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e to the A, in a New Way

Paul Federbush

The support of the Fourier transform of e^{iA} was established by Nelson in [4] (that is, the Fourier transform of each entry of the matrix e^{iA} in terms of the entries of the hermitian matrix A). But I believe this result of Nelson is very little known in the mathematical community at large. In further work [1; 3; 5], the transform was exhibited in the 2×2 case and presented in some unwieldy forms in higher dimensions. Our expressions are new and in practical form for some low-dimensional cases. We exhibit explicit expressions for traceless hermitian matrices of sizes 2×2, 3×3, and 4×4. In d dimensions, where A acts on $\mathbb{C}^d$, we employ matrices W, hermitian projections of rank 1. Thus W may be written as $W_{ij} = v_i \bar{v}_j$, where v_i is a unit vector in $\mathbb{C}^d$ and $\mathrm{Tr}(AW) = \langle v, Av \rangle$. The integral $\int d\Omega$ denotes a normalized integral over all such W defined by integrating the associated v_i over a unit sphere in $\mathbb{C}^d$ with the normalized unitary-invariant measure. (That the support of the Fourier transform lies on the complex projective space of such W is the content of Nelson's theorem.) The expressions we give are not unique among similar forms, and it is clear that one can derive such expressions in any number of dimensions. Substantial progress in the general case is made in [2].

2-*dimensional case*:

$$e^A = \int d\Omega e^{\mathrm{Tr}(AW)} \left[I + 4\left(W - \frac{1}{2}I\right) + \mathrm{Tr}(AW)I + 2\,\mathrm{Tr}(AW)\left(W - \frac{1}{2}I\right)\right]. \quad (1)$$

3-*dimensional case*:

$$e^A = \int d\Omega e^{\mathrm{Tr}(AW)} \left[I + 9\left(W - \frac{1}{3}I\right) - \mathrm{Tr}(AW)I + 9\,\mathrm{Tr}(AW)W - \frac{1}{2}A + \left[\frac{3}{2}(\mathrm{Tr}(AW))^2 - \frac{1}{4}\mathrm{Tr}(A^2)\right]W\right]. \quad (2)$$

4-*dimensional case*:

$$e^A = \int d\Omega e^{\mathrm{Tr}(AW)} \left[I + \frac{52}{3}\left(W - \frac{1}{4}I\right) - \frac{8}{3}\mathrm{Tr}(AW)I + \frac{68}{3}\mathrm{Tr}(AW)W - A + \frac{1}{6}[-4\,\mathrm{Tr}(A^2) + 46(\mathrm{Tr}(AW))^2 + 2A\,\mathrm{Tr}(AW)]W\right.$$

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$$
\begin{aligned}
&-\frac{1}{6}\left[-\frac{1}{2}\operatorname{Tr}(A^2)I + 3(\operatorname{Tr}(AW))^2 I + A^2 + 2A\operatorname{Tr}(AW)\right]\\
&+\frac{2}{3}(\operatorname{Tr}(AW))^3 W - \frac{1}{18}\operatorname{Tr}(A^3)W\\
&-\frac{1}{6}\operatorname{Tr}(A^2)\operatorname{Tr}(AW)W\Bigg]. \qquad (3)
\end{aligned}
$$

These expressions were derived using trickery and chicanery that will likely not be useful to one proving a general theory: extracting leading and subleading asymptotic behaviors as one eigenvalue of A approaches ∞ (see Appendix B); finding the simple rational numbers involved in the "angular integrals" of W by numerical integration (see Appendix A). The expressions were also checked by numerical integration. The 2-dimensional expression of equation (1) is intimately related to the expression derived in [1] for this dimension.

Appendix A

In this appendix we collect the most useful of the "angular integrals" computed. All of these results were derived numerically, but analytic derivation should be straightforward, if tedious. We denote

$$Av(f) \equiv \int d\Omega f \qquad (4)$$

for any function of W, f.

2 dimensions:

$$Av(\operatorname{Tr}(AW)W) = \frac{1}{6}A. \qquad (5)$$

3 dimensions:

$$Av(\operatorname{Tr}(AW)W) = \frac{1}{12}A, \qquad (6)$$

$$Av\big((\operatorname{Tr}(AW))^2 W\big) = \frac{1}{30}A^2 + \frac{1}{60}\operatorname{Tr}(A^2), \qquad (7)$$

$$Av\big((\operatorname{Tr}(AW))^3 W\big) = \frac{1}{30}A^3. \qquad (8)$$

4 dimensions:

$$Av(\operatorname{Tr}(AW)W) = \frac{1}{20}A, \qquad (9)$$

$$Av\big((\operatorname{Tr}(AW))^2 W\big) = \frac{1}{120}\operatorname{Tr}(A^2) + \frac{1}{60}A^2, \qquad (10)$$

$$Av\big((\operatorname{Tr}(AW))^3 W\big) = \frac{1}{420}\operatorname{Tr}(A^3) + \frac{1}{280}\operatorname{Tr}(A^2)A + \frac{1}{140}A^3. \qquad (11)$$

Appendix B

In this appendix we sketch the computations used to derive equation (2) in 3 dimensions. The derivation of the 4-dimensional case, equation (3), is similar but more complex. We let A have the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \tag{12}$$

with

$$1 + \lambda_1 + \lambda_2 = 0 \tag{13}$$

and

$$1 > \lambda_1, \qquad 1 > \lambda_2. \tag{14}$$

We also set

$$W_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

We then find (via steepest descent or Laplace's method) that

$$I(s) \equiv \int d\Omega e^{s\,\mathrm{Tr}(AW)} P(W) \tag{16}$$

as s goes to $+\infty$ and has leading asymptotic behavior coming entirely from an arbitrarily small neighborhood of W_0, and that

$$I(s) \sim \frac{2}{s^2(1-\lambda_1)(1-\lambda_2)} e^s P(W_0). \tag{17}$$

A little calculation then shows

$$(1-\lambda_1)(1-\lambda_2) = 3(\mathrm{Tr}(AW_0))^2 - \tfrac{1}{2}\,\mathrm{Tr}(A^2). \tag{18}$$

Equations (16)–(18) explain the choice of terms in (2) quadratic in A. We found terms linear in A by writing the most general invariant linear expression and checking the supposed identity (2) order-by-order in A, using the formulas in Appendix A. (If we allow the invariant $\mathrm{Tr}(A^2W)$ to appear in the quadratic terms, say by adding the expression $\mathrm{Tr}(A^2W) - (\mathrm{Tr}(AW))^2$ to the quadratic terms and appropriately modifying the linear terms, then alternate expressions for the identity equation (2) are obtained.)

References

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Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1109

pfed@math.lsa.umich.edu