1 is proved in Section 4.

We use the lemma to construct the sequence

$$\chi_n = (X_n : D^* \to \mathbb{R}^3,\ (P_n, Q_n), \varepsilon_n, \xi_n, k_n),$$

where X_n is a conformal minimal immersion, (P_n, Q_n) is a polygonal pair, and $\{\varepsilon_n\}$, $\{\xi_n\}$, $\{k_n\}$ are decreasing sequences of nonvanishing terms that converge to zero. The sequence $\{\chi_n\}$ must verify the following properties:

($\mathrm{A_n}$) $(1-k_n)\rho_n < \operatorname{dist}_{(X_n,T_n)}(z, S_{2/3}) < \rho_n$ for all $z \in P_n \cup Q_n$, where $\rho_n = \sum_{i=1}^n 1/i$;

($\mathrm{B_n}$) $(1-k_{n-1})\rho_{n-1} < \operatorname{dist}_{(X_{n-1}, T_{n-1}^{\xi_n})}(z, S_{2/3}) < \rho_{n-1}$ for all $z \in P_{n-1}^{\xi_n} \cup Q_{n-1}^{\xi_n}$;

($\mathrm{C_n}$) $X_n(T_n) \subset B_{r_n}$, where $r_1 > 1$ and $r_n = \sqrt{r_{n-1}^2 + (2/n)^2} + \varepsilon_n$;

($\mathrm{D_n}$) $S(X_n) = 0$;

($\mathrm{E_n}$) $X_n(z) = \operatorname{Re}\big(\int_{2/3}^z \phi^n(w)\,dw\big) + c_n$, where $c_n \in \mathbb{R}^3$ and $\phi^n : D^* \to \mathbb{C}^3$ is of z^2-type;

($\mathrm{F_n}$) $0 < k_n < 1$, $\rho_n k_n < 1/(n+1)$, and $\varepsilon_n < 1/n^2$;

($\mathrm{G_n}$) $\|X_n - X_{n-1}\| < \varepsilon_n$ in $T_{n-1}^{\xi_n}$;

($\mathrm{H_n}$) $\lambda_{X_n} \geq \alpha_n \lambda_{X_{n-1}}$ in $T_{n-1}^{\xi_n}$, where $\{\alpha_i\}_{i\in\mathbb{N}}$ is a sequence of real numbers such that $0 < \alpha_i < 1$ and $\big\{\prod_{i=1}^n \alpha_i\big\}_n$ converges to $1/2$ (e.g., take $\alpha_1 = \frac{1}{2}e^{1/2}$ and $\alpha_n = e^{-1/2^n}$ for $n > 1$);

($\mathrm{I_n}$) $T_n \subset \mathrm{I}(T_{n-1})$;

($\mathrm{J_n}$) $T_{n-2}^{\xi_{n-1}} \subset \mathrm{I}(T_{n-1}^{\xi_n})$;

($\mathrm{K_n}$) $T_{n-1}^{\xi_n} \subset \mathrm{I}(T_n)$.

For instance, we can take

$$\chi_1 = (X_1,\ (P_1, Q_1),\ \varepsilon_1 = 1/2,\ \xi_1,\ k_1 = 1/3),$$

where $X_1 : D^* \to \mathbb{R}^3$ is given by $X_1(u+iv) = 5/2(u, -v, 0)$ and where (P_1, Q_1) is a suitable polygonal pair. Suppose that we have $\chi_1, \ldots, \chi_n$.

Now we construct the $n+1$ term. Choose k_{n+1} verifying ($\mathrm{F_{n+1}}$) and ξ_{n+1} verifying ($\mathrm{B_{n+1}}$) and ($\mathrm{J_{n+1}}$) (the choice of ξ_{n+1} is possible because χ_n satisfies ($\mathrm{A_n}$) and ($\mathrm{K_n}$)). Moreover, we choose two decreasing and convergent sequences to zero, $\{\hat\varepsilon_m\}$ and $\{\hat\xi_m\}$, with $\hat\xi_m < \xi_{n+1}$ and $\hat\varepsilon_m < 1/(n+1)^2$ for all m. For each m we consider $Y_m : D^* \to \mathbb{R}^3$ and $(\tilde P_m, \tilde Q_m)$, as given by Lemma 1, for the following data:

$$X = X_n, \quad (P, Q) = (P_n, Q_n), \quad k' = k_{n+1}, \quad k = k_n, \quad \rho = \rho_n, \quad r = r_n,$$

$$s = 1/(n+1), \quad \varepsilon = \hat\varepsilon_m, \quad \xi = \hat\xi_m.$$

From assertion (5) of the lemma, we deduce that the sequence $\{Y_m\}$ converges to X_n on the space $\operatorname{Har}(\overline{T_n})$ of harmonic maps from $\overline{T_n}$ in $\mathbb{R}^3$. This implies that $\{\lambda_{Y_m}\}$ converges uniformly to λ_{X_n} in $T_n^{\xi_{n+1}}$ and hence there is a $m_0 \in \mathbb{N}$ such that

$$\lambda_{Y_{m_0}} \geq \alpha_{n+1}\lambda_{X_n} \ \text{ in } T_n^{\xi_{n+1}}. \tag{9}$$

We define $X_{n+1} = Y_{m_0}$, $(P_{n+1}, Q_{n+1}) = (\tilde P_{m_0}, \tilde Q_{m_0})$, and $\varepsilon_{n+1} = \hat\varepsilon_{m_0}$. Observe that k_{n+1}, ξ_{n+1}, and ε_{n+1} could be chosen sufficiently small so that the sequences

$\{k_i\}$, $\{\xi_i\}$, and $\{\varepsilon_i\}$ decrease and converge to zero. Because of the way in which we have chosen the term χ_{n+1}, it is easy to check (using Lemma 1) that χ_{n+1} verifies $(\mathrm{A_{n+1}})$, $(\mathrm{B_{n+1}})$, ..., $(\mathrm{K_{n+1}})$. This concludes the construction of the sequence $\{\chi_i\}$.

Now we define

$$A = \mathrm{I}\Big(\bigcap_{n\in\mathbb{N}} T_n\Big).$$

The open set A has the following properties.

(1) $A = \bigcup_n T_n^{\xi_{n+1}}$. To prove this, first note that properties $(\mathrm{I_n})$, $(\mathrm{J_n})$, and $(\mathrm{K_n})$ imply $\bigcup_n T_n^{\xi_{n+1}} \subseteq A$. On the other hand, suppose that $z \in A \setminus \bigcup_n T_n^{\xi_{n+1}}$. Then $z \in T_n \setminus T_n^{\xi_{n+1}}$ for all $n \in \mathbb{N}$. This implies that $z \in \partial A$, which is absurd (recall that A is open). This contradiction proves the equality.

(2) A is an open arc-connected set.

(3) $\mathbb{C} \setminus A$ has two connected components; one of them contains 0 and the other one is not bounded. Indeed, any point of $\mathbb{C} \setminus \bar{A}$ could be connected with 0 or ∞ by a continuous curve in $\mathbb{C} \setminus T_n$ if n is large enough. Then, $\mathbb{C} \setminus A$ has two connected components because $\mathbb{C} \setminus \bar{A}$ has two arc-connected components.

Therefore, A is a domain in $\mathbb{C}$ such that $\mathbb{C} \cup \{\infty\} \setminus A$ consists of two connected components; thus A is biholomorphic to $\mathbb{C} \setminus \{0\}$, $D \setminus \{0\}$, or $C_\vartheta = \{z \in \mathbb{C} : \vartheta < |z| < 1\}$ (see [5, Thm. IV.6.9]). But A is a hyperbolic domain, so $A \not\equiv \mathbb{C} - \{0\}$. Furthermore, A is a subset of the annulus $C_{1/3}$ and a generator of the homology of A also generates the homology of $C_{1/3}$. Therefore, $A \equiv C_\vartheta$ for a $\vartheta \in]0, 1[$.

Let K be a compact set that is a subset of A. Then there is an n_0 such that $K \subset T_{n-1}^{\xi_n}$ for all $n > n_0$. From $(\mathrm{G_n})$, we have:

$$\|X_N - X_{n-1}\| < \sum_{i=n}^{\infty} \varepsilon_i < \sum_{i=n}^{\infty} \frac{1}{i^2} \ \text{ in } K, \quad N > n > n_0.$$

Thus, the sequence of minimal immersion $\{X_n\}$ is a Cauchy sequence in $\mathrm{Har}(A)$. Consequently, Harnack's theorem implies that $\{X_n\}$ converges in $\mathrm{Har}(A)$.

Let $X : A \to \mathbb{R}^3$ be the limit of $\{X_n\}$. Then X has the following properties.

(i) X is minimal and conformal.

(ii) X is an immersion. Indeed, for any $z \in A$ there exists $n \in \mathbb{N}$ such that $z \in T_n^{\xi_{n+1}}$. From property $(\mathrm{H_i})$ it follows that, for all $k > n$,

$$\lambda_{X_k}(z) \geq \alpha_k \lambda_{X_{k-1}}(z) \geq \cdots \geq \alpha_k \cdots \alpha_{n+1} \lambda_{X_n}(z) \geq \alpha_k \cdots \alpha_1 \lambda_{X_{n(z)}}.$$

Taking the limit as $k \to \infty$, we deduce that

$$\lambda_X(z) \geq \tfrac{1}{2} \lambda_{X_n}(z) > 0 \tag{10}$$

and so X is an immersion.

(iii) $X(A)$ is bounded in $\mathbb{R}^3$. Let $z \in A$ and $n \in \mathbb{N}$ be such that $z \in T_n^{\xi_{n+1}}$; then

$$\|X(z)\| \leq \|X(z) - X_n(z)\| + \|X_n(z)\| \leq \tfrac{1}{2} + r_n$$

for an n large enough. The sequence $\{r_n\}$ is bounded in $\mathbb{R}$.

(iv) The annulus A is complete with the metric induced by X. Indeed, if n is large enough and taking (10) into account, one has:

$$\mathrm{dist}_{(X,T_n^{\xi_{n+1}})}(2/3,\partial T_n^{\xi_{n+1}}) > \tfrac{1}{2}\,\mathrm{dist}_{(X_n,T_n^{\xi_{n+1}})}(2/3,\partial T_n^{\xi_{n+1}}).$$

The right-hand side of this inequality is controlled by $(\mathrm{B_n})$, so we infer that

$$\mathrm{dist}_{(X,T_n^{\xi_{n+1}})}(2/3,\partial T_n^{\xi_{n+1}}) > \tfrac{1}{2}(1-k_n)\rho_n.$$

The completeness is due to the fact that $\left\{\frac{1}{2}(1-k_n)\rho_n\right\}_{n\in\mathbb{N}}$ diverges.

This completes the proof of the theorem.

4. Proof of the Lemma

This section is devoted to proving Lemma 1. As we mentioned before, it is a generalized version of that used by Nadirashvili in [10] and by Collin and Rosenberg in [4]. Although the proof is similar, we have introduced some new techniques that permit us to apply Nadirashvili's methods to non–simply connected planar domains.

The following proposition is a direct consequence of Runge's theorem and plays a crucial role in this section.

PROPOSITION 1. *Let $\tau > 1$ and let E_1, E_2 be two disjoint compact sets of $\mathbb{C}$ such that*:

(a) $E_i = -E_i$, $i = 1, 2$;

(b) $\mathbb{C}\setminus(E_1\cup E_2)$ *has two arc-connected components, one that contains zero and one that is not bounded*.

Then there exists $h\colon \mathbb{C}\setminus\{0\}\to\mathbb{C}$, a holomorphic nonnull function, such that:

(i) $|h-1| < 1/\tau$ *in* E_1;

(ii) $|h-\tau| < 1/\tau$ *in* E_2;

(iii) $h(z) = \hat{h}(z^2)$, *where $\hat{h}$ is a holomorphic function in* $\mathbb{C}\setminus\{0\}$.

Proof. Let $E_i^2 = \{z^2 : z \in E_i\}$, $i = 1, 2$. It is clear that E_1^2 and E_2^2 are disjoint and that $\mathbb{C}\setminus(E_1^2\cup E_2^2)$ has two connected components: one contains zero and the other is not bounded. Thanks to Runge's theorem, for any $\varepsilon > 0$ there exists a holomorphic function $\mu\colon \mathbb{C}\setminus\{0\}\to\mathbb{C}$ (with pole in zero) such that:

(a) $|\mu| < \varepsilon$ on E_1^2;

(b) $|\mu - a| < \varepsilon$ on E_2^2, where $e^a = \tau$.

We define $h(z) = e^{\mu(z^2)}$ for ε small enough. □

The main idea in the proof of Lemma 1 is to use Proposition 1 successively over a labyrinth constructed in a neighborhood of the boundary of T. We thus modify the intrinsic metric of our immersion near the boundary without increasing in excess the distance in $\mathbb{R}^3$. Hence, the next step is to describe some subsets of D^* that we use to construct the aforementioned labyrinth.

Consider (P, Q), the polygonal pair given in the statement of Lemma 1. Let s and s' be the number of sides of P and Q, respectively, and let N be a nontrivial multiple of s and s'.

REMARK 1. Along the proof of the lemma, a set of real positive constants $\{r_i, i = 1, \dots, 13\}$ depending on X, (P, Q), k, ρ, r, ε, s, ξ, and k' will appear. It is important to note that the choice of these constants does not depend on the integer N.

Let r_1 and r_2 be a lower and an upper bound (respectively) for the length of the sides of polygons P^ζ and Q^ζ for all $\zeta \le 2/N$. Let $v_1, \dots, v_{2N}$ be points in the polygon P that divide each side of P into $2N/s$ equal parts. We can transfer this partition to the polygon $P^{2/N}$: $v'_1, \dots, v'_{2N}$ (see Figure 1). We define the following sets:

$L_i =$ the segment that joins v_i and v'_i, $i = 1, \dots, 2N$;
$P_i = P^{i/N^3}$, $i = 0, \dots, 2N^2$;
$\mathcal{A} = \bigcup_{i=0}^{N^2-1} \overline{\mathrm{Int}(P_{2i}) \setminus \mathrm{Int}(P_{2i+1})}$ and $\tilde{\mathcal{A}} = \bigcup_{i=1}^{N^2} \overline{\mathrm{Int}(P_{2i-1}) \setminus \mathrm{Int}(P_{2i})}$;
$R = \bigcup_{i=0}^{2N^2} P_i$;

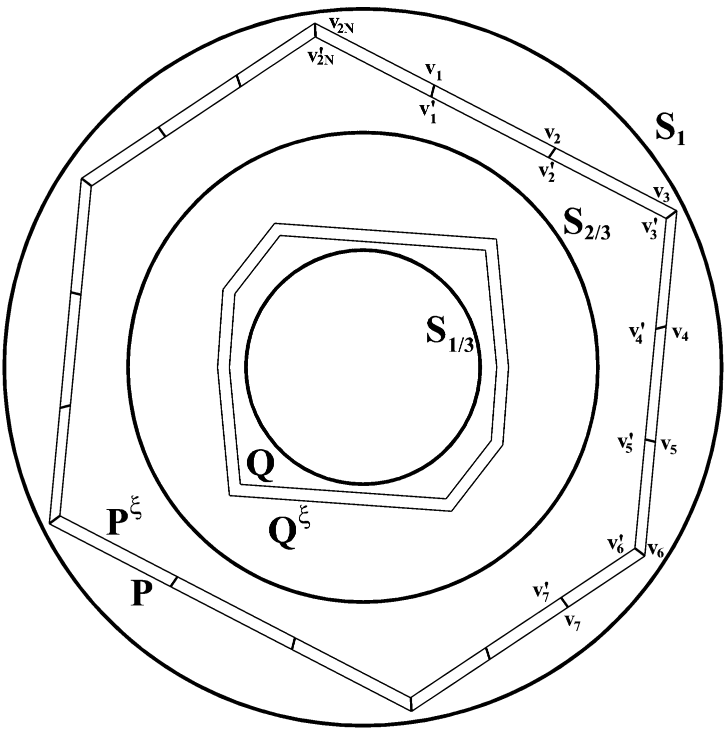

Figure 1 Polygonal pairs (P, Q) and (P^ξ, Q^ξ)

$\mathcal{B} = \bigcup_{i=1}^{N} L_{2i}$ and $\tilde{\mathcal{B}} = \bigcup_{i=0}^{N-1} L_{2i+1}$;
$L = \mathcal{B} \cap \mathcal{A}$, $\tilde{L} = \tilde{\mathcal{B}} \cap \tilde{\mathcal{A}}$, and $H = R \cup L \cup \tilde{L}$;
$\Omega_N^P = \{z \in \mathrm{Int}(P_0) \setminus \mathrm{Int}(P_{2N^2}) : \mathrm{dist}(z, H) \geq \min\{1/(4N^3), r_1/N^2\}\}$.

We define ω_i^1 as the union of the segment L_i and those connected components of Ω_N^P that have nonempty intersection with L_i for $i = 1, \dots, N$. Similarly, we define ω_i^2 as the union of the segment L_{N+i} and those connected components of Ω_N^P that intersect L_{N+i} for $i = 1, \dots, N$. Finally,

$\varpi_i^j = \{z \in \mathbb{C} : \mathrm{dist}(z, \omega_i^j) < \delta\}$, where $j = 1, 2$, $i = 1, \dots, N$, and $\delta > 0$ is chosen in such a way that the sets ϖ_i^j ($j = 1, 2$, $i = 1, \dots, N$) are pairwise disjoint (see Figure 2); and
$\omega_i = \omega_i^1 \cup \omega_i^2$ and $\varpi_i = \varpi_i^1 \cup \varpi_i^2$ for $i = 1, \dots, N$.

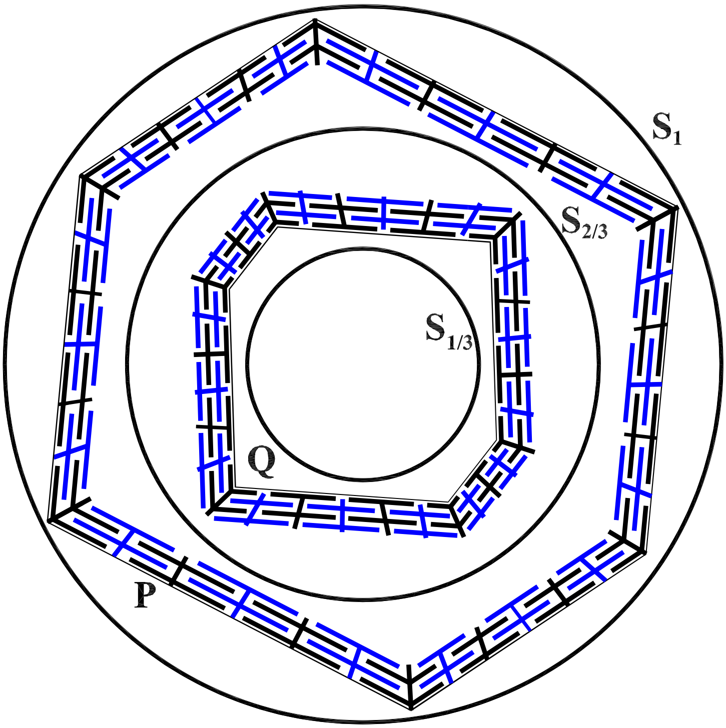

Figure 2 Distribution of the sets ϖ_i^j

Because P is symmetric (i.e., $P = -P$), the construction of the sets just described leads us to $\omega_i^1 = -\omega_i^2$ and $\varpi_i^1 = -\varpi_i^2$.

For the polygon Q we define, in the same way, the sets

$$\Omega_N^Q,\ \omega_{N+1}^j, \dots, \omega_{2N}^j,\ \varpi_{N+1}^j, \dots, \varpi_{2N}^j, \quad j = 1, 2.$$

Finally, we define $\Omega_N = \Omega_N^P \cup \Omega_N^Q$.

The aim of all this construction is to guarantee the following claims for an N large enough.

Claim A. There is a constant r_3 such that $\operatorname{diam}(\varpi_i^j) \leq r_3/N$.

Claim B. If $\lambda^2\langle\cdot,\cdot\rangle$ is a metric in D^* that is conformal to the Euclidean metric verifying

$$\lambda \geq \begin{cases} c & \text{in } T, \\ cN^4 & \text{in } \Omega_N, \quad c \in \mathbb{R}^+, \end{cases}$$

and if α is a curve in T from $S_{2/3}$ to the boundary of T, then the length of α with this metric is greater than $cr_1N/2$. This is a consequence of the fact that each piece α_i $(i = 0, \ldots, N^2 - 1)$ of α connecting P_{2i} with P_{2i+2} verifies the fact that either the Euclidean length of α_i is greater than $r_1/(2N)$ or α_i goes through a connected component of Ω_N.

Now, our purpose is to construct (for an N large enough) a sequence of conformal minimal immersions, $F_0 = X, F_1, \ldots, F_{2N}$ in D^* such that:

(P1$_\text{i}$) $F_i(z) = \operatorname{Re}\left(\int_{2/3}^z \phi^i(w)\,dw\right) + c$, where $c = X(2/3)$ and $\phi^i : D^* \to \mathbb{C}^3$ is of z^2-type;
(P2$_\text{i}$) $\|\phi^i(z) - \phi^{i-1}(z)\| \leq 1/N^2$ for all $z \in T \setminus \varpi_i$;
(P3$_\text{i}$) $\|\phi^i(z)\| \geq N^{7/2}$ for all $z \in \omega_i$;
(P4$_\text{i}$) $\|\phi^i(z)\| \geq 1/\sqrt{N}$ for all $z \in \varpi_i$;
(P5$_\text{i}$) $\operatorname{dist}_{\mathbb{S}^2}(G_i(z), G_{i-1}(z)) < 1/\left(N\sqrt{N}\right)$ for all $z \in T \setminus \varpi_i$, where $\operatorname{dist}_{\mathbb{S}^2}$ is the intrinsic distance in $\mathbb{S}^2$ and G_i represents the Gauss map of the immersion F_i;
(P6$_\text{i}$) there exists a set $S_i = \{e_1, e_2, e_3\}$ of orthogonal coordinates in $\mathbb{R}^3$ and a real constant $r_4 > 0$ such that:
(P6.1$_\text{i}$) if $z \in \overline{\varpi_i}$ and $\|F_{i-1}(z)\| \geq 1/\sqrt{N}$ then

$$\|((F_{i-1}(z))_1, (F_{i-1}(z))_2)\| < \frac{r_4}{\sqrt{N}}\|F_{i-1}(z)\|,$$

(P6.2$_\text{i}$) $(F_i(z))_3 = (F_{i-1}(z))_3$ for all $z \in \bar{T}$,
where $(\cdot)_k$ is the kth coordinate function with respect to $\{e_1, e_2, e_3\}$.

Suppose that we have $F_0, \ldots, F_{j-1}$ verifying the claims (P1$_\text{i}$), …,(P6$_\text{i}$), $i = 1, \ldots, j-1$. Then, for an N large enough, there are positive constants $r_5, \ldots, r_9$ such that the following statements hold.

(L1) $\|\phi^{j-1}\| \leq r_5$ in $T \setminus \bigcup_{k=1}^{j-1} \varpi_k$.
This follows easily from (P2$_\text{i}$) for $i = 1, \ldots, j-1$.
(L2) $\|\phi^{j-1}\| \geq r_6$ in $T \setminus \bigcup_{k=1}^{j-1} \varpi_k$.
To obtain this property, it suffices to apply (P2$_\text{i}$) for $i = 1, \ldots, j-1$ once again.
(L3) The diameter in $\mathbb{R}^3$ of $F_{j-1}(\varpi_j^i)$ is less than r_7/N.
This is a consequence of (L1), the bound of $\operatorname{diam}(\varpi_j^i)$ in Claim A, and equation (5).
(L4) The diameter in $\mathbb{S}^2$ of $G_{j-1}(\varpi_j^i)$ is less than $r_8/\sqrt{N}$.

Indeed, from the bound of $\operatorname{diam}(\varpi_j^i)$, we have a bound of diameter of $G_0(\varpi_j^i)$. The bound is $\sup\{\|(dG_0)_p\| : p \in T\}(r_3/N)$. From successive applications of ($\mathrm{P5_i}$) we have

$$\operatorname{diam}(G_{j-1}(\varpi_j^i)) < r_8/\sqrt{N}.$$

(L5) $\|S(F_{j-1})\| \le r_9/N$.

This is a consequence of ($\mathrm{P1_i}$) and ($\mathrm{P2_i}$) for $i = 1, \dots, j-1$.

We shall now construct F_j. We look for a set of orthogonal coordinates $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ and a constant $r_{10} > 0$ such that:

(D1) if $z \in \varpi_j$ and $\|F_{j-1}(z)\| \ge 1/\sqrt{N}$, then

$$\angle(e_3, F_{j-1}(z)) \le r_{10}/\sqrt{N} \quad \text{or} \quad \angle(-e_3, F_{j-1}(z)) \le r_{10}/\sqrt{N};$$

(D2) $\angle(\pm e_3, G_{j-1}(z)) \ge \nu/\sqrt{N}$ for all $z \in \varpi_j$,

where $\angle(a, b) \in [0, \pi[$ is the angle formed by a and b in $\mathbb{R}^3$ and where $\nu > 1/r_6$. We denote

$$\operatorname{Con}(q, r) = \{x \in \mathbb{S}^2 : \angle(x, q) \le r\}.$$

Let $g_1 \in G_{j-1}(\varpi_j^1)$ and $g_2 \in G_{j-1}(\varpi_j^2)$. Taking (L4) into account, the condition (D2) holds if e_3 is chosen in $\mathbb{S}^2 \setminus R$, where

$$R = \operatorname{Con}\left(g_1, \frac{r_8+\nu}{\sqrt{N}}\right) \cup \left[-\operatorname{Con}\left(g_1, \frac{r_8+\nu}{\sqrt{N}}\right)\right] \cup \operatorname{Con}\left(g_2, \frac{r_8+\nu}{\sqrt{N}}\right) \cup \left[-\operatorname{Con}\left(g_2, \frac{r_8+\nu}{\sqrt{N}}\right)\right].$$

The next step is to find $e_3 \in \mathbb{S}^2 \setminus R$ satisfying (D1) for a suitable $r_{10} > 0$.

To do this, we define

$$F = \{p/\|p\| : p \in F_{j-1}(\varpi_j^1) \text{ and } \|p\| \ge 1/\sqrt{N} - r_9/N\}.$$

From the diameter bound of $F_{j-1}(\varpi_j^1)$, we have that $F \subset \operatorname{Con}\big(q, 2r_7/\big(\sqrt{N} - r_9\big)\big)$ for any $q \in F$. Consider r_{10} such that

$$\frac{2(r_8+\nu)}{\sqrt{N}} + \frac{2r_7}{\sqrt{N}-r_9} + \frac{2r_9}{\sqrt{N}-r_9} < \frac{r_{10}}{\sqrt{N}}.$$

If $(\mathbb{S}^2 \setminus R) \cap F \neq \emptyset$, we take $e_3 \in (\mathbb{S}^2 \setminus R) \cap F$. On the other hand, if $(\mathbb{S}^2 \setminus R) \cap F = \emptyset$ then we take $e_3 \in \mathbb{S}^2 \setminus R$ such that $\angle(e_3, q) < 2(r_8+\nu)/\sqrt{N}$ for some $q \in F$.

We now check the property (D1) in both cases.

Case 1: $(\mathbb{S}^2 \setminus R) \cap F \neq \emptyset$. Take $z \in \varpi_j$ verifying $\|F_{j-1}(z)\| \ge 1/\sqrt{N}$. If $z \in \varpi_j^1$ then a straightforward computation leads to $\angle(e_3, F_{j-1}(z)) \le r_{10}/\sqrt{N}$. If $z \in \varpi_j^2$ then, taking into account that $\|S(F_{j-1})\| \le r_9/N$, we have $F_{j-1}(-z)/\|F_{j-1}(-z)\| \in F$ and $\angle(F_{j-1}(-z), -F_{j-1}(z)) \le 2r_9/(\sqrt{N} - r_9)$. Therefore,

$$\begin{aligned}\angle(-e_3, F_{j-1}(z)) &= \angle(e_3, -F_{j-1}(z)) \\ &\le \angle(e_3, F_{j-1}(-z)) + \angle(F_{j-1}(-z), -F_{j-1}(z)) \\ &\le \left(\frac{2(r_8+\nu)}{\sqrt{N}} + \frac{2r_7}{\sqrt{N}-r_9}\right) + \frac{2r_9}{\sqrt{N}-r_9} \le \frac{r_{10}}{\sqrt{N}}.\end{aligned}$$

Case 2: $(\mathbb{S}^2 \setminus R) \cap F = \emptyset$. In this case, if $p \in F$ then

$$\angle(e_3, p) \leq \angle(e_3, q) + \angle(q, p) \leq \frac{2(r_8 + \nu)}{\sqrt{N}} + \frac{2r_7}{\sqrt{N} - r_9} < \frac{r_{10}}{\sqrt{N}}.$$

This proves (D1) for $z \in \varpi_j^1$. If $z \in \varpi_j^2$, the proof is the same as in Case 1.

Finally, we take e_1, e_2 such that $S_j = \{e_1, e_2, e_3\}$ is a set of orthogonal coordinates in $\mathbb{R}^3$.

Let (f, g) be the Weierstrass data of the immersion F_{j-1} in the coordinate system S_j. Let h be the function given by Proposition 1 for $E_1 = \bar{T} \setminus \varpi_j$, $E_2 = \omega_j$, and τ large enough in order N. We define $\tilde{f} = fh$ and $\tilde{g} = g/h$. Now $\tilde{\phi}_k^j$ $(k = 1, 2, 3)$ are the functions defined by (3) for $(\tilde{f}, \tilde{g})$; they are holomorphic and have no periods in zero because they are of z^2-type, too. Therefore, the minimal immersion F_j is well-defined and its expression in the set of coordinates S_j is

$$F_j(z) = \mathrm{Re}\left(\int_{2/3}^{z} \tilde{\phi}^j(w)\, dw \right) + F_{j-1}(2/3).$$

We shall now see that F_j verifies the properties $(\mathrm{P1_j}), \ldots, (\mathrm{P6_j})$. (Note that claims $(\mathrm{P1_j}), \ldots, (\mathrm{P6_j})$ do not depend on changes of coordinates in $\mathbb{R}^3$.) Claim $(\mathrm{P1_j})$ easily holds. Making some calculations, we get $(\mathrm{P2_j})$ and $(\mathrm{P3_j})$ for τ large enough, as follows:

$$\|\phi^j - \phi^{j-1}\| = \frac{1}{\sqrt{2}}\left(|f(h-1)| + \left| fg^2 \frac{1-h}{h} \right| \right)$$
$$\leq \frac{\|\phi^{j-1}\|}{\tau - 1} \leq \frac{\sup_{\bar{T}}\{\|\phi^{j-1}\|\}}{\tau - 1} \quad \text{in } T \setminus \varpi_j;$$

$$\|\phi^j\| = \frac{1}{\sqrt{2}}\left(|fh| + \left| \frac{fg^2}{h} \right| \right) \geq \frac{1}{\sqrt{2}} |f||h| \geq \frac{1}{\sqrt{2}} \sup_{\bar{T}}\{|f|\}(\tau - 1) \quad \text{in } \omega_j.$$

From (D2) we have

$$\frac{\sin(\nu/\sqrt{N})}{1 + \cos(\nu/\sqrt{N})} \leq |g| \leq \frac{\sin(\nu/\sqrt{N})}{1 - \cos(\nu/\sqrt{N})} \quad \text{in } \varpi_j,$$

and so

$$\|\phi^j\| = \frac{1}{\sqrt{2}} |fg| \left(\frac{|h|}{|g|} + \frac{|g|}{|h|} \right) \geq \frac{2}{\sqrt{2}} |fg| \geq 2\|\phi^{j-1}\| \frac{|g|}{1 + |g|^2}$$
$$\geq r_6 \sin(\nu/\sqrt{N}) \geq 1/\sqrt{N} \quad \text{in } \varpi_j$$

for an N large enough. Therefore, the property $(\mathrm{P4_j})$ is true.

Property $(\mathrm{P5_j})$ is a consequence of the following inequality:

$$2 \sin\left(\frac{\mathrm{dist}_{\mathbb{S}^2}(G_j(z) - G_{j-1}(z))}{2} \right)$$
$$= \|G_j(z) - G_{j-1}(z)\|_{\mathbb{R}^3} < 2|\tilde{g}(z) - g(z)|$$
$$= 2|g(z)||h(z) - 1| \leq 2 \frac{\sup_{\bar{T}}\{|g|\}}{\tau} \quad \forall z \in T \setminus \varpi_j.$$

Using (D1), we get ($\mathrm{P6.1_j}$) for $r_4 = r_{10}$. And ($\mathrm{P6.2_j}$) is true because, in the coordinate system S_j, we have that

$$\phi_3^{j-1} = fg = fh\frac{g}{h} = \phi_3^j.$$

Hence, we have constructed the immersions $F_0, F_1, \dots, F_{2N}$ verifying claims ($\mathrm{P1_j}$), ...,($\mathrm{P6_j}$) for $j = 1, \dots, 2N$. In particular, we have the following.

PROPOSITION 2. *If N is large enough, then F_{2N} verifies that*:

(i) $\rho + s < \mathrm{dist}_{(F_{2N},T)}(z, S_{2/3})$ *for all* $z \in P \cup Q$;
(ii) $\mathrm{dist}_{(F_{2N},T^\xi)}(z, S_{2/3}) < (1-k')(\rho+s)$ *for all* $z \in P^\xi \cup Q^\xi$;
(iii) *there is a* $r_{11} > 0$ *such that* $\|F_j(z) - F_{j-1}(z)\| \le r_{11}/N^2$ *in* $T \setminus \varpi_j$;
(iv) $\|F_{2N} - X\| \le 2r_{11}/N$ *in* $T \setminus \bigcup_{j=1}^{2N} \varpi_j$;
(v) *there is a polygonal pair* $(\tilde{P}, \tilde{Q})$ *such that*

$$(1-k')(\rho+s) < \mathrm{dist}_{(F_{2N},\tilde{T})}(z, S_{2/3}) < \rho + s \quad \forall z \in \tilde{P} \cup \tilde{Q};$$

(vi) *if* $\tilde{T}$ *is the set associated to* $(\tilde{P}, \tilde{Q})$, *then* $\tilde{T} \subset \mathrm{I}(T)$ *and* $T^\xi \subset \mathrm{I}(\tilde{T})$;
(vii) $F_{2N}(\tilde{T}) \subset B_{R-\varepsilon/2}$, *where* $R = \sqrt{r^2 + (2s)^2} + \varepsilon$.

Here the minimal immersion X and the constants $\varepsilon, \rho, s, r, \xi$ are as in Lemma 1.

Proof. To prove assertion (i), notice that (L2) implies

$$\lambda_{F_{2N}} = \frac{\|\phi^{2N}\|}{\sqrt{2}} \ge \frac{r_6}{\sqrt{2}} > \frac{1}{2\sqrt{N}} \ \text{ in } T \setminus \textstyle\bigcup_{k=1}^{2N} \varpi_k.$$

Taking into account ($\mathrm{P4_j}$) and ($\mathrm{P2_i}$) for $i = j+1, \dots, 2N$, we have

$$\lambda_{F_{2N}} \ge \frac{\|\phi^j\| - \|\phi^{2N} - \phi^j\|}{\sqrt{2}} \ge \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{N}} - \frac{2}{N}\right) \ge \frac{1}{2\sqrt{N}} \ \text{ in each } \varpi_j.$$

From ($\mathrm{P3_j}$) and ($\mathrm{P2_i}$) for $i = j+1, \dots, 2N$, we obtain

$$\lambda_{F_{2N}} \ge \frac{\|\phi^j\| - \|\phi^{2N} - \phi^j\|}{\sqrt{2}} \ge \frac{1}{\sqrt{2}}\left(N^{7/2} - \frac{2}{N}\right) \ge \frac{1}{2\sqrt{N}} N^4 \ \text{ in each } \omega_j.$$

Using the three displayed inequalities together with Claim B, we conclude the proof of the first assertion in this proposition.

To obtain assertion (ii), consider $z \in P^\xi \cup Q^\xi$. From inequality (6), there is a curve α with origin z and ending at $z' \in S_{2/3}$ that verifies $\alpha \subset T^\xi$ and $\mathrm{length}(\alpha, X) < \rho$. Since $T^\xi \subset T \setminus \bigcup_{l=1}^{2N} \varpi_l$ (if N is large enough), we can apply ($\mathrm{P2_j}$) for $j = 1, \dots, 2N$ to obtain $|\mathrm{length}(\alpha, F_{2N}) - \mathrm{length}(\alpha, X)| \le (2/\sqrt{2}N)\,\mathrm{length}(\alpha)$. Bearing in mind (L2), we get

$$\mathrm{length}(\alpha, X) \ge \frac{r_6}{\sqrt{2}}\,\mathrm{length}(\alpha),$$

and then

$$|\mathrm{length}(\alpha, F_{2N}) - \mathrm{length}(\alpha, X)| \le \frac{2}{r_6 N}\rho.$$

Therefore,

$$\text{length}(\alpha, F_{2N}) < \text{length}(\alpha, X) + \frac{2}{r_6 N}\rho < \rho + \frac{2}{r_6 N}\rho < (1-k')(\rho+s).$$

Now we shall prove (iii). First observe that, if N is large enough and ϖ_j is a set in the labyrinth Ω_N, then it is possible to find a positive constant r_{11} depending only on T such that, for all $z \in T \setminus \varpi_j$, there exists a curve α_z in $T \setminus \varpi_j$ from $2/3$ to z satisfying $\text{length}(\alpha_z) < r_{11}$. This comes from the fact that the Euclidean diameter of ϖ_j is uniformly bounded. Using the former, we obtain

$$\|F_j(z) - F_{j-1}(z)\| = \left\| \text{Re} \int_{\alpha_z} (\phi^j(w) - \phi^{j-1}(w))\, dw \right\| \le r_{11} \frac{1}{N^2},$$

which proves assertion (iii). From (iii), it is not hard to deduce (iv).

Concerning (v), we will construct only the polygon $\tilde{P}$; the other polygon $\tilde{Q}$ can be constructed in a similar way. Let

$$\mathcal{S} = \{z \in \mathbb{C} \setminus D_{2/3} : (1-k')(\rho+s) < \text{dist}_{(F_{2N},T)}(z, S_{2/3}) < \rho + s\}.$$

Note that $\mathcal{S}$ is a nonempty open subset of T. For $\zeta > 0$ satisfying $(1-k')(\rho+s) < \zeta < \rho + s$, consider

$$\mathcal{S}_\zeta = \{z \in \mathbb{C} \setminus D_{2/3} : \text{dist}_{(F_{2N},T)}(z, S_{2/3}) = \zeta\}.$$

Since $\mathcal{S}_\zeta$ is a compact subset of $\mathcal{S}$, there are closed balls $B_1, \dots, B_d$ of $\mathbb{R}^2$ such that $\mathcal{S}_\zeta \subset \bigcup_{i=1}^d B_i \subset \mathcal{S}$. Note that 0 and ∞ are in disjoint arc-connected components of $\mathbb{C} \setminus \bigcup_{i=0}^d B_i$. We can then construct a polygonal line $\tilde{P}$ in $\bigcup_{i=0}^d B_i$ such that $\overline{D_{2/3}} \subset \text{Int}(\tilde{P})$. As ϕ^{2N} is of z^2-type, we have $\lambda_{F_{2N}}(z) = \lambda_{F_{2N}}(-z)$. This means that $\text{dist}_{F_{2N}}(z, S_{2/3}) = \text{dist}_{F_{2N}}(-z, S_{2/3})$ for all $z \in D^*$. Therefore, $\tilde{P}$ can be chosen in such a way that $\tilde{P} = -\tilde{P}$, because $\mathcal{S} = -\mathcal{S}$ and $\mathcal{S}_\zeta = -\mathcal{S}_\zeta$.

As a consequence of assertions (i), (ii), and (v), we obtain $\tilde{P} \subset \text{Int}(P)$, $\tilde{Q} \subset \text{Ext}(Q)$, $P^\xi \subset \text{Int}(\tilde{P})$, and $Q^\xi \subset \text{Ext}(\tilde{Q})$. Thus we have $\tilde{T} \subset \text{I}(T)$ and $T^\xi \subset \text{I}(\tilde{T})$, which concludes our proof of (vi).

Finally, we prove assertion (vii). Thanks to the maximum modulus theorem, we only need to check that

$$F_{2N}(\tilde{P} \cup \tilde{Q}) \subset B_{R-\varepsilon/2}.$$

Let $\eta \in \tilde{P} \cup \tilde{Q}$. If $\eta \in T \setminus \bigcup_{j=1}^{2N} \varpi_j$, we have

$$\|F_{2N}(\eta)\| \le \|F_{2N}(\eta) - X(\eta)\| + \|X(\eta)\| \le \frac{2r_{11}}{N} + r \le R - \frac{\varepsilon}{2}.$$

On the other hand, if $\eta \in \varpi_j$ for $j \in \{1, \dots, 2N\}$ then the reasoning is slightly more complicated. From (v), it is possible to find a curve $\gamma : [0,1] \to T$ such that $\gamma(0) \in S_{2/3}$, $\gamma(1) = \eta$, and $\text{length}(\gamma, F_{2N}) \le \rho + s$. We define:

$$\bar{t} = \sup\{t \in [0,1] : \gamma(t) \in \partial\varpi_j\}, \qquad \tilde{t} = \inf\{t \in [0,1] : \gamma(t) \in P^\xi\};$$
$$\bar{\eta} = \gamma(\bar{t}), \qquad \tilde{\eta} = \gamma(\tilde{t}).$$

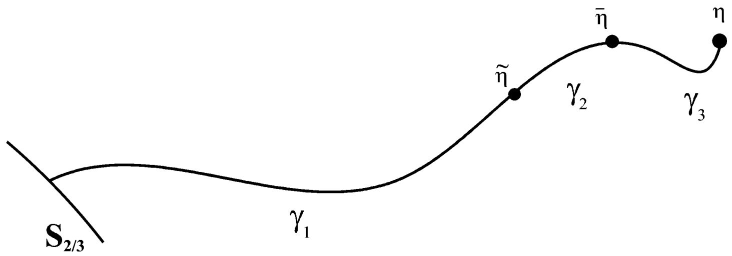

Figure 3 The partition of γ

For an N large enough, one has $\varpi_j \subset \mathrm{Int}(P) \setminus \mathrm{Int}(P^\xi)$ and so $\tilde{t} < \bar{t}$. Therefore, γ is divided into three disjoint pieces: γ_1 from $S_{2/3}$ to $\tilde{\eta}$, γ_2 from $\tilde{\eta}$ to $\bar{\eta}$, and γ_3 from $\bar{\eta}$ to η (see Figure 3). To continue, we need to demonstrate the existence of a constant r_{12}, not depending on N, such that

$$\|F_j(\bar{\eta}) - F_j(\eta)\| \le \frac{r_{12}}{N} + 2s. \tag{11}$$

Indeed,

$$\begin{aligned}\|F_j(\bar{\eta}) - F_j(\eta)\| &\le \|F_j(\bar{\eta}) - F_{2N}(\bar{\eta})\| + \|F_{2N}(\bar{\eta}) - F_{2N}(\eta)\| \\ &\quad + \|F_{2N}(\eta) - F_j(\eta)\| \\ &\le 2\frac{2r_{11}}{N} + \|F_{2N}(\bar{\eta}) - F_{2N}(\eta)\| \le 4\frac{r_{11}}{N} + \mathrm{length}(\gamma_3, F_{2N}) \\ &\le 4\frac{r_{11}}{N} + \rho + s - \mathrm{length}(\gamma_1, F_{2N}). \end{aligned} \tag{12}$$

Taking into account that $\mathrm{length}(\gamma_1, F_{2N}) \le \rho + s$, we reason as in assertion (ii) and obtain

$$|\mathrm{length}(\gamma_1, F_{2N}) - \mathrm{length}(\gamma_1, F_0)| \le \frac{2}{r_6 N}(\rho + s). \tag{13}$$

Using (12) and (13), it follows that

$$\|F_j(\bar{\eta}) - F_j(\eta)\| \le 4\frac{r_{11}}{N} + \rho + s - \mathrm{length}(\gamma_1, F_0) + \frac{2(\rho + s)}{r_6 N};$$

by (6) in the hypotheses of Lemma 1, we have

$$\le 4\frac{r_{11}}{N} + \rho + s - (1 - k)\rho + \frac{2(\rho + s)}{r_6 N}.$$

Thus, inequality (11) holds for $r_{12} = 4r_{11} + 2(\rho + s)/r_6$.

At this point, we distinguish two cases.

Case 1: $\|F_{j-1}(\bar{\eta})\| < 1/\sqrt{N}$. Then

$$\begin{aligned}\|F_{2N}(\eta)\| &\le \|F_{2N}(\eta) - F_j(\eta)\| + \|F_j(\eta) + F_j(\bar{\eta})\| \\ &\quad + \|F_j(\bar{\eta}) - F_{j-1}(\bar{\eta})\| + \|F_{j-1}(\bar{\eta})\| \\ &\le \frac{2r_{11}}{N} + \frac{r_{12}}{N} + 2s + \frac{r_{11}}{N^2} + \frac{1}{\sqrt{N}} \le R - \frac{\varepsilon}{2}\end{aligned}$$

for an N large enough.

Case 2: $\|F_{j-1}(\bar\eta)\| > 1/\sqrt{N}$. In this case, from (P6.2$_\mathrm{j}$) we have, in the set of Cartesian coordinates given by S_j,

$$|(F_j(\eta))_3| = |(F_{j-1}(\eta))_3| \le |(F_{j-1}(\eta))_3 - (X(\eta))_3| + |(X(\eta))_3| \le \frac{2r_{11}}{N} + r.$$

Using inequality (11), the fact that $\bar\eta \in T \setminus \varpi_j$, assertion (iii), and property (P6.1$_\mathrm{j}$), one has

$$\begin{aligned}
\|((F_j(\eta))_1, (F_j(\eta))_2)\| &\le \|((F_j(\eta))_1, (F_j(\eta))_2) - ((F_j(\bar\eta))_1, (F_j(\bar\eta))_2)\| \\
&\quad + \|((F_j(\bar\eta))_1, (F_j(\bar\eta))_2) - ((F_{j-1}(\bar\eta))_1, (F_{j-1}(\bar\eta))_2)\| \\
&\quad + \|((F_{j-1}(\bar\eta))_1, (F_{j-1}(\bar\eta))_2)\| \\
&\le \frac{r_{12}}{N} + 2s + \frac{r_{11}}{N^2} + \frac{r_4}{\sqrt{N}}\|F_{j-1}(\bar\eta)\| \\
&\le \frac{r_{12}}{N} + 2s + \frac{r_{11}}{N^2} + \frac{r_4}{\sqrt{N}}\left(\frac{2r_{11}}{N} + r\right) \le 2s + \frac{r_{13}}{\sqrt{N}},
\end{aligned}$$

where $r_{13} = r_{12} + r_{11} + r_4(2r_{11} + r)$. By Pythagoras' theorem,

$$\begin{aligned}
\|F_{2N}(\eta)\| &\le \|F_{2N}(\eta) - F_j(\eta)\| + \|F_j(\eta)\| \\
&\le \frac{2r_{11}}{N} + \sqrt{|(F_j(\eta))_3|^2 + \|((F_j(\eta))_1, (F_j(\eta))_2)\|^2} \\
&< \sqrt{r^2 + (2s)^2} + \frac{\varepsilon}{2} = R - \frac{\varepsilon}{2}
\end{aligned}$$

for an N large enough. □

In order to finish the proof of the lemma, we define Y as $Y = F_{2N} - S(F_{2N})/2$. It is straightforward to check that Y verifies all the claims in Lemma 1.

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