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Isometric Actions and Harmonic Morphisms

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Introduction

It is well known that a Riemannian foliation with minimal leaves has the property that it produces harmonic morphisms, that is, *its leaves are locally fibers of submersive harmonic morphisms*. This is an immediate consequence of the fact that Riemannian submersions with minimal fibers are harmonic morphisms.

More generally, a Riemannian foliation (of codimension not equal to 2) produces harmonic morphisms if and only if the vector field determined by the mean curvatures of the leaves is locally a gradient vector field. This is a consequence of the fundamental equation of Baird and Eells [1] (see Proposition 1.2 in the next section). Although this condition is quite simple, few examples of such Riemannian foliations were known; our work will provide many new ones.

For a 1-dimensional Riemannian foliation, the condition just stated is equivalent to the fact that the foliation is locally generated by Killing fields (a result due to Bryant [6]), but this is not true for foliations of dimension greater than 1. In this paper we show that, for a foliation locally generated by Killing fields, the condition depends only on the integrability tensor of the horizontal distribution and the induced local action. Thus we obtain a useful criterion for a foliation locally generated by Killing fields to produce harmonic morphisms. This is done in Section 1 (Theorem 1.13). In Section 2 we derive a few consequences, thus obtaining the following classes of Riemannian foliations (of codimension $\neq 2$) that produce harmonic morphisms:

(a) foliations locally generated by Killing fields and with integrable orthogonal complement;
(b) foliations generated by the local action of an abelian Lie group of isometries;
(c) foliations generated by the action of a unimodular closed subgroup of the isometry group;
(d) foliations generated by the action of a Lie group of isometries whose orbits are naturally reductive homogeneous Riemannian manifolds;
(e) foliations formed by the fibers of principal bundles for which the total space is endowed with a metric such that the structural group acts as an isometry

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group and the connection induced on the determinant bundle of the adjoint bundle is flat.

We remark that the omitted case of codimension 2 is much simpler. In this case a foliation produces harmonic morphisms if and only if it is conformal and its leaves are minimal (see [23]).

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1. Characterization of the Isometric Actions That Produce Harmonic Morphisms

Foliations whose leaves are locally fibers of (submersive) harmonic morphisms were introduced in [23]. We recall the following definition.

DEFINITION 1.1. Let (M, g) be a (connected) Riemannian manifold and let $\mathcal{V}$ be (the tangent bundle of) a foliation on it.

We will say that $\mathcal{V}$ *produces harmonic morphisms on* (M, g) if each point of M has an open neighborhood O that is the domain of a submersive harmonic morphism $\varphi\colon \big(O, g|_O\big) \to (N, h)$ whose fibers are open subsets of the leaves of $\mathcal{V}$.

Since harmonic morphisms are harmonic maps that are horizontally (weakly) conformal [9; 14], a foliation that produces harmonic morphisms is, in particular, a conformal foliation. Riemannian foliations that produce harmonic morphisms can be easily characterized as follows. (For the general characterization of conformal foliations that produce harmonic morphisms, see [3; 6; 17].)

PROPOSITION 1.2. *Let $\mathcal{V}$ be a Riemannian foliation on (M, g) of* $\operatorname{codim}\mathcal{V} \neq 2$. *Then the following assertions are equivalent*:

(i) *$\mathcal{V}$ produces harmonic morphisms*;
(ii) *the mean curvature form* $\operatorname{trace}({}^{\mathcal{V}}B)^{\flat}$ *of $\mathcal{V}$ is closed*.

Proof. Suppose (i) holds; let ${}^{\mathcal{V}}B$ be the second fundamental form of $\mathcal{V}$ and let $\varphi\colon \big(O, g|_O\big) \to (N, h)$ be a submersive harmonic morphism whose fibers are open subsets of leaves of $\mathcal{V}$. Let λ be the dilation of φ. Recall the following fundamental formula (due to Baird and Eells [1]):

$$\operatorname{trace}({}^{\mathcal{V}}B) + (n-2)\mathcal{H}(\operatorname{grad}(\log\lambda)) = 0,$$

where $\mathcal{H}$ is the "horizontal" distribution (i.e., the orthogonal complement of $\mathcal{V}$) and we have denoted by the same letter the projection onto it.

Since $\mathcal{V}$ is Riemannian, λ must be basic (i.e., constant along the leaves of $\mathcal{V}$), so $\mathcal{H}(\operatorname{grad}(\log\lambda)) = \operatorname{grad}(\log\lambda)$ and (ii) follows. Conversely, if (ii) holds, let λ be a positive smooth function defined on an open subset U of M such that $\operatorname{trace}({}^{\mathcal{V}}B) = -(n-2)\operatorname{grad}(\log\lambda)$. Then λ is basic.

By restricting λ, if necessary, to an open subset of U, we can suppose that there exists a Riemannian submersion $\varphi\colon \big(U, g|_U\big) \to (N, h)$ whose fibers are connected open subsets of leaves of $\mathcal{V}$. Since λ is basic, it descends to a positive smooth function $\check{\lambda}$ on N. Then $\varphi\colon \big(U, g\big|_U\big) \to (N, \check{\lambda}^2 h)$ is a harmonic morphism and (i) follows. □

REMARK 1.3. (1) If a Riemannian foliation produces harmonic morphisms then its mean curvature is a basic vector field. This follows from the fundamental equation and from the fact that the dilation of a horizontally conformal submersion whose fibers form a Riemannian foliation is basic.

Note also that, as a horizontal 1-form, $\operatorname{trace}(^{\mathcal{V}}B)^\flat$ is basic if and only if

$$\mathrm{d}(\operatorname{trace}(^{\mathcal{V}}B)^\flat)(V, X) = 0$$

for any vertical V and horizontal X.

(2) If $\mathcal{V}$ produces harmonic morphisms and if its leaves are the fibers of the Riemannian submersion $\varphi\colon (M, g) \to (N, h)$, then φ can be lifted to a harmonic morphism whose domain is a Riemannian covering space of (M, g) (see the proof of Corollary 1.15).

It is well known [19] that the foliations whose mean curvatures are locally gradient vector fields can be characterized in terms of the associated Bott connection, defined as follows (see [22]).

DEFINITION 1.4. Let $\mathcal{V}$ be a foliation on (M, g) and let $\mathcal{H}$ be its orthogonal complement. The (*adapted*) *Bott connection* $\overset{\circ}{\nabla}$ on $\mathcal{V}$ is defined by

$$\overset{\circ}{\nabla}_E V = \mathcal{V}[\mathcal{H}E, V] + \mathcal{V}(\nabla_{\mathcal{V}E} V)$$

for $E \in \Gamma(TM)$ and $V \in \Gamma(\mathcal{V})$, where ∇ is the Levi–Civita connection of (M, g).

The following proposition is well known (see [19]).

PROPOSITION 1.5. *Let $\mathcal{V}$ be a foliation of* $\dim \mathcal{V} = r$ *on* (M, g). *Then the following assertions are equivalent.*

(i) *The mean curvature form of $\mathcal{V}$ is closed.*
(ii) *The connection induced by $\overset{\circ}{\nabla}$ on $\Lambda^r(\mathcal{V})$ is flat.*
(iii) *The identity component of the holonomy group of $\overset{\circ}{\nabla}$ is contained in* $\mathrm{SL}(r)$.

Proof. Recall [19] that $\operatorname{trace}(\overset{\circ}{R}) = \mathrm{d}(\operatorname{trace}(^{\mathcal{V}}B))^\flat$, where $\overset{\circ}{R}$ is the curvature form of $\overset{\circ}{\nabla}$. The equivalence (i) $\Longleftrightarrow$ (ii) is now obvious. The equivalence (ii) $\Longleftrightarrow$ (iii) follows from the holonomy theorem (see [15]). □

REMARK 1.6. It is obvious that a Riemannian foliation $\mathcal{V}$ is locally generated by Killing fields if and only if it is locally generated by infinitesimal automorphisms of $\mathcal{H}$ $(= \mathcal{V}^\perp)$ that, when restricted to any leaf of $\mathcal{V}$, are Killing fields.

Motivated by the last remark, from now on we shall suppose that we have a foliation $\mathcal{V}$ on (M, g) that is locally generated by infinitesimal automorphisms of its orthogonal complement $\mathcal{H}$—that is, vertical vector fields V whose local flows preserve $\mathcal{H}$; equivalently, $\mathcal{L}_V(\Gamma(\mathcal{H})) \subseteq \Gamma(\mathcal{H})$ or $[V, X] = 0$ for any basic vector field X (see [20; 22]). It seems difficult to characterize such foliations, but we give some examples (see [20; 22]): (i) foliations for which $\mathcal{H}$ is integrable; (ii) foliations locally generated by Killing fields (or, more generally, by conformal vector fields); and (iii) foliations formed by the connected components of the fibers of a principal bundle endowed with a principal connection $\mathcal{H}$ (for which, as is well known, the fundamental vector fields are infinitesimal automorphisms).

Let I be the integrability tensor field of $\mathcal{H}$ that is the $\mathcal{V}$-valued horizontal tensor field characterized by $I(X, Y) = -\mathcal{V}[X, Y]$ with $X, Y \in \Gamma(\mathcal{H})$, where the negative sign is included for convenience. For later use we prove the following lemma.

LEMMA 1.7. *Let V be an infinitesimal automorphism of $\mathcal{H}$ tangent to $\mathcal{V}$. Then $\mathcal{H}^*(\mathcal{L}_V I) = 0$.*

Proof. Let ψ be a local diffeomorphism from the local flow of V. Recall that if X and Y are basic then $\mathcal{H}[X, Y]$ is basic and that $\psi_* X = X$ for any basic X. Hence

$$\begin{aligned}\psi_*(I(X, Y)) &= -\psi_*(\mathcal{V}[X, Y]) = -\big(\psi_*[X, Y] - \psi_*(\mathcal{H}[X, Y])\big)\\ &= -([\psi_* X, \psi_* Y] - \mathcal{H}[X, Y])\\ &= -([\psi_* X, \psi_* Y] - \mathcal{H}[\psi_* X, \psi_* Y])\\ &= -\mathcal{V}[\psi_* X, \psi_* Y] = I(\psi_* X, \psi_* Y)\end{aligned}$$

for any basic X and Y. □

Let $\{X_1, \dots, X_n, V_1, \dots, V_r\}$ be a local frame field on M such that $X_j \in \Gamma(\mathcal{H})$ are basic and $V_\alpha \in \Gamma(\mathcal{V})$ are infinitesimal automorphisms of $\mathcal{H}$ (i.e., $[V_\alpha, X] = 0$ for any basic X) that locally generate $\mathcal{V}$ ($r = \dim \mathcal{V}$, $n = \operatorname{codim} \mathcal{V}$). We shall always denote "horizontal" indices by j, k, l and "vertical" indices by α, β, γ.

LEMMA 1.8 [19]. *Let $\overset{\circ}{R}$ be the curvature form of $\overset{\circ}{\nabla}$. Then*

$$\overset{\circ}{R}{}^{\beta}_{\alpha jk} = \Gamma^{\beta}_{\alpha\gamma} I^{\gamma}_{jk},$$

$$\overset{\circ}{R}{}^{\beta}_{\alpha j\gamma} = X_j(\Gamma^{\beta}_{\alpha\gamma}),$$

$$\overset{\circ}{R}{}^{\delta}_{\alpha\beta\gamma} = R^{\delta}_{\alpha\beta\gamma}.$$

Here $\{\Gamma^{\beta}_{\alpha\gamma}\}$ are Christoffel symbols of the Levi–Civita connection of (M, g) defined by $\mathcal{V}(\nabla_{V_\alpha} V_\beta) = \Gamma^{\gamma}_{\beta\alpha} V_\gamma$, and R is the curvature form of the Levi–Civita connection of (M, g).

Proof. Since V_α is an infinitesimal automorphism of $\mathcal{H}$, we have that $[V_\alpha, X] = 0$ for any basic vector field X. Then

$$\begin{aligned}\overset{\circ}{R}{}^{\beta}_{\alpha jk} V_\beta &= \overset{\circ}{R}(X_j, X_k) V_\alpha\\ &= \overset{\circ}{\nabla}_{X_j}(\overset{\circ}{\nabla}_{X_k} V_\alpha) - \overset{\circ}{\nabla}_{X_k}(\overset{\circ}{\nabla}_{X_j} V_\alpha) - \overset{\circ}{\nabla}_{[X_j, X_k]} V_\alpha\\ &= \mathcal{V}[X_j, \mathcal{V}[X_k, V_\alpha]] - \mathcal{V}[X_k, \mathcal{V}[X_j, V_\alpha]] - \mathcal{V}[\mathcal{H}[X_j, X_k], V_\alpha]\\ &\quad - \mathcal{V}(\nabla_{\mathcal{V}[X_j, X_k]} V_\alpha)\\ &= -\mathcal{V}(\nabla_{\mathcal{V}[X_j, X_k]} V_\alpha) = \mathcal{V}(\nabla_{I(X_j, X_k)} V_\alpha) = I^{\gamma}_{jk} \mathcal{V}(\nabla_{V_\gamma} V_\alpha) = I^{\gamma}_{jk} \Gamma^{\beta}_{\alpha\gamma} V_\beta.\end{aligned}$$

Also,

$$\begin{aligned}
\mathring{R}^{\beta}_{\alpha j\gamma}V_\beta &= \mathring{R}(X_j, V_\gamma)V_\alpha \\
&= \mathring{\nabla}_{X_j}(\mathring{\nabla}_{V_\gamma}V_\alpha) - \mathring{\nabla}_{V_\gamma}(\mathring{\nabla}_{X_j}V_\alpha) - \mathring{\nabla}_{[X_j, V_\gamma]}V_\alpha \\
&= \mathcal{V}[X_j, \mathcal{V}(\nabla_{V_\gamma}V_\alpha)] - \mathring{\nabla}_{V_\gamma}(\mathcal{V}[X_j, V_\alpha]) \\
&= \mathcal{V}[X_j, \Gamma^{\beta}_{\alpha\gamma}V_\beta] \\
&= X_j(\Gamma^{\beta}_{\alpha\gamma})V_\beta + \Gamma^{\beta}_{\alpha\gamma}\mathcal{V}[X_j, V_\beta] \\
&= X_j(\Gamma^{\beta}_{\alpha\gamma})V_\beta.
\end{aligned}$$

The lemma is proved. □

In order to give the necessary and sufficient condition for a foliation locally generated by Killing fields to produce harmonic morphisms, we need the following definition.

Definition 1.9. Let $\mathcal{V}$ be an orientable foliation of dimension r on a smooth manifold M. Let $\mathcal{H}$ be a complementary distribution (i.e., $\mathcal{V}\oplus\mathcal{H} = TM$), and let I be its integrability tensor. Let ω be a volume form on $\mathcal{V}$ (i.e., a vertical nonvanishing r-form).

Suppose that $\mathcal{V}$ is locally generated by local frames $\{V_\alpha\}$ such that:

(1) $\mathcal{V}^*(\mathcal{L}_{V_\alpha}\omega) = 0$; and
(2) V_α is an infinitesimal automorphism of $\mathcal{H}$ for any α.

We define the 2-form $\operatorname{trace}(\operatorname{ad} I)$ on M by

$$\operatorname{trace}(\operatorname{ad} I) = c^{\alpha}_{\beta\alpha}I^{\beta},$$

where $I = V_\alpha \otimes I^\alpha$ and $[V_\alpha, V_\beta] = c^{\gamma}_{\alpha\beta}V_\gamma$.

Example 1.10. Let $\mathcal{V}$ be an orientable foliation on (M, g), let $\mathcal{H} = \mathcal{V}^\perp$, and let ω be a volume form for $\mathcal{V}$ with respect to g. Let $\{V_\alpha\}$ be a local frame for $\mathcal{V}$ made up of Killing fields. Then:

(i) the $\{V_\alpha\}$ satisfy (1) and (2) of Definition 1.9;
(ii) more generally, the $\{V_\alpha\}$ satisfy (1) of Definition 1.9 if and only if their restrictions to each leaf is divergence-free.

It can be shown directly that Definition 1.9 is independent of the local frame $\{V_\alpha\}$ of $\mathcal{V}$ such that (1) and (2) hold. This also follows from the following proposition.

Proposition 1.11. *Let $\mathcal{V}, \mathcal{H}, I, \omega, \{V_\alpha\}$ be as in Definition 1.9, and let $\mathring{R}$ be the curvature form of $\mathring{\nabla}$. Then*:

(a) $\operatorname{trace}(\operatorname{ad} I) = \operatorname{trace}(\mathring{R})$;
(b) $\mathcal{V}^*(\mathcal{L}_{I(X,Y)}\omega) = \operatorname{trace}(\operatorname{ad} I)(X, Y)\omega$ *for any basic vector fields X and Y.*

Moreover, the following assertions are equivalent:

(i) trace(ad I) $= 0$;
(ii) *at least locally there can be defined smooth positive basic functions ρ such that $\mathcal{V}^*(\mathcal{L}_X(\rho\omega)) = 0$ for any horizontal field X*;
(iii) $\mathcal{V}^*(\mathcal{L}_{I(X,Y)}\omega) = 0$ *for any basic vector fields X and Y.*

In particular, if the first Betti number of M is 0 *and* (i) *holds, then $\mathcal{V}$ is taut* (*i.e., there exists a Riemannian metric on M with respect to which the leaves of $\mathcal{V}$ are minimal*).

Proof. To prove (a), let g be a Riemannian metric on M such that $\mathcal{H} = \mathcal{V}^{\perp}$ and ω is equal to the induced volume form on $\mathcal{V}$, and let $\overset{\circ}{\nabla}$ be the adapted Bott connection on $\mathcal{V}$ corresponding to g. Because $[V_\alpha, V_\beta]$ is an infinitesimal automorphism of $\mathcal{H}$, we have that $X(c^{\gamma}_{\alpha\beta}) = 0$ for any $X \in \mathcal{H}$.

Also note that (1) is equivalent to the fact that V_α, when viewed as a vector field tangent to the leaves of $\mathcal{V}$, is divergence-free. Hence, using the same notation as in Lemma 1.8, we have that $\Gamma^{\beta}_{\alpha\beta} = 0$. But the Levi–Civita connection is torsion-free and hence $\Gamma^{\beta}_{\beta\alpha} = c^{\beta}_{\alpha\beta} + \Gamma^{\beta}_{\alpha\beta} = c^{\beta}_{\alpha\beta}$.

Now, from Lemma 1.8 we have that

$$\overset{\circ}{R}{}^{\alpha}_{\alpha jk} = \Gamma^{\alpha}_{\alpha\gamma} I^{\gamma}_{jk} = c^{\alpha}_{\gamma\alpha} I^{\gamma}_{jk},$$

$$\overset{\circ}{R}{}^{\alpha}_{\alpha j\gamma} = X_j(\Gamma^{\alpha}_{\alpha\gamma}) = X_j(c^{\alpha}_{\gamma\alpha}) = 0.$$

Also, since $\overset{\circ}{\nabla}$ restricted to any leaf is equal to the Levi–Civita connection of the leaf, which is a metric connection, we have that $\overset{\circ}{R}{}^{\alpha}_{\alpha\beta\gamma} = 0$ and so (a) follows. Note that assertion (a) is equivalent to the fact that trace(ad I) is the curvature form of the connection induced by $\overset{\circ}{\nabla}$ on $\Lambda^r(\mathcal{V})$.

To prove (b), first show that the left-hand side is equal to $V_\alpha(I^\alpha(X, Y))\omega$, where $\{V_\alpha\}$ is a local frame for $\mathcal{V}$ satisfying (1) and (2). Then, apply Lemma 1.7 to prove that trace(ad I)$(X, Y) = V_\alpha(I^\alpha(X, Y))$.

It is easy to see that (ii) is equivalent to the fact that the mean curvature of $\mathcal{V}$ with respect to g is closed. The equivalence (i) $\Longleftrightarrow$ (ii) now follows from Proposition 1.5. The equivalence (i) $\Longleftrightarrow$ (iii) follows from (b). □

REMARK 1.12. (1) Let $\mathcal{V}$ be a foliation on M endowed with a volume form ω and a complementary distribution $\mathcal{H}$. Define η to be the horizontal 1-form characterized by $\mathcal{V}^*(\mathcal{L}_X\omega) = -\eta(X)\omega$ for any horizontal field X. Then η is the mean curvature form of $\mathcal{V}$ with respect to any Riemannian metric g on M such that $g(\omega, \omega) = 1$.

If (1) and (2) of Definition 1.9 are satisfied, then a straightforward calculation shows that η is basic (which is equivalent to $\mathrm{d}\eta(V, X) = 0$ for any vertical V and horizontal X). This gives a more direct argument for the facts that (a) any Riemannian foliation locally generated by Killing fields has basic mean curvature and (b) the differential of the mean curvature form is zero when evaluated on a pair consisting of a vertical and a horizontal vector.

(2) Note that trace(ad I) is well-defined also for nonorientable foliations (in which case ω is defined just up to the sign).

We now state the main result of this section.

THEOREM 1.13. *Let $\mathcal{V}$ be a Riemannian foliation on* (M, g) *of* $\operatorname{codim} \mathcal{V} \neq 2$, *and let I be the integrability tensor field of its orthogonal complement. Suppose that $\mathcal{V}$ is locally generated by Killing fields. Then the following assertions are equivalent*:

(i) *$\mathcal{V}$ produces harmonic morphisms*;
(ii) $\operatorname{trace}(\operatorname{ad} I) = 0$.

Proof. This follows from Proposition 1.2, Proposition 1.5, and Proposition 1.11. □

REMARK 1.14. It is easy to see that Theorem 1.13 holds more generally for Riemannian foliations locally generated by infinitesimal automorphisms of the horizontal distribution that, when restricted to any leaf, are divergence-free (cf. Example 1.10).

When the foliation is simple, the result of Theorem 1.13 takes a more concrete form.

COROLLARY 1.15. *Let* $\varphi\colon (M, g) \to (N, h)$, $\dim N \neq 2$, *be a Riemannian submersion whose fibers are connected and locally generated by Killing fields, and let I be the integrability tensor of the horizontal distribution. Then the following assertions are equivalent*:

(i) *φ lifts to a harmonic morphism* $\tilde{\varphi}\colon (\tilde{M}, \tilde{g}) \to (\tilde{N}, \tilde{h})$, *where* $(\tilde{M}, \tilde{g}) \to (M, g)$ *is a Riemannian regular covering and $\tilde{N} \to N$ is a regular covering such that $\tilde{h}$ and the pull-back of h to $\tilde{N}$ are conformally equivalent*;
(ii) $\operatorname{trace}(\operatorname{ad} I) = 0$.

(See [21, Part I, Sec. 14.6] for the definition of regular coverings.)

Proof. From Theorem 1.13 it follows that it is sufficient to prove that if $\mathcal{V}$ ($= \ker \varphi_*$) produces harmonic morphisms then (i) holds.

Let $\tilde{M} \to M$ be the regular covering that corresponds to the cohomology class $[a] \in H^1(M; \mathbb{R})$ induced by the differentials of the logarithms of the dilations of the (local) harmonic morphisms produced by $\mathcal{V}$ ($= \ker \varphi_*$). (From the fundamental equation it follows that a can be also defined as the 1-form obtained by applying the musical isomorphism $\flat$ to $\frac{-1}{n-2} \operatorname{trace}({}^{\mathcal{V}}B)$.) It is obvious that the pull-back of $[a]$ to $\tilde{M}$ is zero; let λ be a smooth positive function on $\tilde{M}$ such that $\mathrm{d}(\log \lambda)$ is equal to the pull-back of a to $\tilde{M}$.

Since a is basic (see Remark 1.3(1)), there exists a regular covering $\tilde{N} \to N$ whose pull-back by φ is $\tilde{M} \to M$. It is obvious that φ lifts to a smooth map $\tilde{\varphi}\colon \tilde{M} \to \tilde{N}$.

Let $\tilde{g}$ be the pull-back of g to $\tilde{M}$ and let $\bar{h}$ be the pull-back of h to $\tilde{N}$. Then $\tilde{\varphi}\colon (\tilde{M}, \tilde{g}) \to (\tilde{N}, \lambda^2 \bar{h})$ is a harmonic morphism and the corollary is proved. □

REMARK 1.16. If in Corollary 1.15 we have that $H^1(M; \mathbb{R}) = 0$ or $H^1(N; \mathbb{R}) = 0$, then assertion (i) can be replaced by the following stronger assertion:

(i′) *there exists a Riemannian metric* h_1 *on* N *that is conformally equivalent to* h *such that* $\varphi\colon (M, g) \to (N, h_1)$ *is a harmonic morphism.*

The same improvement can be made if the foliation formed by the fibers is generated by a *commuting* family of Killing fields $\{V_1, \dots, V_r\}$ (in particular, if the foliation is generated by an abelian Lie group of isometries). To see this, define λ by $g(V_1 \wedge \dots \wedge V_r,\ V_1 \wedge \dots \wedge V_r) = \lambda^{2n-4}$, $n = \dim N$. Then λ is the dilation of the induced harmonic morphism. (Note that here the leaves are flat, since they are locally generated by parallel vector fields.)

In some cases, $\operatorname{trace}(\operatorname{ad} I)$ can be defined in a different way. To show this we need the following.

DEFINITION 1.17. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra of it. We shall denote the induced projection by $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ and the adjoint representation of $\mathfrak{g}$ by ad.

Suppose that $\operatorname{trace}(\operatorname{ad} A) = 0$ for any $A \in \mathfrak{h}$. Then $\operatorname{trace} \circ \operatorname{ad}\colon \mathfrak{g} \to \mathbb{R}$ naturally descends to a linear functional on $\mathfrak{g}/\mathfrak{h}$, which we shall denote by the same $\operatorname{trace} \circ \operatorname{ad}$.

LEMMA 1.18. *Let* $\mathfrak{h}$ *be a Lie subalgebra of* $\mathfrak{g}$. *Suppose that* $\operatorname{trace}(\operatorname{ad} A) = 0$ *for any* $A \in \mathfrak{h}$. *Also, suppose that there exists a linear subspace* $\mathfrak{m} \subseteq \mathfrak{g}$ *such that* $\mathfrak{h} \oplus \mathfrak{m} = \mathfrak{g}$ *and* $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. *Then*

$$\operatorname{trace}(\operatorname{ad} A) = c^{\beta}_{\alpha\beta} A^{\alpha}$$

for any $A = A^{\alpha} U_{\alpha} \in \mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$ *and where* $\{U_a\}_a = \{U_j\}_j \cup \{U_{\alpha}\}_{\alpha}$ *is a basis of* $\mathfrak{g}$ *such that* $\{U_j\}_j \subseteq \mathfrak{h}$, $\{U_{\alpha}\}_{\alpha} \subseteq \mathfrak{m}$, *and* c^c_{ab} *are the corresponding structural constants of* $\mathfrak{g}$ *given by* $[U_a, U_b] = c^c_{ab} U_c$.

REMARK 1.19. Note that, if $\mathfrak{h}$ is a compact Lie subalgebra of $\mathfrak{g}$, then both of the assumptions of Lemma 1.18 are satisfied.

The following lemma is immediate.

LEMMA 1.20. *Let* $\mathcal{V}$ *be a foliation on* (M, g) *that is generated by the action of a closed subgroup* G *of the isometry group of* (M, g). *Also, let* $\mathfrak{g}$ *be the Lie algebra of* G *and for* $x \in M$ *let* $\mathfrak{h}_x \subseteq \mathfrak{g}$ *be the Lie algebra of the isotropy group at* $x \in M$.

Let $(\operatorname{trace} \circ \operatorname{ad})_x$ *be the descended linear functional on* $\mathfrak{g}/\mathfrak{h}_x \cong \mathcal{V}_x$, *and let* $\operatorname{trace} \circ \operatorname{ad}$ *be the induced vertical* 1*-form on* M. *Then* $\operatorname{trace}(\operatorname{ad} I) = (\operatorname{trace} \circ \operatorname{ad})(I)$.

REMARK 1.21. Let $\mathcal{V}$ be a foliation on (M, g) that is locally generated by Killing fields. Some special circumstances are needed for $\operatorname{trace} \circ \operatorname{ad}$ to be well-defined, but to simplify the exposition we shall write $\operatorname{trace} \circ \operatorname{ad} = 0$ to mean that, in the neighborhood of each point, a local frame $\{V_{\alpha}\}$ for $\mathcal{V}$ can be found that is made up of Killing fields and is such that $c^{\alpha}_{\beta\alpha} = 0$, where the $c^{\alpha}_{\beta\gamma}$ are defined by $[V_{\alpha}, V_{\beta}] = c^{\gamma}_{\alpha\beta} V_{\gamma}$.

2. Applications

The following two corollaries follow immediately from Theorem 1.13.

COROLLARY 2.1. *A foliation of codimension* $\neq 2$ *that is locally generated by Killing fields and has integrable orthogonal complement produces harmonic morphisms.*

COROLLARY 2.2. *A foliation of codimension* $\neq 2$ *that is locally generated by Killing fields and for which* $\operatorname{trace} \circ \operatorname{ad} = 0$ *produces harmonic morphisms.*

Corollary 2.2 admits the following partial converse.

PROPOSITION 2.3. *Let* $\mathcal{V}$ *be a foliation of codimension* $\neq 2$ *that produces harmonic morphisms on* (M, g) *and is locally generated by Killing fields. Let* I *be the integrability tensor of the orthogonal complement* $\mathcal{H}$ *of* $\mathcal{V}$.

Suppose that, on each leaf L *of* $\mathcal{V}$, *a point* $x \in L$ *can be found such that* $\mathcal{V}_x$ *is spanned by* $\{I(X, Y) \mid X, Y \in \mathcal{H}_x\}$. *Then* $\operatorname{trace} \circ \operatorname{ad} = 0$.

Proof. From Theorem 1.13 it follows that we need only prove that $\mathcal{V}_x$ is spanned by $\{I(X, Y) \mid X, Y \in \mathcal{H}_x\}$ at each point $x \in M$. Obviously this holds on an open (nonempty) subset of each leaf L of $\mathcal{V}$. By Lemma 1.7, these subsets are also closed; the proof follows from the fact that the leaves are connected. □

Next we give an example of a Riemannian foliation locally generated by Killing fields and for which $\operatorname{trace}(\operatorname{ad} I) = 0$ but $I \neq 0$ and $\operatorname{trace} \circ \operatorname{ad} \neq 0$.

EXAMPLE 2.4. Let $\mathcal{F}$ be a Riemannian foliation locally generated by Killing fields that produces harmonic morphisms on (M, g). Suppose that the orthogonal complement of $\mathcal{F}$ is not integrable (see the examples that follow). Let G be the Lie group defined by

$$G = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \,\middle|\, a > 0,\ b \in \mathbb{R} \right\}.$$

Endow G with a right invariant metric γ and consider the Riemannian product manifold $(M \times G, g + \gamma)$. Let $\mathcal{V} = \mathcal{F} \times TG$. It is obvious that $\mathcal{V}$ is a foliation locally generated by Killing fields that produces harmonic morphisms on $(M \times G, g + \gamma)$. Notice, however, that the orthogonal complement $\mathcal{H}$ of $\mathcal{V}$ is nonintegrable and also that $\operatorname{trace} \circ \operatorname{ad} \neq 0$.

For the next application, we recall the following definition (cf. [4, 7.84]).

DEFINITION 2.5. Let (L^p, h) be a locally homogeneous Riemannian manifold (i.e., a Riemannian manifold whose tangent bundle admits, in a neighborhood of each point, local frames made up of Killing fields). Then (L^p, h) is called *naturally reductive* if each point $x \in L$ has an open neighborhood on which a local frame $\{V_\alpha\}_{\alpha=1,\dots,p}$ made up of Killing fields can be found such that $h([V_\alpha, V_\beta], V_\gamma) + h(V_\beta, [V_\alpha, V_\gamma]) = 0$ at x.

Since any skew-symmetric endomorphism is trace-free, we have the following.

LEMMA 2.6. *Let* (L, h) *be a naturally reductive locally homogeneous Riemannian manifold. Then* $\mathrm{trace} \circ \mathrm{ad} = 0$.

Our next result follows from Corollary 2.2 and Lemma 2.6.

PROPOSITION 2.7. *A foliation of codimension* $\neq 2$ *that is locally generated by Killing fields and whose leaves are naturally reductive produces harmonic morphisms.*

REMARK 2.8. Recall that any locally homogeneous Riemannian manifold that is locally symmetric is naturally reductive; thus, Proposition 2.7 holds also for foliations that are locally generated by Killing fields and whose leaves are locally symmetric.

Using Proposition 2.7, we obtain another proof for the following result from [17].

THEOREM 2.9. *Let* G *be a Lie group that acts as an isometry group on the Riemannian manifold* (M, g), *and suppose that the following conditions are satisfied*:

(i) *the orbits of the action of* G *on* M *have the same codimension* $\neq 2$;
(ii) *there exists on* G *a bi-invariant Riemannian metric*;
(iii) *the canonical representation of an isotropy group is irreducible.*

Then the connected components of the orbits form a Riemannian foliation with umbilical leaves that produces harmonic morphisms.

Proof. It is well known (see [20, Chap. IV, 4.10]) that (i) implies that the connected components of the orbits form a Riemannian foliation. Let $\mathcal{V}$ be this foliation.

By choosing an $\mathrm{Ad}\, G$ invariant metric on the Lie algebra of G and restricting it to the orthogonal complement of the Lie algebra of the isotropy group at $x \in M$, we can induce a metric $\bar{h}_x$ on $\mathcal{V}_x$ that by (iii) must be homothetic to $g_x\big|_{\mathcal{V}_x}$ (see [15, Vol. I, Apx. 5]). Then $\bar{h}$ is a metric on $\mathcal{V}$ that can be extended to a metric h on M such that $h\big|_{\mathcal{H}} = g\big|_{\mathcal{H}}$, where $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$.

Because $h\big|_{\mathcal{V}}$ is induced by an $\mathrm{Ad}\, G$ invariant metric, $\mathcal{V}$ has naturally reductive leaves with respect to h. But g and h are homothetic when restricted to a leaf and hence the leaves of $\mathcal{V}$ are also naturally reductive with respect to g. Moreover, since

(a) $\mathcal{V}$ has totally geodesic leaves with respect to h and
(b) g and h are conformal when restricted to $\mathcal{V}$ and equal when restricted to $\mathcal{H}$,

it follows that the leaves of $\mathcal{V}$ are umbilical with respect to g. □

REMARK 2.10. Note that the same argument can be applied to show that the Ricci tensor of each leaf is proportional with the induced metric (see [4, 7.44]). Hence, each leaf is an Einstein manifold in Theorem 2.9.

THEOREM 2.11. *Let* G *be a closed subgroup of the isometry group of* (M, g) *that generates a foliation* $\mathcal{V}$ *of codimension* $\neq 2$.

(i) *Suppose that the Lie algebra* $\mathfrak{g}$ *of* G *satisfies* $\operatorname{trace}(\operatorname{ad}\mathfrak{g}) = 0$. *Then* $\mathcal{V}$ *produces harmonic morphisms.*

(ii) *Conversely, if* $\mathcal{V}$ *produces harmonic morphisms and if, on each orbit* Q, *a point* $x \in Q$ *can be found such that* $\mathcal{V}_x$ *is spanned by* $\{I(X, Y) \mid X, Y \in \mathcal{H}_x\}$, *where* I *is the integrability tensor of the orthogonal complement of* $\mathcal{V}$, *then* $\operatorname{trace}(\operatorname{ad}\mathfrak{g}) = 0$.

Proof. (i) It is sufficient to prove that $\operatorname{trace} \circ \operatorname{ad} = 0$. Let H be the isotropy group of G at $x \in M$. Since G is a closed subgroup of the isometry group, we have that H is compact and hence we can find $\mathfrak{m} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

Let $\{A_1, \dots, A_r\}$ be a basis of $\mathfrak{h}$ and $\{A_{r+1}, \dots, A_s\}$ a basis of $\mathfrak{m}$. Let $\{c^{\gamma}_{\alpha\beta}\}$ be the corresponding structural constants of $\mathfrak{g}$. Because $\operatorname{trace}(\operatorname{ad}\mathfrak{g}) = 0$, we have $\sum_{\alpha=1}^{s} c^{\alpha}_{\alpha\beta} = 0$ for $\beta = 1, \dots, s$. Also, $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ implies that $c^{\gamma}_{\alpha\beta} = 0$ for $\alpha, \gamma = 1, \dots, r$ and $\beta = r+1, \dots, s$.

Let $\beta \in \{r+1, \dots, s\}$. Then $\sum_{\alpha=r+1}^{s} c^{\alpha}_{\alpha\beta} = -\sum_{\alpha=1}^{r} c^{\alpha}_{\alpha\beta} = 0$ and it follows that $(\operatorname{trace} \circ \operatorname{ad})_x = 0$.

(ii) Note that if a point of an orbit has the assumed property then, on each component of that orbit, a point can be found with the same property. The proof now follows from Proposition 2.3. □

COROLLARY 2.12. *Let* G *be a compact Lie group of isometries of* (M, g). *If the principal orbits of* G *have codimension* $\neq 2$ *then their connected components form a Riemannian foliation that produces harmonic morphisms.*

In particular, if (M, g) *is compact and if the principal orbits of the isometry group are of codimension* $\neq 2$, *then their connected components form a Riemannian foliation that produces harmonic morphisms.*

REMARK 2.13. Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Recall that if G is connected then $\operatorname{trace}(\operatorname{ad}\mathfrak{g}) = 0$ if and only if G is unimodular (i.e., if its left and right invariant Haar measures—which are unique up to multiplicative constants—are equal). Also, the condition $\operatorname{trace}(\operatorname{ad}\mathfrak{g}) = 0$ is automatically satisfied in the following cases:

(1) G is nilpotent (in particular, G is Abelian);
(2) G is semisimple;
(3) G is compact (more generally, $\operatorname{Ad} G$ is relatively compact in $Gl(\mathfrak{g})$);
(4) G is a general linear group.

For (1) this follows from the fact that if $A \in \mathfrak{g}$ then $\operatorname{ad} A : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent linear endomorphism. For (2) and (3) this follows from the fact that, in these cases, the adjoint group $\operatorname{Ad} G$ preserves a nondegenerate symmetric bilinear form. For (4) it can be checked directly that $\operatorname{trace}(\operatorname{ad}(\mathfrak{gl}_n)) = 0$.

Note that if G is as in (2), (3), or (4) then it is a reductive Lie group (i.e., its derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is semisimple). Since for any reductive Lie algebra $\mathfrak{g}$ we have $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple, it follows that any reductive Lie group G with Lie algebra $\mathfrak{g}$ satisfies $\operatorname{trace}(\operatorname{ad}\mathfrak{g}) = 0$. Moreover, any foliation generated by a reductive Lie group G of isometries whose isotropy group is a reductive Lie subgroup of G produces harmonic morphisms.

THEOREM 2.14. *Let $\xi = (P, N, G)$ be a principal bundle,* $\dim N \neq 2$, *whose total space P is endowed with a Riemannian metric g that is invariant by the action of G. Let $\mathcal{H}$ be the induced principal connection on ξ, and let h be the* (*unique*) *Riemannian metric on N such that the projection $\pi : (P, g) \to (N, h)$ is a Riemannian submersion.*

Then the following assertions are equivalent.

(i) *The connection induced by $\mathcal{H}$ on the determinant bundle of the adjoint bundle* $\mathrm{Ad}\,\xi$ *is flat.*
(ii) *The identity component H of the holonomy group of $\mathcal{H}$ satisfies* $\det(\mathrm{Ad}_G H) = 1$.
(iii) *The projection π lifts to a harmonic morphism $\tilde{\pi} : (\tilde{P}, \tilde{g}) \to (\tilde{N}, \tilde{h})$, where $(\tilde{P}, \tilde{g}) \to (P, g)$ is a Riemannian regular covering and $\tilde{N} \to N$ is a regular covering such that $\tilde{h}$ and the pull-back of h to $\tilde{N}$ are conformally equivalent. Moreover, $(\tilde{P}, \tilde{N}, G)$ is in a natural way a principal bundle.*

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) follows from the holonomy theorem (see [15]).

It is obvious that $\mathrm{trace}(\mathrm{ad}\, I)$ is a basic form that is the pull-back of the curvature form of the determinant bundle of $\mathrm{Ad}\,\xi$. Hence, (i) is equivalent to the fact that the fibers of π form a (Riemannian) foliation that produces harmonic morphisms.

Now, from Corollary 1.15 it follows that it is sufficient to prove that $(\tilde{P}, \tilde{N}, G)$ is in a natural way a principal bundle. Using the same notation as in Corollary 1.15 (with $P = M$), this follows from the fact that $\tilde{P}$ is the total space of $\xi + \eta \in H^1(N, G \times K)$, where $\eta \in H^1(N, K)$ is the regular covering corresponding to $[b] \in H^1(N, \mathbb{R})$ and $[b]$ is such that $\pi^*[b] = [a]$. □

REMARK 2.15. Let $\mathcal{V}$ be a foliation on (M, g) generated by the action of the closed subgroup G of the isometry group of (M, g).

Then it can be proved, directly by using the mass invariance chracteristic property of harmonic morphisms (see [17]), that $\mathcal{V}$ produces harmonic morphisms if and only if the identity component H of the holonomy group at x of the orthogonal complement of $\mathcal{V}$ satisfies $\det(\mathrm{Ad}_G H) = 1$. (The holonomy group of $\mathcal{H}$ $(= \mathcal{V}^{\perp})$ at $x \in M$ is formed of those $a \in G$ such that x and xa can be joined by a horizontal path; cf. [5].) In this way another proof can be obtained for the result of Theorem 1.13 applied to foliations *globally* generated by closed subgroups of the isometry group.

COROLLARY 2.16. *Let (P, N, G) be a principal bundle,* $\dim N \geq 3$, *whose total space P is endowed with a Riemannian metric g that is invariant by the action of G. Let $H \subseteq G$ be a closed subgroup and suppose that* $\mathrm{trace}(\mathrm{ad}\,\mathfrak{h}) = 0$ *and* $\mathrm{trace}(\mathrm{ad}\,\mathfrak{g}) = 0$. *Let $E = P \times_G G/H$ be the total space of the associated bundle.*

Then there exists, at least locally, Riemannian metrics on E and on N with respect to which the natural projection $E \to N$ is (*suitably restricted*) *a harmonic morphism.*

Proof. It is well known that (P, E, H) is in a natural way a principal bundle (see [15]). From Theorem 2.14 it follows that both of the foliations induced on P by the actions of G and H produce harmonic morphisms. Thus, at least locally, we can find metrics on E and N with respect to which the projections $P \to E$ and $P \to N$ are harmonic morphisms. Since the first of these is surjective this implies that the projection $E \to N$ can also be made, at least locally, a harmonic morphism (see [8, 2.31] and [11, Prop. 1.1]). □

It is obvious that Corollary 2.16 still holds if $\dim N = 1 \neq \dim G - \dim H$.

COROLLARY 2.17. *Let* (G, g) *be a Lie group endowed with a right* (*left*) *invariant Riemannian metric. Let* $K \subseteq H \subseteq G$ *be closed subgroups such that* $\operatorname{trace}(\operatorname{ad}\mathfrak{k}) = 0$, $\operatorname{trace}(\operatorname{ad}\mathfrak{h}) = 0$, *and* $\dim G - \dim H \geq 3$.

Then there exists, at least locally, Riemannian metrics on G/H *and* G/K *with respect to which the natural projection* $G/K \to G/H$ *is* (*suitably restricted*) *a harmonic morphism*.

It is obvious that Corollary 2.17 still holds if

$$\dim G - \dim H = 1 \neq \dim H - \dim K.$$

EXAMPLE 2.18. (1) A 1-dimensional Riemannian foliation produces harmonic morphisms if and only if it is locally generated by Killing fields. This result is due to Bryant [6] and the "if" part follows also from Corollary 2.2 (see also [17, Prop. 2.3]).

(2) The foliation formed on (an open subset of) a hypersphere S^n by the intersections with it of a parallel family of planes in $\mathbb{R}^{n+1}$ of codimension $\neq 2$ produces harmonic morphisms. This follows from Corollary 2.1 or Theorem 2.9, and can also be proved by noting that the foliation is induced by one of the projections of a warped product (see [17, Ex. 1.26(1)]). Similar examples can be obtained on Euclidean spaces and hyperbolic spaces.

(3) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and consider on $Gl_n(\mathbb{K})$, $n \geq 2$, the following well-known right invariant Riemannian metric:

$$g = \sum_{i,j=1}^{n} |\mathrm{d}x_k^i \cdot (x^{-1})_j^k|^2.$$

Let $K \subseteq H \subseteq G \subseteq Gl_n(\mathbb{K})$ be closed subgroups such that $\operatorname{trace}(\operatorname{ad}\mathfrak{k}) = 0$, $\operatorname{trace}(\operatorname{ad}\mathfrak{h}) = 0$, and $\dim G - \dim H \neq 2 \neq \dim G - \dim K$. Then, at least locally, a metric can be found on G/H (which is unique up to homotheties) such that the projection $G \to G/H$ becomes, suitably restricted, a harmonic morphism. (If G or G/H has zero first Betti number then this metric can be defined globally on G/H.) Also, at least locally, a metric can be found on the total space of the projection $G/K \to G/H$ such that the induced foliation produces harmonic morphisms. (If G/K and G/H both have zero first Betti number then there can be defined (global) metrics on them such that $G/K \to G/H$ becomes a harmonic morphism.)

For instance, the foliations formed by the fibers of the following natural maps produce harmonic morphisms:

$$Gl_{p+q}(\mathbb{K}) \to G_{p+q,p}(\mathbb{K}) \times G_{p+q,q}(\mathbb{K}),$$
$$Gl_{p+q}(\mathbb{K}) \to V_{p+q,p}(\mathbb{K}) \times G_{p+q,q}(\mathbb{K}),$$
$$Gl_{p+q}(\mathbb{K}) \to PGl_{p+q}(\mathbb{K}),$$

where $G_{p+q,p}(\mathbb{K})$ for $p, q \geq 1$ is the Grassmanian manifold of p-dimensional subspaces of $\mathbb{K}^{p+q}$, $V_{p+q,p}(\mathbb{K})$ is the Stiefel manifold of p-frames on $\mathbb{K}^{p+q}$, and for the first projection $p + q \geq 3$ if $\mathbb{K} = \mathbb{R}$. If $\mathbb{K} = \mathbb{H}$ then the first Betti number of $Gl_{p+q}(\mathbb{K})$ is zero and hence in these cases, on the image of each of the three maps just displayed, a metric can be found such that the induced map becomes a harmonic morphism.

In particular, consider on $Gl_2^+(\mathbb{R})$ the coordinates given by $x = \left(\begin{smallmatrix} x_1 & x_3 \\ x_2 & x_4 \end{smallmatrix}\right)$. With respect to these coordinates we have

$$g = \frac{1}{(x_1x_4 - x_2x_3)^2}\{(x_3^2 + x_4^2)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) + (x_1^2 + x_2^2)(\mathrm{d}x_3^2 + \mathrm{d}x_4^2) - 2(x_1x_3 + x_2x_4)(\mathrm{d}x_1\mathrm{d}x_3 + \mathrm{d}x_2\mathrm{d}x_4)\}.$$

Hence on the images of the following two maps there exist Riemannian metrics, unique up to homotheties, with respect to which the induced maps are harmonic morphisms:

$$\varphi_1 \colon Gl_2^+(\mathbb{R}) \to PGl_2^+(\mathbb{R}),$$

given by the natural projection; and

$$\varphi_2 \colon Gl_2^+(\mathbb{R}) \to \mathbb{R}^2 \times \mathbb{R}P^1,$$

given by $\varphi_2(x_1, x_2, x_3, x_4) = ((x_1, x_2), [x_3 : x_4])$. If $\mathbb{K} = \mathbb{C}, \mathbb{H}$ then we can also consider the foliation induced by the map $Gl_2(\mathbb{K}) \to \mathbb{K}P^1 \times \mathbb{K}P^1$.

Other examples can be obtained by considering other linear Lie groups.

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