A General Splitting Formula for the Spectral Flow

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with an Appendix by K. P. Wojciechowski

1. Introduction

Several articles have been written containing formulas expressing the spectral flow of a path of self-adjoint Dirac operators on a closed split manifold M ($M = X \cup_\Sigma Y$) in terms of quantities determined by each piece in the decomposition and "interaction" terms (see e.g. [4; 7; 8; 15; 18]). The article of Nicolaescu [15] is perhaps the most elegant and conceptually appealing. Additionally, a large number of articles consider the closely related but more delicate problem of splitting theorems for the Atiyah–Patodi–Singer invariant. The bibliography of Bunke's article [4] contains a long list of citations. Most of these articles, with the exception of Nicolaescu's, use delicate analytical methods and estimates such as heat kernel methods, and the results apply only after one has stretched the collar neighborhood of the separating hypersurface. Nicolaescu instead treats the problem largely from the point of view of linear algebra in a symplectic Hilbert space, and his main result is appealing in the simplicity of its statement: the spectral flow of the path equals the Maslov index $\mu(\Lambda_X, \Lambda_Y)$. Here Λ_X and Λ_Y denote the paths of Cauchy data spaces consisting of the restrictions of nullspace elements of the operators on X and Y to their common boundary Σ. Unfortunately, Nicolaescu's formulation does not lend itself easily to computation. What is needed is a splitting formula that isolates the contribution from each of the two pieces of the decomposition to the spectral flow. This is especially important when studying spectral flow in the context of cut-and-paste constructions.

In this article we prove a general splitting theorem and show how it can be used to derive most of the various splitting theorems in the literature. The proof of our result is quite simple; it uses only elementary properties of the Maslov index in addition to three results of Nicolaescu: the theorem described in the preceding paragraph, a version from his subsequent article [16] for manifolds with boundary, and the calculation of the adiabatic limit of the Cauchy data space from [15].

Our main result, Theorem 5.1, is as follows.

Theorem. *Let $D(t)$ be a continuous path of self-adjoint Dirac operators on a smooth, closed, oriented, odd-dimensional, Riemannian manifold M. Suppose that M can be split along a hypersurface Σ ($M = X \cup_\Sigma Y$) and that each $D(t)$ is*

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cylindrical and neck-compatible with respect to this splitting. Let $B_X(t)$ and $B_Y(t)$ be paths of self-adjoint elliptic boundary conditions for the restriction of $D(t)$ to X and Y, respectively.

Then

$$\mathrm{SF}(D) = \mathrm{SF}(D|_X, B_X) + \mathrm{SF}(D|_Y, B_Y) + \mu(B_Y(1-t), B_X(1-t)) + \sum_{i=1,2,4,5,7,8,10,11} \mu(L_i, M_i).$$

The terms appearing in the sum are certain Maslov indices of explicitly defined paths of Lagrangians.

Notice that this formula, in contrast to the theorems cited previously, holds with neither any preliminary stretching assumptions nor any prescription on what the boundary conditions should be.

Perhaps the method itself is more important than the actual formula, in the sense that in any given application it is probably easier to adapt the method we introduce here to the specific situation than to make the problem fit our formula. (This is the case in [2] on the $SU(3)$ Casson invariant.) For that reason we include a lengthy "user's guide" (Section 6), which indicates how various additional hypotheses can be used to force some of the $\mu(L_i, M_i)$ terms to vanish. We also show how to derive with ease many of the different versions of the splitting theorems cited here. In particular, we derive the splitting theorem of Bunke, give a generalization of this theorem and the splitting theorem of Yoshida and Nicolaescu, and indicate the relation between our formula and the formula of [7].

Our results are stated and proven for Dirac operators on odd-dimensional manifolds, since these include most of the geometrically important classes of self-adjoint elliptic operators—for example, the odd-signature operator and the spin Dirac operator.

We finish this introduction with a brief example of the method for those readers who are familiar with this subject. Other readers can return to the following paragraphs after finishing Section 4.

Suppose that $D(t)\colon \Gamma(E) \to \Gamma(E)$, $t \in [0,1]$ is a path of self-adjoint Dirac operators on a manifold M decomposed along a hypersurface $M = X \cup_\Sigma Y$. Let $\Lambda_X(t)$ and $\Lambda_Y(t)$ be the Cauchy data spaces associated to the restrictions of $D(t)$ to X and Y, respectively. These are Lagrangian subspaces of the symplectic Hilbert space $L^2(E|_\Sigma)$. Assume further that each $D(t)$ is cylindrical ($D(t) = J(\partial/\partial s + S(t))$ on a collar neighborhood of Σ) and neck-compatible (for each t, $S(t)$ is self-adjoint). Furthermore, suppose that the kernels of the tangential operators $S(t)$ are trivial for all t and denote by $P^{\pm}(t)$ the positive/negative eigenspace of $S(t)$.

Finally, suppose that $\Lambda_X(0) = P^-(0)$, $\Lambda_X(1) = P^-(1)$, $\Lambda_Y(0) = P^+(0)$, and $\Lambda_Y(1) = P^+(1)$. These four equalities rarely hold except in artificial examples, but Nicolaescu's adiabatic limit theorem says these conditions are asymptotically true; compensating for this leads to the extra terms in our formula.

The path $\Lambda_X(t)$ is clearly homotopic rel endpoints to the composite of the three paths $P^-(t)$, $P^-(1-t)$, and $\Lambda_X(t)$. Similarly, the path $\Lambda_Y(t)$ is homotopic rel endpoints to the composite of the three paths $\Lambda_Y(t)$, $P^+(1-t)$, and $P^+(t)$. Because the Maslov index is invariant under rel endpoint homotopies and additive with respect to compositions of paths, we conclude that

$$\begin{aligned}\mathrm{SF}(D, M) &= \mu(\Lambda_X, \Lambda_Y) \quad \text{(Nicolaescu's splitting theorem)} \\ &= \mu(P^-, \Lambda_Y) + \mu(P^-(1-t), P^+(1-t)) + \mu(\Lambda_X; P^+) \\ &= \mathrm{SF}(D|_Y; P^-) + \mathrm{SF}(D|_X; P^+)\end{aligned}$$

The last step follows from the version of Nicolaescu's theorem for manifolds with boundary and the fact that P^+ and P^- are transverse. The proof of our main result is no more difficult than this. The extra terms come about by moving to the adiabatic limits at the endpoints and from allowing general boundary conditions.

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2. Dirac Operators

There are many different definitions of *Dirac operator* in the literature. For our purposes, we adopt that of [15]. Briefly, a Dirac operator is determined by a Clifford module over a manifold along with a compatible connection. More precisely, suppose we are given the following.

(1) An *oriented Riemannian manifold* (M, g).

(2) A *self-adjoint Clifford module* $E \to M$. Hence E is a vector bundle over M with an action $c\colon C(M) \to \mathrm{End}(E)$. Here $C(M)$ is the bundle of Clifford algebras over M generated by the cotangent bundle using the metric. The adjective "self-adjoint" means that c carries each element of T^*M to a *skew-adjoint* endomorphism. Together with the Clifford relations, this implies that each element of T^*M acts orthogonally. For convenience we assume that the vector bundle E is a complex vector bundle.

(3) A *Clifford compatible covariant derivative* ∇^E on E. Thus,

$$\nabla^E\colon \Gamma(E) \to \Gamma(E \otimes T^*M)$$

is a differential operator that (a) satisfies the Leibniz rule

$$\nabla^E(fs) = df \otimes s + f\nabla^E s$$

for any $f \in C^\infty(M)$ and $s \in \Gamma(E)$, and (b) is compatible with the Clifford action in the sense that

$$[\nabla^E, c(a)] = c(\nabla a),$$

where $a \in \Gamma(C(M))$ and ∇ is the Levi–Civita connection (naturally extended from TM to $C(M)$).

These data determine a Dirac operator as the composition

$$\Gamma(E) \xrightarrow{\nabla^E} \Gamma(E \otimes T^*M) \xrightarrow{CC} \Gamma(E),$$

where CC denotes contraction with respect to the Clifford action (denoted by c previously). Incidentally, this definition agrees with that of a *Dirac operator on a Dirac bundle* as defined in [14]. In this article, we consider only self-adjoint Dirac operators over odd-dimensional manifolds.

We are particularly interested in Dirac operators over split manifolds. A manifold M is *split along a hypersurface* Σ if it can be expressed as the union of two manifolds (X and Y) with boundary such that $\partial X = -\partial Y = \Sigma = X \cap Y$. In this case we also require the existence of a neighborhood U of Σ in M that is isometric to $\Sigma \times (-1, 1)$. Over this neighborhood, all relevant structures (e.g., the Clifford bundle E) should decompose similarly.

Thus we are led to consider Dirac operators on manifolds with boundary, and in this context we impose two further restrictions. *First*, such a Dirac operator must be *cylindrical*. This means that, in a neighborhood of the boundary (of the form $\Sigma \times (-1, 0]$ or $\Sigma \times [0, 1)$ as just described), D can be written as

$$D = c(du)(\partial/\partial u + S), \tag{2.1}$$

where u is the second factor in $\Sigma \times (-1, 0]$ (or $\Sigma \times [0, -1)$), chosen so that $\|du\| = 1$, and S is a Dirac operator on $E|_{\partial M}$, referred to as the *tangential operator*. (Note that S is assumed to be constant in that it does not depend on the coordinate u.) *Second*, we require that D be *neck-compatible*, meaning that the tangential operator S is self-adjoint.

In what follows we consider only Dirac operators satisfying these conditions. Although these conditions may appear restrictive, most important geometrically defined self-adjoint operators are of this type—*for example*, the spin Dirac and odd-signature operators.

The Clifford relation ($v \otimes w + w \otimes v = -2\langle v, w\rangle$) implies that the algebraic operator $c(du)\colon \Gamma(E|_{\partial M}) \to \Gamma(E|_{\partial M})$ is a fiberwise isometry satisfying $c(du)^2 = -\mathrm{Id}$ and so induces a complex structure on $L^2(E|_{\partial M})$, which we rename (suggestively)

$$J\colon L^2(E|_{\partial M}) \to L^2(E|_{\partial M}). \tag{2.2}$$

Thus $J^2 = -\mathrm{Id}$. Moreover, $SJ = -JS$ and so the spectrum of the elliptic self-adjoint operator $S\colon L^2(E|_{\partial M}) \to L^2(E|_{\partial M})$ is symmetric, and its λ and $-\lambda$ eigenspaces are interchanged by J.

Define a hermitian symplectic structure on $L^2(E|_{\partial M})$ by

$$\omega(x, y) = \langle x, Jy\rangle,$$

where $\langle\ ,\ \rangle$ denotes the L^2 inner product.

DEFINITION 2.1. Two closed subspaces L_1, L_2 of a Hilbert space form a *Fredholm pair* if $L_1 \cap L_2$ is finite dimensional and $L_1 + L_2$ is closed with finite codimension.

DEFINITION 2.2.

(1) A closed subspace $L \subset L^2(E|_{\partial M})$ is called *isotropic* if L and JL are orthogonal. Thus, $\omega(l, m) = 0$ for all $l, m \in L$.
(2) A closed subspace $L \subset L^2(E|_{\partial M})$ is called *Lagrangian* if JL is the orthogonal complement of L. Thus, $\omega(l, m) = 0$ for all $l, m \in L$ and $L + JL = L^2(E|_{\partial M})$.

Since $SJ = -JS$, the Hilbert space $L^2(E|_{\partial M})$ has an orthogonal decomposition into the orthogonal direct sum of the negative eigenspace, kernel, and positive eigenspace of S:

$$L^2(E|_{\partial M}) = P^-(S) \oplus \ker S \oplus P^+(S). \tag{2.3}$$

In this decomposition, $\ker S$ is finite-dimensional because S is elliptic on the closed manifold ∂M. Moreover, J preserves $\ker S$ and so $\ker S$ is a symplectic subspace. The spaces $P^+(S)$ and $P^-(S)$ are interchanged by J since $JS = -SJ$, so $P^+(S)$ and $P^-(S)$ are infinite-dimensional and isotropic.

If $L \subset \ker S$ is a (finite-dimensional) Lagrangian subspace (defined just as before but substituting $\ker S$ for $L^2(E|_{\partial M})$), then the spaces $P^-(S) \oplus L$ and $L \oplus P^+(S)$ are easily seen to be Lagrangian subspaces of $L^2(E|_{\partial M})$. An important case occurs when $\ker S = 0$, in which case $P^\pm(S)$ are themselves Lagrangian subspaces.

It will be convenient to have a slightly more general decomposition of $L^2(E|_{\partial M})$ than Equation 2.3. Toward this end, let ν be any nonnegative real number and define

$$H_\nu(S) = \operatorname{span}_{L^2}\{\phi \mid S\phi = \lambda\phi \text{ and } |\lambda| \le \nu\}, \tag{2.4}$$

$$P_\nu^-(S) = \operatorname{span}_{L^2}\{\phi \mid S\phi = \lambda\phi \text{ and } \lambda < -\nu\}, \tag{2.5}$$

$$P_\nu^+(S) = \operatorname{span}_{L^2}\{\phi \mid S\phi = \lambda\phi \text{ and } \lambda > \nu\}. \tag{2.6}$$

Then, as before, the $P_\nu^\pm(S)$ are infinite-dimensional isotropic subspaces and H_ν is a finite-dimensional symplectic subspace. Moreover, the decomposition of Equation 2.3 is a special case ($\nu = 0$) of the decomposition

$$L^2(E|_{\partial M}) = P_\nu^-(S) \oplus H_\nu(S) \oplus P_\nu^+(S). \tag{2.7}$$

It is proven in [13] that if S is taken to vary continuously over some parameter space T—that is, if the map $t \mapsto S(t) - S(t_0)$ is a continuous map from T into the space of bounded operators (here t_0 is some fixed base point in T)—and if $\nu(t)$ is a continuous nonnegative function on T such that $S(t)$ has a spectral gap at $\nu(t)$ (i.e., $\nu(t)$ misses the spectrum of $S(t)$), then the decomposition of Equation 2.7 is continuous in T. Continuity for subspaces will always be taken in the gap topology [11].

3. Cauchy Data Spaces

For a given Dirac operator D on a manifold X with nonempty boundary Σ, its *Cauchy data space* $\Lambda_X(D)$ is a Lagrangian subspace of $L^2(E|_{\partial M})$ consisting roughly of boundary values of its kernel elements. We give a definition suitable for our purposes, referring to [15] for a careful construction.

In [3] it is shown that in the present context there is a well-defined, bounded, injective restriction map

$$R\colon \ker(D\colon L^2_{1/2}(E) \to L^2_{-1/2}(E)) \longrightarrow L^2(E|_\Sigma) \tag{3.1}$$

(see [15, Prop. 2.2]). Here $L^2_s(E)$ means the Sobolev space of sections of E with s derivatives in L^2, extended in the usual way to real s.

The image of R is a closed infinite-dimensional Lagrangian subspace of $L^2(E|_\Sigma)$. It will be denoted by

$$\Lambda_X(D) := R\big(\ker(D\colon L^2_{1/2}(E) \to L^2_{-1/2}(E))\big) \tag{3.2}$$

and called the *Cauchy data space* of the operator D on X. Sometimes we will abbreviate $\Lambda_X(D)$ to Λ_X or even Λ when D or X are clear from the context. Thus the Cauchy data space is space of boundary values of solutions to $D\sigma = 0$. In [15] it is proven that, if D varies regularly (smooth is sufficient but not necessary) in the space of Dirac operators with respect to some parameter space T, then the Cauchy data spaces $\Lambda_X(D(t))$ vary regularly (at least C^1) in $t \in T$. Regularity for closed subspaces may be interpreted in terms of the norm topology of the associated projections. The resulting topology is equivalent to the gap topology [11].

An important property of the Cauchy data space of a Dirac operator D of the form $J(\partial/\partial u + S)$ on the collar $\Sigma \times [-1, 0]$ of the boundary of X is that the pair $(\Lambda_X(D), P^+(S))$ forms a Fredholm pair of subspaces [15]. Since $P^+_\nu(S) \subset P^+(S)$ has finite codimension, it follows that if B is any closed subspace of $L^2(E|_\Sigma)$ that contains $P^+_\nu(S)$ for some ν with finite codimension, then $(\Lambda_X(D), B)$ form a Fredholm pair.

The proof of our main theorem will require *stretching*, which we now describe. Given a manifold X with boundary Σ and (open) collar $\Sigma \times (-1, 0]$, define

$$X^r = X \cup_{\Sigma\times(-1,0]} \Sigma \times (-1, r] \quad \text{for } r \geq 0. \tag{3.3}$$

Thus $X = X^0$. Using Equation 2.1 to define D on $\Sigma \times (-1, r]$ gives a natural extension of D to X^r. In this way one obtains a 1-parameter family of Cauchy data spaces $\Lambda_{X^r}(D)$. The limit of $\Lambda_{X^r}(D)$ as r approaches infinity is identified in Theorem 4.9 of [15]. We elaborate on this important and interesting result.

For notational convenience we write Λ^r_X for $\Lambda_{X^r}(D)$ and P^+_ν for $P^+_\nu(S)$. Since $\Lambda^0_X \cap P^+_0$ is finite-dimensional, and since $\bigcap_{\nu\to\infty} P^+_\nu = 0$, there exists a number $\nu_0 \geq 0$ such that

$$\Lambda^0_X \cap P^+_{\nu_0} = 0. \tag{3.4}$$

Following Nicolaescu, the set of all nonnegative real numbers that satisfy Equation 3.4 is a nonempty, closed, unbounded interval called the *nonresonance range of D*. The smallest such ν_0 is called the *nonresonance level of D*. Fix some ν_0 in the nonresonance range of D.

The symplectic reduction of Λ_X^0 to H_{ν_0} is the Lagrangian subspace

$$\tilde{\Lambda}_X(D) = \mathrm{proj}_{H_{\nu_0}}(\Lambda_X^0 \cap (H_{\nu_0} \oplus P_{\nu_0}^+)) = \frac{\Lambda_X^0 \cap (H_{\nu_0} \oplus P_{\nu_0}^+)}{\Lambda_X^0 \cap P_{\nu_0}^+} \subset H_{\nu_0}. \quad (3.5)$$

The decomposition of Equation 2.7 is preserved by S, since this is a decomposition in terms of eigenspaces of S. In particular, S preserves H_{ν_0} and the restriction of S to H_{ν_0} is self-adjoint, with all eigenvalues in $[-\nu_0, \nu_0]$. Thus we can form the 1-parameter family of (finite-dimensional) operators

$$e^{-rS}\colon H_{\nu_0} \to H_{\nu_0}. \quad (3.6)$$

It is not too hard to see that the limit

$$L_X(D) := \lim_{r\to\infty} e^{-rS}\tilde{\Lambda}_X(D) \quad (3.7)$$

exists and is a Lagrangian subspace of H_{ν_0}.

We may now state Nicolaescu's adiabatic limit theorem [15].

THEOREM 3.1. *As $r \to \infty$,*

$$\Lambda_X^r(D) \to P_{\nu_0}^- \oplus L_X(D). \quad (3.8)$$

The limiting subspace is called the *adiabatic limit* of Λ_X^r. Thus the adiabatic limit is determined, up to a finite-dimensional piece, by the tangential operator.

The identification of the adiabatic limit is an important ingredient in the proof of our splitting formula, but we require a little bit more. We complement the previous theorem with a lemma stating that the adiabatic deformation is in fact regular.

LEMMA 3.2. *Let $r(t) = 1/(1-t)$ for $t \in [0,1)$. The path of Lagrangian subspaces*

$$t \mapsto \begin{cases} \Lambda^{r(t)}, & t < 1, \\ P_{\nu_0}^- \oplus L_X(D), & t = 1 \end{cases}$$

is continuous.

The proof of Lemma 3.2 was provided to us by K. P. Wojciechowski and can be found in the Appendix.

One warning is in order here. It is not true that the adiabatic limits of the Cauchy data spaces vary continuously when D is varying continuously over some parameter space, even if ν_0 is larger than the nonresonance level for every operator D in this family. The reason for this is that the dynamics of e^{-rS} acting on subspaces of H_{ν_0} is quite sensitive to the initial subspace. See [2] for an explicit example of an analytic path of Dirac operators $D(t)$ for which the path of adiabatic limits of the Cauchy data spaces $P^-(t) \oplus L_X(D(t))$ is *not* continuous. There are some special circumstances under which one can conclude that the adiabatic limits vary continuously, and in those cases a splitting theorem can be proven easily. One such example is our Theorem 6.8.

We now let M be a closed manifold split along a hypersurface Σ into two pieces X and Y ($M = X \cup_\Sigma Y$). As before, we identify a closed neighborhood of Σ in M as $\Sigma \times (-1,1)$, with $\Sigma = \partial X = -\partial Y$. In the previous paragraphs we have stated various facts about Dirac operators from the point of view of "the X side". For convenience we state the analogous facts for "the Y side". The main thing to keep in mind here is that the complex structure J on $L^2(E|_\Sigma)$ and the cylindrical decomposition (2.1) in the collar both use the outward normal to X, which is the inward normal to Y. This generally has the effect of switching signs. Chasing down the repercussions, we have the following facts (see [15]).

(1) $(P^-(S), \Lambda_Y(D))$ is a Fredholm pair.
(2) The limit as $r \to \infty$ of $\Lambda_{Y^r}(D)$ is $L_Y(D) \oplus P^+_{\nu_1}(S)$, where ν_1 is in the nonresonance range of D acting on Y and $L_Y(D)$ is defined similarly to $L_X(D)$ but by taking $r \to -\infty$.
(3) The pair $(\Lambda_X(D), \Lambda_Y(D))$ is a Fredholm pair.
(4) The kernel of $D\colon \Gamma(E) \to \Gamma(E)$ is taken isomorphically to the intersection $\Lambda_X(D) \cap \Lambda_Y(D)$ by restricting to Σ.

4. Spectral Flow Equals Maslov Index

The theorems of Nicolaescu presented in this section establish the equality of two a priori different invariants that can be associated to a path of Dirac operators. Accordingly, we begin with a description of these two invariants.

The spectrum of a Dirac operator D on a closed manifold M consists of discrete eigenvalues of finite multiplicity. The *spectral flow* of a continuous path $D(t)$ ($t \in [0,1]$) of self-adjoint Dirac operators is (roughly) defined to be the algebraic count (with multiplicity) of the number of eigenvalues crossing through zero. Although this definition is somewhat imprecise, it suffices for our purposes—particularly because we never actually work with the spectral flow directly. Instead, we use Nicolaescu's theorems to convert the spectral flow to the Maslov index. In any case, precise definitions of the spectral flow can be found in [7; 12; 15].

An important technical point is appropriate here. One must set conventions so that the spectral flow is well defined on paths $D(t)$ for which $D(0)$ and/or $D(1)$ have nontrivial kernel. One must decide whether or not an eigenvalue that *starts* or *ends* at 0 counts as crossing *through* 0. It is important to be precise here, because different conventions appear in the literature, and the particular choice affects the properties of the invariant. Among the several such conventions that can be found, we use the following. Given a path $D(t)$ ($t \in [0,1]$) of Dirac operators, let $\varepsilon > 0$ be a number small enough so that the operators $D(0)$ and $D(1)$ have no eigenvalues in the interval $[-\varepsilon, 0)$. We define the spectral flow of the path $D(t)$ to be the spectral flow of the path $D(t) + \varepsilon\,\mathrm{Id}$:

$$\mathrm{SF}(D(t)) := \mathrm{SF}(D(t) + \varepsilon\,\mathrm{Id}).$$

Effectively, we count the eigenvalues that cross $-\varepsilon$ rather than those that cross 0. Notice that this avoids the issue of starting or ending at the crossing value because, by definition, no eigenvalues start or end at $-\varepsilon$.

Given a continuous path of Fredholm pairs of Lagrangians $(\Lambda_1(t), \Lambda_2(t))$ in a symplectic vector space, the *Maslov index* $\mu(\Lambda_1, \Lambda_2)$ is the integer defined to be the algebraic count of how many times $\Lambda_1(t)$ passes through $\Lambda_2(t)$ along the path. The complex structure J is used to specify the signs in this algebraic count. In particular, the normalization is chosen so that $\mu(e^{tJ}\Lambda_1, \Lambda_2)$, $t \in [-\varepsilon, \varepsilon]$, equals $\dim(\Lambda_1, \Lambda_2)$ when Λ_1 and Λ_2 are constant paths. See [5; 8; 15] for the precise definition.

Note: the condition that the Lagrangians be Fredholm is vacuous in the finite-dimensional case but critical in our context $(L^2(E|_\Sigma))$. Typically, the Fredholm property is easily verified—for any pair of paths we consider—by appealing to facts about Cauchy data spaces and related Lagrangians, as discussed in Section 2.

As with the spectral flow, a convention must be chosen to define the Maslov index for paths of pairs that are not transverse at the endpoints. Again, it is important to be explicit here because there are a number of possibilities. We use a convention defined in terms of the complex structure J, as explained in [5]. Choose a small positive ε such that:

(1) $(e^{sJ}L_1(t), L_2(t))$ form a Fredholm pair for each t and each $0 \le s \le \varepsilon$ (this is possible because Fredholm pairs form an open subspace of the space of closed pairs [11]); and
(2) $e^{sJ}L_1(0)$ is transverse to $L_2(0)$ and $e^{sJ}L_1(1)$ is transverse to $L_2(1)$ for all $0 < s \le \varepsilon$ (the proof that such an ε exists can be found in [5]).

Thus the path of pairs $(e^{J\varepsilon}L_1(t), L_2(t))$ forms a path of Fredholm pairs which are transverse at the endpoints. One then defines the Maslov index of L_1 and L_2 by taking

$$\mu(L_1, L_2) := \mu(e^{J\varepsilon}L_1, L_2). \tag{4.1}$$

We will use the following two elementary properties of the Maslov index.

(1) *Path Additivity.* Let L_1, L_2, K_1, K_2 be paths of Lagrangians such that $L_i(1) = K_i(0)$ for $i = 1, 2$, and let M_i be the path obtained by concatenating L_i and K_i (we write $M_i = L_i * K_i$). Then
$$\mu(M_1, M_2) = \mu(L_1, L_2) + \mu(K_1, K_2).$$
(2) *Homotopy Invariance.* Let L_1, L_2, K_1, K_2 be paths of Lagrangians such that L_i is homotopic rel endpoints to K_i. Then
$$\mu(L_1, L_2) = \mu(K_1, K_2).$$

Proofs of these facts follow from the interpretation of the Maslov index as an intersection number (see [5]). It is worth noting that the proof of our main theorem requires only these elementary properties of the Maslov index, eschewing more technical tools such as symplectic reduction.

Path additivity does not hold with all possible conventions, and it is for this property that we use the chosen convention. There are other conventions. To go back and forth between conventions, one need only know that, if μ' is another convention, then there exist numbers σ_0 and σ_1 in $\{-1, 0, 1\}$ and $e \in \{1, -1\}$ such that

$$\mu'(L, M) = e \cdot \mu(L, M) + \sigma_0 \cdot \dim(L(0) \cap M(0)) + \sigma_1 \cdot \dim(L(1) \cap M(1)).$$

A similar remark applies to the spectral flow, and it is not hard to see that the formula of our main result, Theorem 5.1, remains true provided one chooses the spectral flow and Maslov index conventions compatibly—after perhaps adding a correction term depending only on the dimensions of $\ker D(0)$ and $\ker D(1)$. The main result of [9] states that our choices of spectral flow and Maslov index conventions are compatible.

Two further simple facts, which we will use in Section 6 without explicit mention, are that (with our chosen conventions):

(1) $\mu(L, M) = 0$ if L and M are constant paths; and
(2) if L, M are paths of Lagrangians in $H_\nu(S)$, then

$$\mu(P_\nu^- \oplus L, M \oplus P_\nu^+) = \mu(L, M).$$

These are easy consequences of the definitions.

The following remarkable theorem of Nicolaescu will be the basis of what follows.

THEOREM 4.1. *Let $D(t)$, $t \in [0,1]$, be a smooth path of (cylindrical, neck-compatible, self-adjoint) Dirac operators on a smooth, oriented, closed, odd-dimensional Riemannian manifold M that splits as $M = X \cup_\Sigma Y$. Then*

$$\mathrm{SF}(D) = \mu(\Lambda_X(D), \Lambda_Y(D)).$$

The theorem explicitly states the intuitively appealing notion that counting kernel elements along the path (i.e., counting eigenvalues that cross through zero) is equivalent to counting pairs of boundary values that match up (i.e., nontrivial intersections between the Cauchy data spaces). Theorem 4.1 was first proved by Nicolaescu [15] for paths of Dirac operators whose endpoints have trivial kernel; the restriction to trivial kernel at the endpoints was removed in [9].

A similar theorem may be stated for manifolds with boundary. In this case we must impose boundary conditions for the spectral flow to be well-defined. This is the subject of the next definition.

DEFINITION 4.2. Let X be a manifold with boundary $\partial X = \Sigma$, and let D be a self-adjoint Dirac operator on X in cylindrical form with tangential operator S. A *self-adjoint elliptic boundary condition* is a Lagrangian subspace $B \subset L^2(E|_\Sigma)$ that contains $P_\nu^+(S)$ as a finite codimensional subspace for some ν. (See [3] and [16] for details.)

The condition that B be Lagrangian implies that the operator D on X with boundary conditions B is self-adjoint. The requirement that B contain P^+ with finite codimension ensures that the operator D—acting on sections over X whose restriction to the boundary lies in B—is elliptic. Thus, given a path $D(t)$ of Dirac operators on X and a path of elliptic self-adjoint boundary conditions $B(t)$, the spectral flow $\mathrm{SF}(D, B)$ is defined.

Then Nicolaescu's theorem extends to the bounded case as follows.

THEOREM 4.3. *Let $D(t)$, $t \in [0,1]$, be a smooth path of (cylindrical, neck-compatible, self-adjoint) Dirac operators on a smooth, oriented, odd-dimensional Riemannian manifold X with nontrivial boundary $\partial X = \Sigma$. Let $B(t)$ be a smooth path of elliptic boundary conditions for $D(t)$. Then*

$$\mathrm{SF}(D, B) = \mu(\Lambda_X(D), B).$$

5. The General Splitting Formula

In this section we state and prove the general splitting formula. The formula expresses the spectral flow of a path of Dirac operators on a closed manifold in terms of the spectral flows of the restricted paths (with associated elliptic boundary conditions). Whereas other results of this type have many additional hypotheses and produce more succinct formulas, our result requires only the minimal hypotheses but produces a longer formula. In Section 6 we discuss additional conditions that may be imposed to make various terms in our formula vanish or cancel.

The set-up is as follows. Let $D(t)$ be a smooth path of Dirac operators on a smooth, oriented, closed, odd-dimensional Riemannian manifold M. Suppose that M can be split along a hypersurface Σ ($M = X \cup_\Sigma Y$) and that each $D(t)$ is cylindrical and neck-compatible with respect to this splitting. Let $B_X(t)$ and $B_Y(t)$ be paths of elliptic boundary conditions for $D(t)$ restricted to X and Y, respectively. Then we will show that there is an 11-term formula

$$\begin{aligned}\mathrm{SF}(D) &= \mathrm{SF}(D_X, B_X) + \mathrm{SF}(D_Y, B_Y)\\ &\quad + \mu(B_Y(1-t), B_X(1-t)) + \sum_{i=1,2,4,5,7,8,10,11} \mu(L_i, M_i). \qquad (5.1)\end{aligned}$$

The $\mu(L_i, M_i)$ are certain Maslov indices; they will be defined shortly and discussed at length in the following section.

Theorem 4.1 allows us to replace $\mathrm{SF}(D)$ by $\mu(\Lambda_X(D), \Lambda_Y(D))$. We have at our disposal the path additivity and the homotopy invariance of the Maslov index. We will describe paths L and M that are homotopic rel endpoints to $\Lambda_X(D)$ and $\Lambda_Y(D)$, respectively. These new paths will each be the concatenation of eleven pieces (L_i and M_i, resp.). Each piece will contribute a term to the right-hand side of Equation 5.1.

To begin, let $\nu_0 \geq 0$ and $\nu_1 \geq 0$ be numbers chosen such that:

(1) ν_0 is in the nonresonance range for $D(0)$ on X and the tangential operator $S(0)$ has a spectral gap at ν_0; and
(2) ν_1 is in the nonresonance range for $D(1)$ on Y and the tangential operator $S(1)$ has a spectral gap at ν_1.

We abbreviate the notation for the Cauchy data spaces using the symbol $\Lambda^r_X(t)$ for $\Lambda_{X^r}(D(t))$. Moreover, $\Lambda_X(t)$ means $\Lambda_{X^0}(D(t)) = \Lambda_X(D(t))$; similar notation applies to Y. Nicolaescu's adiabatic limit theorem (Theorem 3.1) shows that there exists a Lagrangian $L_X(0) \subset H_{\nu_0}$ (and gives a recipe for constructing it) such that

$$\lim_{r\to\infty} \Lambda^r_X(0) = P^-_{\nu_0}(S(0)) \oplus L_X(0),$$

and there exists a Lagrangian $L_Y(1) \subset H_{\nu_0}$ such that

$$\lim_{r\to\infty} \Lambda^r_Y(1) = L_Y(1) \oplus P^+_{\nu_1}(S(1)).$$

We can now enumerate the eleven pieces of each path.

(1) Let L_1 be the path starting at $\Lambda^0_X(0)$ and ending at

$$\lim_{r\to\infty} \Lambda^r_X(0) = P^-_{\nu_0}(S(0)) \oplus L_X(0)$$

and obtained by stretching. An explicit formula is given in the statement of Lemma 3.2. Let M_1 be the constant path at $\Lambda_Y(0)$.

(2) Let L_2 be any path of Lagrangians starting at $P^-_{\nu_0}(S(0)) \oplus L_X(0)$ and ending at $B_Y(0)$ so that, for all t, $L_2(t)$ is a self-adjoint elliptic boundary condition for the restriction of $D(0)$ to Y (more generally, it suffices to assume that $(L_2(t), \Lambda_Y(0))$ are a Fredholm pair). Let M_2 be the constant path $\Lambda_Y(0)$.

(3) Let $L_3(t)$ be $B_Y(t)$ and let $M_3(t)$ be $\Lambda_Y(t)$. Theorem 4.3 applied to Y implies that

$$\mu(L_3, M_3) = \mathrm{SF}(D|_Y, B_Y). \tag{5.2}$$

(4) Take L_4 to be the constant path $B_Y(1)$, and let M_4 be the path from $\Lambda_Y(1)$ to $\lim_{r\to\infty} \Lambda^r_Y(1) = L_Y(1) \oplus P^+_{\nu_1}(S(1))$ obtained by stretching as in Lemma 3.2.

(5) Let L_5 be the constant path $B_Y(1)$. For M_5, choose a path of Lagrangians starting at $L_Y(1) \oplus P^+_{\nu_1}(S(1))$ and ending at $B_X(1)$ so that, for all t, $M_5(t)$ is a self-adjoint elliptic boundary condition for the restriction of $D(1)$ to X (or, more generally, so that $(\Lambda_X(1), M_5(t))$ form a Fredholm pair).

(6) Let L_6 be the path L_3 run backward (i.e., $L_6(t) = L_3(1-t)$), and let M_6 be B_X run backward. Thus,

$$\mu(L_6, M_6) = \mu(B_Y(1-t), B_X(1-t)). \tag{5.3}$$

(7) Let L_7 be L_2 run backward and let M_7 be the constant path $B_X(0)$.

(8) Let L_8 be L_1 run backward and M_8 the constant path $B_X(0)$.

(9) Let L_9 be the path $\Lambda_X(t)$ and let M_9 be the path $B_X(t)$. Theorem 4.3 states that

$$\mu(L_9, M_9) = \mathrm{SF}(D|_X, B_X). \tag{5.4}$$

(10) Take L_{10} to be the constant path $\Lambda_X(1)$ and M_{10} to be M_5 run backward.

(11) Finally, let L_{11} be the constant path $\Lambda_X(1)$ and let M_{11} be M_4 run backward.

The reader may verify that the composite path $L = L_1 * L_2 * \cdots * L_{11}$ is defined and is homotopic rel endpoints to the path Λ_X. Similarly $M = M_1 * M_2 * \cdots * M_{11}$ is homotopic rel endpoints to Λ_Y. Hence

$$\mathrm{SF}(D) = \mu(\Lambda_X, \Lambda_Y) = \mu(L, M) = \sum_{i=1}^{11} \mu(L_i, M_i),$$

using homotopy invariance of the Maslov index and additivity of the Maslov index under composition of paths.

We summarize our conclusions in the following theorem.

THEOREM 5.1. *Let $D(t)$ be a continuous path of self-adjoint Dirac operators on a smooth, closed, oriented, odd-dimensional Riemannian manifold M. Suppose that M can be split along a hypersurface Σ $(M = X \cup_\Sigma Y)$ and that each $D(t)$ is cylindrical and neck-compatible with respect to this splitting. Let $B_X(t)$ and $B_Y(t)$ be paths of self-adjoint elliptic boundary conditions for the restriction of $D(t)$ to X and Y.*

Then

$$\begin{aligned}\mathrm{SF}(D) = {} & \mathrm{SF}(D|_X, B_X) + \mathrm{SF}(D|_Y, B_Y) \\ & + \mu(B_Y(1-t), B_X(1-t)) + \sum_{i \neq 3,6,9} \mu(L_i, M_i).\end{aligned}$$

6. User's Guide to Theorem 5.1

In this section we explain how to use Theorem 5.1. Specifically, we show how various natural hypotheses simplify the formula and then derive some earlier theorems as consequences. We will not exhaust all the possibilities, but we hope to give some indication of the formula's utility.

The authors' background concerns the application of this subject to the odd-signature operator coupled to a path of connections starting and ending at flat connections. This is the kind of operator considered in topological applications of spectral flow, such as computations of Atiyah–Patodi–Singer ρ_α invariants, Casson's invariant, and Floer homology. The methods we describe are particularly well suited for this class of problem.

6.1. Transversality at Endpoints and Stretching

First some notation. We have defined X^r and Y^r to be the manifolds obtained by adding a collar of length r to X and Y. Let M^r be the closed manifold obtained by stretching M along Σ, so that

$$M^r = X^r \cup_\Sigma Y^r.$$

HYPOTHESIS 1. *The adiabatic limits of the Cauchy data spaces are transverse at the endpoints*:

$$\lim_{r\to\infty} \Lambda^r_X(i) \cap \lim_{r\to\infty} \Lambda^r_Y(i) = 0, \quad i = 0, 1.$$

PROPOSITION 6.1. *Suppose that Hypothesis 1 holds. Then there exists an $r_0 \geq 0$ such that, replacing M by M^r for $r \geq r_0$ in Theorem 5.1, the terms $\mu(L_1, M_1)$ and $\mu(L_{11}, M_{11})$ vanish.*

Proof. Continuity of the path of Lemma 3.2 implies that there exists some r_0 such that the Lagrangians $\Lambda^r_X(i)$ and $\Lambda^r_Y(i)$ are transverse for $r \geq r_0$ and $i = 0, 1$. Then the Lagrangians $L_1(t)$ and $M_1(t)$ are transverse for all $t \in [0, 1]$ and hence

$\mu(L_1, M_1) = 0$. The same argument applies at the other end of the path to show that $\mu(L_{11}, M_{11}) = 0$. $\square$

Notice that the two cases are independent. That is, if the limits of the Cauchy data spaces are transverse at the initial point then $\mu(L_1, M_1) = 0$ for r large enough, and if they are transverse at the terminal point then $\mu(L_{11}, M_{11}) = 0$ for r large enough.

A slight generalization of this can be obtained by using the following hypothesis.

HYPOTHESIS 2. *For $i = 0, 1$, the dimension of $\Lambda^r_X(i) \cap \Lambda^r_Y(i)$ is independent of r for $r \geq r_0$ and equals the dimension of the intersection of the limits of the Cauchy data spaces*:

$$\dim(\Lambda^r_X(i) \cap \Lambda^r_Y(i)) = \dim\left(\lim_{r\to\infty} \Lambda^r_X(i) \cap \lim_{r\to\infty} \Lambda^r_Y(i)\right).$$

Notice that the intersection $\Lambda^r_X(i) \cap \Lambda^r_Y(i)$ is isomorphic to the kernel of $D(i)$ on M^r, so Hypothesis 2 implies that the dimension of this kernel is independent of r.

PROPOSITION 6.2. *If Hypothesis 2 holds then, after replacing M by M^r for $r \geq r_0$ in Theorem 5.1, the terms $\mu(L_1, M_1)$ and $\mu(L_{11}, M_{11})$ vanish.*

Proof. Let $\Lambda^\infty_X(0)$ denote the adiabatic limit of $\Lambda^r_X(0)$, with similar notation for Y.

Fix $r \geq r_0$ and let $u \geq r$. Since $\dim(\Lambda^u_X(0) \cap \Lambda^r_Y(0))$ is isomorphic to the kernel of D on M^{u+r}, which in turn is isomorphic to $\dim(\Lambda_X^{(u+r)/2}(0) \cap \Lambda_Y^{(u+r)/2}(0))$, Hypothesis 2 implies that

$$\dim(\Lambda^u_X(0) \cap \Lambda^r_Y(0)) = \dim(\Lambda^\infty_X(0) \cap \Lambda^\infty_Y(0)).$$

Thus the dimension of the intersection of $L_1(t)$ with $M_1(t)$ is independent of t. This implies that $\mu(L_1, M_1) = 0$. A similar argument shows that $\mu(L_{11}, M_{11})$ vanishes. $\square$

6.2. *Choice of Boundary Conditions*

The boundary conditions B_X and B_Y can be restricted to simplify the splitting formula. The most direct way to do this is just to kill the terms $\mu(L_2, M_2)$, $\mu(L_5, M_5)$, $\mu(L_7, M_7)$, and $\mu(L_{10}, M_{10})$ by choosing the boundary conditions $B_Y(0)$ and $B_X(1)$ as follows.

HYPOTHESIS 3. $B_Y(0) = P^-_{\nu_0}(0) \oplus L_X(0)$ *and* $B_X(1) = L_Y(1) \oplus P^+_{\nu_1}(1)$.

PROPOSITION 6.3. *Assume that Hypothesis 3 holds. Then one can choose the paths L_2 and M_5 (and their reverses L_7 and M_{10}) so that*

$$\mu(L_2, M_2) = \mu(L_5, M_5) = \mu(L_7, M_7) = \mu(L_{10}, M_{10}) = 0.$$

Proof. Take L_2 and M_5 to be constant paths. Then L_7 and M_{10} are also constant. By definition, M_2, L_5, M_7, and L_{10} are constant. Hence the four terms are Maslov indices of constant paths, and thus all vanish. $\square$

We could have taken the point of view in Theorem 5.1 that only boundary conditions satisfying Hypothesis 3 are allowed. This would have given a formula with four fewer terms, but the result would have been less flexible. The decision to state the theorem as we did was made in order to decouple the choice of boundary conditions from the analysis of the adiabatic limits of the Cauchy data spaces.

6.3. The Nonresonance Range, Limiting Values of Extended L^2 Solutions, and Adiabatic Limits

We next give a more detailed description of the adiabatic limit $\lim_{r\to\infty}\Lambda_X^r$, which can be useful in controlling some of the terms.

DEFINITION 6.4. Let D be a cylindrical Dirac operator (as before) on a manifold X with boundary. The Lagrangian subspace

$$\tilde{L}_X(D) \subset \ker S$$

is defined to be the symplectic reduction of the Cauchy data space to the kernel of S,

$$\tilde{L}_X(D) = \mathrm{proj}_{\ker S}\big(\Lambda_X(D) \cap (\ker S \oplus P^+(S))\big)$$

and is called the *limiting values of extended L^2 solutions.* (This terminology comes from [1].)

For convenience, we recall the notation for several Lagrangians that appear in this section.

(1) Λ_X^r, the Cauchy data space on X^r. This is an infinite-dimensional Lagrangian subspace of $L^2(E|_\Sigma)$.

(2) $\tilde{\Lambda}_X$, the symplectic reduction of the (length-0) Cauchy data space Λ_X^0 to H_{ν_0} (Equation 3.5) where $\nu_0 \geq 0$ is greater than or equal to the nonresonance level of D and where S has a spectral gap at ν_0. This is a finite-dimensional Lagrangian subspace of the symplectic vector space H_{ν_0} defined in Equation 2.4.

(3) L_X, the limit of $e^{-rS}\tilde{\Lambda}_X$ as $r \to \infty$, a Lagrangian subspace of H_{ν_0} (Equation 3.7). Thus, the adiabatic limit $\lim_{r\to\infty}\Lambda_X^r = P_{\nu_0}^- \oplus L_X$.

(4) $\tilde{L}_X$, the limiting values of extended L^2 solutions, defined as the symplectic reduction of the Cauchy data space Λ_X^0 to the kernel of S in Definition 6.4.

The following theorem relates these Lagrangians and indicates the structure of L_X. It is convenient to extend the notation slightly so that, for the statement and proof of this theorem, we will allow $\nu < 0$ in the definition of P_ν^+ (Equation 2.6). For example, if ν is positive and in the complement of the spectrum of S, then $H_\nu \oplus P_\nu^+ = P_{-\nu}^+$.

Notice that there is a descending filtration of $H_{\nu_0} \oplus P_{\nu_0}^+$ corresponding to the increasing list of eigenvalues $-\lambda_{n+1} < -\lambda_n < \cdots < 0 < \lambda_1 < \cdots$, assuming that $\lambda_n \leq \nu_0 < \lambda_{n+1}$:

$$P_{-\lambda_{n+1}}^+ \supset P_{-\lambda_n}^+ \supset \cdots \supset P_0^+ \supset P_{\lambda_1}^+ \supset \cdots. \tag{6.1}$$

THEOREM 6.5. *With notation as described before, the following statements are true.*

(1) $L_X = \lim_{r\to\infty} e^{-rS}\tilde{\Lambda}_X$.
(2) $\tilde{L}_X \subset L_X$.
(3) *If* $\nu_0 = 0$ *then* $L_X = \tilde{L}_X = \tilde{\Lambda}_X$.
(4) *Let* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1}$ *denote the ordered list of positive eigenvalues of the tangential operator* S, *so that* $\lambda_n \leq \nu_0 < \lambda_{n+1}$. *Let* $E_i^{\pm}$ *denote the* $\pm\lambda_i$ *eigenspace. Then*

$$H_{\nu_0} = E_n^- \oplus E_{n-1}^- \oplus \cdots \oplus \ker S \oplus E_1^+ \oplus \cdots \oplus E_n^+, \tag{6.2}$$

and the Lagrangian L_X *decomposes in this direct sum as*

$$L_X = W_n \oplus W_{n-1} \oplus \cdots \oplus \tilde{L}_X \oplus JV_1 \oplus \cdots \oplus JV_n, \tag{6.3}$$

where $W_i \subset E_i^-$ *are subspaces and* $V_i \subset E_i^-$ *are their orthogonal complements in* E_i^- *(and so* $JV_i \subset E_i^+$*). Moreover, this decomposition exhibits* L_X *as the associated graded vector space to the filtration of* $\Lambda_X \cap (H_{\nu_0} \oplus P_{\nu_0}^+) = \Lambda_X \cap P_{-\lambda_{n+1}}^+$ *obtained by intersecting* Λ_X *with the decreasing filtration given in Equation 6.1.*

Proof. The first assertion is the definition of L_X. For the third assertion, if $\nu_0 = 0$ then $\tilde{L}_X = \tilde{\Lambda}_X$ by definition. Since the operator S is zero on its kernel, the restriction of e^{-rS} to $\ker S$ is the identity, so that $L_X = \tilde{\Lambda}_X$. The second assertion follows from the fourth, which we will now prove.

Statement (4) follows from a careful analysis of the flow to the adiabatic limit. Notice that, because the P_ν^+ are defined in terms of strict inequalities, $P_{-\lambda_i}^+$ is the span of the eigenvectors whose eigenvalues are *greater* than $-\lambda_i$. Thus $E_i^- = P_{-\lambda_{i+1}}^+/P_{-\lambda_i}^+$ and $E_i^+ = P_{\lambda_{i-1}}^+/P_{\lambda_i}^+$.

Let

$$W_n = \mathrm{proj}_{E_n^-}(\Lambda_X \cap P_{-\lambda_{n+1}}^+) \subset E_n^-.$$

Hence,

$$W_n = (\Lambda_X \cap P_{-\lambda_{n+1}}^+)/(\Lambda_X \cap P_{-\lambda_n}^+).$$

Next, let

$$W_{n-1} = \mathrm{proj}_{E_{n-1}^-}(\Lambda_X \cap P_{-\lambda_n}^+) \subset E_{n-1}^-.$$

Continue in this fashion, peeling one space off at a time in the decomposition (Equation 6.2) of H_{ν_0}. We change notation when we arrive at $\ker S$ to be consistent with our previous notation. Thus (by definition),

$$\tilde{L}_X = \mathrm{proj}_{\ker S}(\Lambda_X \cap P_{-\lambda_1}^+) \subset \ker S.$$

Continue by letting

$$V_1' = \mathrm{proj}_{E_1^+}(\Lambda_X \cap P_0^+) \subset E_1^+,$$

$$V_2' = \mathrm{proj}_{E_2^+}(\Lambda_X \cap P_{\lambda_1}^+) \subset E_2^+,$$

and so forth until the last step:

$$V_n' = \mathrm{proj}_{E_n^+}(\Lambda_X \cap P_{\lambda_{n-1}}^+) \subset E_n^+.$$

Now, let us suppose that $(w_n, w_{n-1}, \ldots, w_1, h, v_1, \ldots, v_n, q)$ is an element of $\Lambda_X \cap (H_{\nu_0} \oplus P^+_{\nu_0})$ expressed in the decomposition of Equation 6.2 with the additional element $q \in P^+_{\nu_0}$. Then $(w_n, w_{n-1}, \ldots, w_1, h, v_1, \ldots, v_n)$ is in $\tilde{\Lambda}_X$.

Either $w_n = 0$ or else $w_n \in W_n - \{0\}$. Since

$$\frac{1}{e^{r\lambda_n}}(w_n, w_{n-1}, \ldots, w_1, h, v_1, \ldots, v_n) \in \tilde{\Lambda}_X$$

and since e^{-rS} acts on the decomposition of Equation 6.2 diagonally with (decreasing) eigenvalues $e^{r\lambda_n}, e^{r\lambda_{n-1}}, \ldots$, it follows that if $w_n \neq 0$ then

$$\lim_{r\to\infty} e^{-rS}\frac{1}{e^{r\lambda_n}}(w_n, w_{n-1}, \ldots, w_1, h, v_1, \ldots, v_n) = (w_n, 0, 0, \ldots, 0)$$

and so w_n is in $\lim_{r\to\infty} e^{-rS}\tilde{\Lambda}_X = L_X$. Arguing by induction, we obtain

$$L_X = W_n \oplus \cdots \oplus W_1 \oplus \tilde{L}_X \oplus V'_1 \oplus \cdots \oplus V'_n.$$

We must now see that $V'_i = JV_i$, where V_i is the orthogonal complement of W_i in E_i^-. But this follows from dimension counting and the fact that L_X is a Lagrangian subspace. Indeed, since the symplectic structure on H_{ν_0} is given by $\omega(x, y) = \langle x, Jy\rangle$, JV'_i is orthogonal to W_i and so lies in V_i. If, for some i, JV'_i were a proper subspace of V_i, then by counting dimensions (and using that the limiting values of extended L^2 solutions $\tilde{L}_X \subset \ker S$ constitute a Lagrangian subspace of $\ker S$) it would follow that L_X has too small a dimension to be a Lagrangian. Thus $JV'_i = V_i$ and so $V'_i = JV_i$ as claimed.

The assertion that L_X is the associated graded vector space to the filtration is simply a brief description of how the $W_i, \tilde{L}_X, V'_i$ were constructed. □

The fourth statement of Theorem 6.5 suggests a more useful and sophisticated alternative to Hypothesis 3. The underlying motivation comes from the fact that it is much easier to calculate $\tilde{L}_X \subset \ker S$ than to calculate $L_X \subset H_{\nu_0}$. Even getting a handle on the nonresonance level ν_0 can be a difficult problem.

There is a natural choice of path of Lagrangians starting at $P^-_{\nu_0} \oplus L_X$ and ending at $P^- \oplus \tilde{L}_X$, defined as follows. Notice that the symplectic subspaces $E_i^- \oplus E_i^+$ have a further decomposition as a direct sum:

$$E_i^- \oplus E_i^+ = (W_i \oplus V_i) \oplus (JW_i \oplus JV_i) = (W_i \oplus JW_i) \oplus (V_i \oplus JV_i).$$

Use these decompositions to define a path $C(t)$ by the formula

$$C\colon t \mapsto P^-_{\nu_0} \oplus \tilde{L}_X \oplus (W_1 \oplus e^{-(1-t)(\pi/2)J}V_1) \oplus \cdots \oplus (W_n \oplus e^{-(1-t)(\pi/2)J}V_n). \quad (6.4)$$

Then we can make the following hypothesis.

HYPOTHESIS 4. *$B_Y(0) = P^- \oplus \tilde{L}_X$ and $B_X(0) = A \oplus P^+$ for some Lagrangian $A \subset \ker S(0)$, and $L_2(t) = C(t)$.*

LEMMA 6.6. *If Hypothesis 4 holds, then $\mu(L_7, M_7) = 0$.*

Proof. This is a consequence of the conventions we are using for spectral flow. Recall that L_7 is L_2 run backward and that M_7 is the constant path at $B_X(0)$.

Using Theorem 6.5, we see that the Lagrangians $L_7(t) = C(1-t)$ and $M_7(t) = B_X(0)$ intersect in the direct sum

$$L_7(t) \cap M_7(t) = \begin{cases} (\tilde{L}_X \cap A) \oplus JV_1 \oplus JV_2 \oplus \cdots \oplus JV_n & \text{if } t = 1, \\ \tilde{L}_X \cap A & \text{if } 0 \le t < 1. \end{cases}$$

Thus, $e^{\varepsilon J} L_7(t)$ is transverse to $M_7(t)$ for all t, so that $\mu(L_7, M_7) = 0$. $\square$

A similar argument shows that if $B_X(1) = \tilde{L}_Y \oplus P^+$ and $B_Y(1) = P^- \oplus A$ for some Lagrangian $A \subset \ker S(1)$, and if M_5 is chosen in a manner similar to $C(t)$ (formula 6.4), then $\mu(L_5, M_5) = 0$.

Finally, with these choices and some additional transversality conditions, one can sometimes also compute $\mu(L_2, M_2)$ and $\mu(L_{10}, M_{10})$ in terms of the sum of the dimensions of the V_i (which is the same as the dimension of the L^2-kernel of D on X^∞) after a preliminary stretching. Lemma 6.6 will be used in a slightly different context in Theorem 6.13 (to follow).

6.4. *When the L^2 Kernel of $D|_X$ or $D|_Y$ Vanishes at the Endpoints*

The nonresonance level for $D|_X$ is zero if $\Lambda_X \cap P^+ = 0$. This is equivalent (see [1]) to the vanishing of the L^2-kernel of the natural extension of $D|_X$ to $X^\infty = X \cup \Sigma \times [0, \infty)$.

In this context, Nicolaescu's adiabatic limit theorem states that if $\Lambda_X \cap P^+ = 0$ (i.e., if the L^2 kernel of $D|_X$ on X^∞ is zero) then $\lim_{r\to\infty} \Lambda_X^r = P^- \oplus \tilde{L}_X$.

HYPOTHESIS 5. *The operators $D(i)|_{X^\infty}$ and $D(i)|_{Y^\infty}$ have no L^2 kernels for $i = 0, 1$.*

(In the terminology of [15], the operators $D(i)|_{X^\infty}$ and $D(i)|_{Y^\infty}$ are nonresonant.)

If Hypothesis 5 holds, then Hypothesis 1 holds if and only if $\tilde{L}_X(0) \cap \tilde{L}_Y(0) = 0$ and $\tilde{L}_X(1) \cap \tilde{L}_Y(1) = 0$. Moreover, in this case one can satisfy Hypothesis 3 by letting $B_Y(0) = P^-(0) \oplus \tilde{L}_X(0)$ and $B_X(1) = \tilde{L}_Y(1) \oplus P^+(1)$.

Let us use these ideas to give a simple proof of a theorem of Bunke [4] (see also [7, Thm. A]). Consider the case when the tangential operator has no kernel along the path. The following theorem appears (in different notation) in [4]; it also follows from Theorem A of [7].

THEOREM 6.7. *Suppose that the kernel of the tangential operator $S(t)$ vanishes for all t. Suppose that Hypothesis 5 holds* (*at the endpoints*).

Then there exists an r_0 such that, for $r \ge r_0$,

$$\mathrm{SF}(D, M^r) = \mathrm{SF}(D_{X^r}; P^+) + \mathrm{SF}(D_{Y^r}; P^-).$$

Proof. Hypothesis 5, together with the vanishing of the kernels of the tangential operators $S(0)$ and $S(1)$, implies that the adiabatic limits are

$$\lim_{r\to\infty} \Lambda_X^r(i) = P^-(i) \quad \text{and} \quad \lim_{r\to\infty} \Lambda_Y^r(i) = P^+(i)$$

for $i = 0, 1$. Since $P^+(i)$ is transverse to $P^-(i)$, Hypothesis 1 holds. Thus, Proposition 6.1 implies that $\mu(L_1, M_1)$ and $\mu(L_{11}, M_{11})$ vanish after sufficient stretching.

Since the kernel of $S(t)$ is zero for all t, the spaces $P^{\pm}(t)$ vary continuously [3; 13] and so we can take $B_X(t) = P^+(t)$ and $B_Y(t) = P^-(t)$. This immediately implies that $\mu(L_i, M_i)$ vanishes for $i = 2, 5, 7, 10$ according to Proposition 6.3, since Hypothesis 3 holds. We also have $\mu(L_6, M_6) = 0$, since this equals $\mu(P^-(1-t), P^+(1-t))$ and $P^+(t)$ is transverse to $P^-(t)$ for all t.

The path L_4 is the constant path at $B_Y(1) = P^-(1)$. The path M_4 is the path from $\Lambda_Y(1)$ to $\lim_{r\to\infty} \Lambda_Y^r(1) = P^+(1)$. Since $P^-(1)$ is transverse to $P^+(1)$, after perhaps making r larger, $\mu(L_4, M_4) = 0$. Likewise, after perhaps making r larger, $\mu(L_8, M_8) = 0$.

The only terms remaining are $\mu(L_3, M_3) = \mathrm{SF}(D|_{Y^r}; P^-)$ and $\mu(L_9, M_9) = \mathrm{SF}(D|_{X^r}, P^+)$. This completes the proof. $\square$

We next state and prove the theorem of Yoshida and Nicolaescu. The proof we give is identical to Nicolaescu's. We include it for the convenience of the reader and to introduce another useful technique that can be combined with our methods. Theorem 6.10 generalizes both Theorem 6.7 and Theorem 6.8.

Notice that if the operator $D(t)|_{X^\infty}$ has no L^2 kernel for all $t \in [0, 1]$, and if the kernel of the tangential operator $S(t)$ has constant dimension along the path, then the limiting values of extended L^2 solutions $\tilde{L}_X(t)$ constitute a continuous path of Lagrangians. (The symplectic reduction is "clean" in the sense of [15].) This is because, for all t, the projection

$$\Lambda_X(t) \cap (\ker S(t) \oplus P^+(S(t))) \to \ker S(t)$$

(with image $\tilde{L}_X(t)$) has no kernel, and it is easily checked that the image is continuous because the path $\Lambda_X(t)$ is continuous.

Thus, the statement of the following theorem makes sense. This is Corollary 4.4 in [15].

THEOREM 6.8 (Yoshida, Nicolaescu). *Suppose that, for all parameters $t \in [0, 1]$, the operators $D(t)|_{X^\infty}$ and $D(t)|_{Y^\infty}$ have no L^2 kernel. Assume further that the kernel of the tangential operator* $\ker S(t)$ *has constant dimension* (*i.e., independent of $t \in [0, 1]$*)*. Assume that*

$$\tilde{L}_X(0) \cap \tilde{L}_Y(0) = 0 \tag{6.5}$$

and

$$\tilde{L}_X(1) \cap \tilde{L}_Y(1) = 0. \tag{6.6}$$

Then there exists an $r_0 \geq 0$ such that, for $r \geq r_0$,

$$\mathrm{SF}(D, M^r) = \mu(\tilde{L}_X, \tilde{L}_Y).$$

Proof. Since 0 is the nonresonance level, $\lim_{r\to\infty} \Lambda_X^r(t) = P^-(t) \oplus \tilde{L}_X(t)$ and $\lim_{r\to\infty} \Lambda_Y^r(t) = \tilde{L}_Y(t) \oplus P^+(t)$ for all $t \in [0, 1]$. Together with Equations 6.5

and 6.6, this implies that there exists an r_0 such that if $r \geq r_0$ then $\Lambda^r_X(i)$ is transverse to $\Lambda^r_Y(i)$ for $i = 0, 1$. Let r_1 be any number greater than or equal to r_0.

Consider the homotopy from the path of pairs $(\Lambda^{r_1}_X(t), \Lambda^{r_1}_Y(t))$ to $(P^-(t)\oplus\tilde{L}_X(t), \tilde{L}_Y(t) \oplus P^+(t))$ obtained by letting r go to infinity. This homotopy is not a rel endpoint homotopy, but it does exhibit a rel endpoint homotopy from the path of pairs $(\Lambda^{r_1}_X(t), \Lambda^{r_1}_Y(t))$ to the composite of three paths:

$$A(t) = \begin{cases} (\Lambda_X^{r_1/(1-t)}(0), \Lambda_Y^{r_1/(1-t)}(0)) & \text{if } t < 1, \\ (P^-(0) \oplus \tilde{L}_X(0), \tilde{L}_Y(0) \oplus P^+(0)) & \text{if } t = 1; \end{cases}$$

$$B(t) = (P^-(t) \oplus \tilde{L}_X(t), \tilde{L}_Y(t) \oplus P^+(t));$$

$$C(t) = \begin{cases} (P^-(1) \oplus \tilde{L}_X(1), \tilde{L}_Y(1) \oplus P^+(1)) & \text{if } t = 0, \\ (\Lambda_X^{r_1/t}(1), \Lambda_Y^{r_1/t}(1)) & \text{if } t > 0. \end{cases}$$

Therefore, $\mathrm{SF}(D, M^{r_1}) = \mu(A * B * C) = \mu(A) + \mu(B) + \mu(C)$.

Since $\Lambda^r_X(i)$ is transverse to $\Lambda^r_Y(i)$ for $i = 0, 1$ and all $r \geq r_1$, it follows that $\mu(A) = 0 = \mu(C)$. Thus,

$$\mathrm{SF}(D, M^{r_1}) = \mu(B) = \mu(\tilde{L}_X, \tilde{L}_Y). \qquad \square$$

The transversality assumptions in Theorem 6.8 can in some contexts be relaxed by requiring only assumptions similar to Hypothesis 2. Some care must be taken with the steps used to calculate $\mu(L_3, M_3)$ and $\mu(L_9, M_9)$.

Next we give a generalization of the two preceding theorems by assuming the existence of a continuously varying spectral gap.

DEFINITION 6.9. A continuous function $\lambda\colon [0, 1] \to [0, \infty)$ is a *spectral gap for the family of tangential operators* $S(t)$ if, for each $t \in [0, 1]$, $\lambda(t)$ is not in the spectrum of $S(t)$.

HYPOTHESIS 6. *The path of tangential operators $S(t)$ has a spectral gap $\lambda(t)$.*

As we remarked earlier, if Hypothesis 6 holds then the decomposition of Equation 2.7 varies continuously. Notice that, by subdividing the path as necessary, Hypothesis 6 can always be arranged to hold. However, this hypothesis by itself is not usually sufficient to simplify the formula of Theorem 5.1. The following theorem gives one possible clean statement that generalizes Theorems 6.7 and 6.8.

Assume that Hypothesis 6 holds. Let $A_X(t)$ and $A_Y(t)$ be continuously varying Lagrangian subspaces of $H_{\lambda(t)}$. Then we can take the self-adjoint boundary conditions to be

$$B_X(t) = A_X(t) \oplus P^+_{\lambda(t)}(t) \quad \text{and} \quad B_Y(t) = P^-_{\lambda(t)}(t) \oplus A_Y(t). \tag{6.7}$$

Then Theorem 5.1 states that

$$\mathrm{SF}(D) = \mathrm{SF}(D|_X, A_X \oplus P_\lambda^+) + \mathrm{SF}(D|_Y, P_\lambda^- \oplus A_Y) \\ + \mu(A_Y(1-t), A_X(1-t)) + \sum_{i \neq 3,6,9} \mu(L_i, M_i).$$

By adding hypotheses, we can make many of the extra terms vanish.

THEOREM 6.10. *Assume that Hypotheses 5 and 6 hold, with spectral gap $\lambda(t)$. Assume that the limiting values of extended L^2 solutions $\tilde{L}_X(i)$ and $\tilde{L}_Y(i)$ are transverse for $i = 0, 1$. Let $A_X(t)$ and $A_Y(t)$ be continuously varying Lagrangian subspaces of $H_{\lambda(t)}$, with $A_Y(0) = (P^-(0) \cap H_{\lambda(0)}) \oplus \tilde{L}_X(0)$ and $A_X(1) = \tilde{L}_Y(1) \oplus (P^+(1) \cap H_{\lambda(1)})$. Assume further that $A_X(0)$ is transverse to $A_Y(0)$ and that $A_X(1)$ is transverse to $A_Y(1)$.*

Then there exists an $r_0 \geq 0$ such that, for all $r \geq r_0$,

$$\mathrm{SF}(D, M^r) = \mathrm{SF}(D|_X, A_X \oplus P_\lambda^+) + \mathrm{SF}(D|_Y, P_\lambda^- \oplus A_Y) + \mu(A_X, A_Y). \quad (6.8)$$

Proof. Since Hypothesis 5 holds,

$$\lim_{r\to\infty} \Lambda_X^r(i) = \tilde{L}_X(i) \oplus P^-(i) \quad \text{and} \quad \lim_{r\to\infty} \Lambda_Y^r(i) = \tilde{L}_Y(i) \oplus P^+(i)$$

for $i = 0, 1$. Since we assumed that $\tilde{L}_X(i)$ is transverse to $\tilde{L}_Y(i)$ for $i = 0, 1$, Hypothesis 1 holds and so by Proposition 6.1 there exists an r_1 such that, after replacing M by M^r for $r \geq r_1$, $\mu(L_1, M_1)$ and $\mu(L_{11}, M_{11})$ vanish.

Take elliptic boundary conditions $B_X(t) = A_X(t) \oplus P^+_{\lambda(t)}(t)$ and $B_Y(t) = P^-_{\lambda(t)}(t) \oplus A_Y(t)$. Since $B_Y(0) = \tilde{L}_X(0) \oplus P^-(0)$ and $B_X(1) = \tilde{L}_Y(1) \oplus P^+(1)$, Hypothesis 3 holds; hence $\mu(L_i, M_i) = 0$ for $i = 2, 5, 7, 10$.

The path L_4 is the constant path at $B_Y(1) = P^-_{\lambda(1)}(1) \oplus A_Y(1)$, and M_4 is obtained by stretching $\Lambda_Y(1)$ to its adiabatic limit $\tilde{L}_Y(1) \oplus P^+(1)$. Since $A_Y(1)$ is transverse to $A_X(1) = \tilde{L}_Y(1) \oplus (P^+(1) \cap H_{\lambda(1)})$ by hypothesis, $\mu(L_4, M_4)$ vanishes—after perhaps replacing M by M^r for large enough r.

The path L_8 is the reverse of stretching $\Lambda_X(0)$ to its adiabatic limit $P^-(0) \oplus \tilde{L}_X(0)$, and M_8 is the constant path at $B_X(0) = A_X(0) \oplus P^+(0)$. By the same argument as in the preceding paragraph, $\mu(L_8, M_8)$ vanishes (after perhaps replacing M by M^r for large enough r).

Now $\mu(B_Y(1-t), B_X(1-t)) = \mu(A_Y(1-t), A_X(1-t))$. Since $A_X(i)$ is transverse to $A_Y(i)$ for $i = 0, 1$ by hypothesis, we have

$$\mu(A_Y(1-t), A_X(1-t)) = \mu(A_X(t), A_Y(t)).$$

Combining these computations proves the theorem. □

The following useful corollary is just the special case of the previous theorem when the path of tangential operators has a spectral gap $\lambda(t) = \varepsilon$ for ε small.

COROLLARY 6.11. *Assume that Hypotheses 1 and 5 hold, that the path of tangential operators has constant-dimensional kernel, and that $A_X(t)$ and $A_Y(t)$ are* *paths of Lagrangians in* $\ker S(t)$ *with* $\tilde{L}_X(i) = A_Y(i)$ *and* $\tilde{L}_Y(i) = A_X(i)$ *for* $i = 0, 1$. *Then, for r large enough,*

$$\mathrm{SF}(D, M^r) = \mathrm{SF}(D|_X, A_X \oplus P^+) + \mathrm{SF}(D|_Y, P^- \oplus A_Y) + \mu(A_X, A_Y).$$

Proof. Hypotheses 1 and 5 together imply that $\tilde{L}_X(i)$ is transverse to $\tilde{L}_Y(i)$ for $i = 0, 1$. Thus, the hypotheses of Theorem 6.10 hold with $\lambda(t) = \varepsilon$, where ε is smaller than the smallest nonzero eigenvalue of $S(t)$ for $t \in [0, 1]$. The corollary follows. □

We finish this section with a few comments about comparing Theorem 5.1 to Theorem C of [7]. Theorem C expresses the spectral flow as a sum of three terms; formally, it looks identical to the formula 6.8, but no transversality hypotheses are assumed in their theorem (although they do assume that preliminary stretching has been done and they restrict the boundary conditions at the endpoints). This might suggest that some of our $\mu(L_i, M_i)$ (in particular $\mu(L_1, M_1)$ and $\mu(L_{11}, M_{11})$) vanish *without* any of the transversality conditions. But this is not true (examples can be concocted). The reason that their formula has only three terms is that their definition of spectral flow differs from ours in the case when transversality hypotheses do not hold at the endpoint. In particular, in Theorem C the "exponentially small" eigenvalues at the endpoints of the path are treated as if they were zero.

To derive Theorem C of [7] from Theorem 5.1 would require a more careful analysis of the rate at which Λ_X^r converges to its adiabatic limit. An examination of Nicolaescu's proof shows that this rate is exponential. We speculate that, by replacing the definition of the Maslov index with the "$(1/r^2)$-Maslov index", one could derive Theorem C of [7] from our Theorem 5.1. The article [6] should be helpful for such a project.

6.5. Spectral Flow around Loops

One nice application of Theorem 5.1 is perhaps of more interest to index theorists than geometric topologists.

THEOREM 6.12. *Let* $D(t)$ *be a loop of cylindrical, neck-compatible Dirac operators on a manifold* $M = X \cup_\Sigma Y$, *and let* B_X, B_Y *be loops of self-adjoint elliptic boundary conditions for the restrictions of D to X and Y, respectively. Then*

$$\mathrm{SF}(D|_X; B_X) + \mathrm{SF}(D|_Y; B_Y) + \mu(B_X, B_Y) = 0. \tag{6.9}$$

Proof. This follows from the formula in Theorem 5.1 after much cancellation. First of all, the collection of all Dirac operators (on a fixed Clifford bundle) is an affine space and hence contractible. It follows that the spectral flow of a loop of Dirac operators on a closed manifold is 0. This is the 0 on the right-hand side of Equation 6.9. Next, one can compute that $\mu(B_Y(1 - t), B_X(1 - t)) = \mu(B_X(t), B_Y(t))$ if B_X and B_Y are loops.

It remains to show that the sum of all the other terms in Theorem 5.1 vanish. This is easy: the composite paths

$$Q_1 = L_1 * L_2 * L_4 * L_5 * L_7 * L_8 * L_{10} * L_{11}$$

and

$$Q_2 = M_1 * M_2 * M_4 * M_5 * M_7 * M_8 * M_{10} * M_{11}$$

are defined *since the path is a loop.* But it is immediate from the definitions of these paths that Q_1 is homotopic to the constant path at $\Lambda_X(0)$ and Q_2 is homotopic to the constant path at $\Lambda_Y(0)$. Thus,

$$\sum_{i \neq 3,6,9} \mu(L_i, M_i) = \mu(Q_1, Q_2) = 0. \qquad \square$$

Notice that there are no hypotheses on stretching, boundary conditions,… etc. in Theorem 6.12.

6.6. Applying the Method to the Spectral Flow on Manifolds with Boundary

We conclude the user's guide with a discussion on how to apply our method to compute the spectral flow of the path of operators on a manifold with boundary obtained by fixing the underlying Dirac operator but varying the boundary conditions.

For simplicity we consider only the case when the boundary conditions are of the special form $B_X(t) = A(t) \oplus P^+$ for $A_X(t) \subset \ker S$ a path of Lagrangians; more general situations can be handled by a similar method. This theorem is very similar to Theorem D of [7].

THEOREM 6.13. *Let D be a cylindrical, neck-compatible Dirac operator on a smooth manifold X with boundary Σ. Let $A(t) \subset \ker S$ be a path of Lagrangian subspaces of the kernel of the tangential operator, and let $B(t) = A(t) \oplus P^+$ be the corresponding path of elliptic boundary conditions. Let ν be in the nonresonance range and let $M(t)$ be the path starting at Λ_X and stretching to the adiabatic limit $P_\nu^- \oplus L_X$ (given in Lemma 3.2). Let $\tilde{L}_X \subset \ker S$ denote the limiting values of extended L^2 solutions.*

Then

$$\mathrm{SF}(D, B) = \mu(\tilde{L}_X, A(t)) + \mu(M(t), A(0) \oplus P^+) + \mu(M(1-t), A(1) \oplus P^+).$$

In particular, if $P_\nu^- \oplus L_X$ is transverse to $A(0) \oplus P^+$ (resp. transverse to $A(1) \oplus P^+$) then, after replacing X by X^r for r sufficiently large, $\mu(M(t), A(0) \oplus P^+) = 0$ (resp. $\mu(M(1-t), A(1) \oplus P^+) = 0$). Hence, if both transversality conditions hold,

$$\mathrm{SF}(D, B) = \mu(\tilde{L}_X, A(t)).$$

Proof. First, $\mathrm{SF}(D, B) = \mu(\Lambda_X, B)$ by Theorem 4.3. Apply the method as follows.

(1) Let $L_1(t) = M(t)$ and let $M_1(t)$ be the constant path at $A(0) \oplus P^+$. Then $\mu(L_1, M_1) = \mu(M(t), A(0) \oplus P^+)$.
(2) Let $L_2(t)$ be the path defined in Equation 6.4 and let M_2 be the constant path at $A(0) \oplus P^+$. Then $\mu(L_2, M_2) = 0$ by Lemma 6.6.
(3) Let L_3 be the constant path at $P^- \oplus \tilde{L}_X$ and let $M_3(t) = B(t) = A(t) \oplus P^+$. Then $\mu(L_3, M_3) = \mu(\tilde{L}_X, A(t))$.
(4) Let L_4 be L_2 run backward, and let M_4 be the constant path at $A(1) \oplus P^+$. Then $\mu(L_4, M_4) = 0$ by Lemma 6.6.
(5) Let L_5 be L_1 run backward, and let M_5 be the constant path at $A(1) \oplus P^+$. Then $\mu(L_5, M_5) = \mu(M(1-t), A(1) \oplus P^+)$.

Thus $L_1 * L_2 * L_3 * L_4 * L_5$ is defined and homotopic rel endpoints to the constant path at Λ_X. Also, $M_1 * M_2 * M_3 * M_4 * M_5$ is defined and homotopic rel endpoints to the path B. Applying the homotopy invariance and additivity of the Maslov index finishes the proof. □

7. Concluding Remarks

We finish with a few comments about Theorem 5.1. First, there is a certain asymmetry in the formula with respect to the roles that X and Y play. This turns out to be useful sometimes—for example, Hypothesis 3 only restricts B_Y at one endpoint and B_X at the other, rather than restricting both at each endpoint.

Another comment is that the sums $\mu(L_2, M_2) + \mu(L_7, M_7)$ and $\mu(L_5, M_5) + \mu(L_{10}, M_{10})$ (the terms depending on the auxiliary choice of the paths L_2 and M_5) depend only on the endpoints of these paths. Thus, each of these sums could be thought of as a single quantity and perhaps expressed in terms of invariants (such as the Maslov triple index) of the endpoints alone, without making any reference to the choice of L_2 and M_5. As we have seen, it is nevertheless convenient for calculation to have the formula expressed the way we did.

Last (but not least), one significant benefit of our formulation is that, since our formula expresses the spectral flow entirely as a sum of Maslov indices with the *ordered* pairs (L_i, M_i) explicitly described, it is easy to keep the signs and conventions under control when carrying out spectral flow calculations.

Appendix

Proof of Lemma 3.2, by K. P. Wojciechowski

The set-up is as follows. We are given a Dirac operator D on manifold X with boundary in cylindrical form $D = J(\partial/\partial u + S)$ on a collar $\Sigma \times [-1, 0]$ of the boundary $\Sigma = \Sigma \times \{0\}$. This extends to an operator on $X^r = X \cup \Sigma \times [0, r]$ in the obvious way. To this extension we associate the Cauchy data spaces Λ^r.

Nicolaescu's adiabatic limits theorem, Theorem 3.1, states that the path (with $r(t) = 1/(1-t)$)

$$t \mapsto \begin{cases} \Lambda^{r(t)} & t < 1, \\ P^-_{\nu_0} \oplus L_X(D) & t = 1, \end{cases}$$

is continuous at $t = 1$. What must be shown is that this path is continuous at finite neck lengths r—in other words, that the Cauchy data spaces Λ^r vary continuously in r. Continuity is measured in the gap topology or (equivalently) in the norm of the associated projections. For notational convenience we will prove continuity at $r = 0$; by reparameterizing, continuity at all r follows easily.

Let ν be a number in the nonresonance range for D on $X = X^0$. Thus $\Lambda^0 \cap P^+_{\nu'} = 0$ for all $\nu' \geq \nu$. We will make frequent use of the splitting $L^2(E|_\Sigma) = P^-_\nu \oplus H_\nu \oplus P^+_\nu$. Notice that the tangential operator S preserves this splitting, since the summands are defined by the eigenspace decomposition of S. The almost complex structure J of Equation 2.2 preserves H_ν and interchanges P^+_ν and P^-_ν.

We will often use the fact that, if α is a section of E on the cylinder $\Sigma \times [-1, r]$ that satisfies $D\alpha = 0$, then writing $\alpha|_{\Sigma\times\{u\}} = \alpha(u)$ for $u \in [-1, r]$ we have

$$\alpha(u) = e^{(t-u)S}\alpha(t).$$

Let $L = \mathrm{proj}_{H_\nu}(\Lambda^0 \cap (H_\nu \oplus P^+_\nu))$. Clearly, $JL \oplus P^+_\nu$ is transverse to Λ^0.

Lemma A.1. *For each $r \geq -1$, Λ^r is transverse to $(e^{-rS}JL) \oplus P^+_\nu$.*

Proof. Suppose that $v \in \Lambda^r \cap ((e^{-rs}JL) \oplus P^+_\nu)$. Then there exists α, a section of E on X^r, such that $D\alpha = 0$ and the restriction of α to $\Sigma \times \{r\}$ equals 0. Thus $\alpha(u) = e^{(r-u)S}v$ for $u \in [-1, r]$. But this formula defines an extension of $\alpha(u)$ for all $u \in [-1, \infty)$, since H_ν is finite-dimensional and since the restriction of $e^{(r-u)S}$ to P^+_ν decays exponentially as $u \to \infty$. Hence α extends to a bounded smooth section on X^u for all $u > -1$, and the extension satisfies $D\alpha = 0$. In particular, $\alpha(0)$ is defined, equals $e^{rS}v$, and lies in Λ^0. Since $v \in (e^{-rS}JL) \oplus P^+_\nu$, it follows that $\alpha(0) \in e^{(r-0)S}((e^{-rS}JL) \oplus P^+_\nu) = JL \oplus P^+_\nu$. By the choice of L, this implies that $\alpha(0) = 0$ and so also $v = 0$. $\square$

For convenience we introduce some notation for certain projections $L^2(E|_\Sigma)$.

(1) The *orthogonal Calderon projection* $\mathcal{P}^r : L^2(E|_\Sigma) \to L^2(E|_\Sigma)$ is the orthogonal projection to the Cauchy data space Λ^r.

(2) The *negative spectral projection* $\pi_- : L^2(E|_\Sigma) \to L^2(E|_\Sigma)$ is the orthogonal projection to the space $P^-(S)$, the negative eigenspan of the tangential operator S.

(3) Fix $\nu \geq 0$ and suppose that $L \subset H_\nu$ is a Lagrangian (thus, $P^-_\nu \oplus L$ is a Lagrangian in $L^2(E|_\Sigma)$). Define $\pi_{-,L} : L^2(E|_\Sigma) \to L^2(E|_\Sigma)$ to be the orthogonal projection to $P^-_\nu \oplus L$.

What must be shown is that the projections $\mathcal{P}^r$ are continuous in norm as r varies. It follows from the results in Chapters 12 and 14 of [3] that $\mathcal{P}^r$, π_-, and $\pi_{-,L}$ are pseudodifferential of order 0.

Let L and ν be as in Lemma A.1. For notational ease, define

$$M_r = e^{rS}L \subset H_\nu$$

and

$$\pi_r = \pi_{-,M_r} : L^2(E|_\Sigma) \to M_r \oplus P_\nu^-.$$

Notice that M_r varies continuously in r and hence so does π_r. The difference $\pi_- - \pi_r$ has image in H_ν, a finite-dimensional space of smooth sections, and hence is a smoothing operator. Corollary 14.3 of [3] shows that the pseudodifferential operators $\mathcal{P}^r$ and π_- have the same principal symbol. Putting these facts together shows that $\mathcal{P}^r - \pi_r$ is a pseudodifferential operator of order at most -1, and in particular is a compact operator.

LEMMA A.2. *The restriction of π_r to Λ^r induces an isomorphism*

$$\pi_r : \Lambda^r \to M_r \oplus P_\nu^-.$$

Proof. It is easy to observe that the operator

$$\pi_r : \Lambda^r \to M_r \oplus P_\nu^-$$

is a Fredholm operator (see [3]); hence, in particular, it has closed range. The kernel of this map is $\Lambda^r \cap (M_r \oplus P_\nu^-)^\perp = \Lambda^r \cap (JM_r \oplus P_\nu^+) = 0$. Since Λ^r and $M_r \oplus P_\nu^-$ are Lagrangians, the isometry J identifies the cokernel with the kernel of π_r and so the map is surjective. □

Since the map of Lemma A.2 is an isomorphism, the Cauchy data space Λ^r can be expressed as a graph of a bounded operator

$$k_r : M_r \oplus P_\nu^- \to JM_r \oplus P_\nu^+;$$

here k_r is the composite of the inverse of $\pi_r : \Lambda^r \to M_r \oplus P_\nu^-$ and the orthogonal projection to $JM_r \oplus P_\nu^+$. Hence,

$$\Lambda^r = \{\, (v, k_r(v)) \mid v \in M_r \oplus P_\nu^- \,\}. \tag{A.1}$$

Let $r > -1$, and choose $v_- \in M_r \oplus P_\nu^-$; hence $v = v_- + k_r(v_-) \in \Lambda^r$. Thus there exists a section α in $\ker D$ on X^r with $\alpha(r) = v$. As observed before, on the cylinder $\Sigma \times [-1, r]$, α has the form

$$\alpha(u) = e^{(r-u)S} v = e^{(r-u)S} v_- + e^{(r-u)S} k_r(v). \tag{A.2}$$

On the other hand, $\alpha(u) \in \Lambda^u$ for $u \le r$ and, taking $u = -1$, we have

$$\alpha(-1) = w_- + k_{-1}(w_-) \tag{A.3}$$

for some $w_- \in M_{-1} \oplus P_\nu^+$. Combining Equations A.2 and A.3 yields

$$w_- = e^{(r+1)S} v_- \quad \text{and} \quad e^{-(r+1)S} k_{-1}(w_-) = k_r(v_-).$$

Therefore,

$$k_r = e^{-(r+1)S_+} k_{-1} e^{(r+1)S_-}, \tag{A.4}$$

where we have denoted the restriction of S to $H_\nu \oplus P_\nu^\pm$ by $S_\pm$ for clarity.

For the next lemma, we recall the standard fact that the operators e^{tS_-}: $H_\nu \oplus P_\nu^- \to H_\nu \oplus P_\nu^-$ and $e^{-tS_+} : H_\nu \oplus P_\nu^+ \to H_\nu \oplus P_\nu^+$ are norm-continuous in t for t away from 0.

LEMMA A.3.

$$\lim_{r\to 0}\|k_r - k_0\| = 0.$$

Proof. Using Equation A.4, for $-\frac{1}{2} \le r \le \frac{1}{2}$ we compute

$$\begin{aligned}
\|k_r - k_0\| &= \|e^{-(r+1)S_+}k_{-1}e^{(r+1)S_-} - e^{-S_+}k_{-1}e^{S_-}\| \\
&\le \|(e^{-(r+1)S_+}k_{-1}e^{(r+1)S_-} - e^{-(r+1)S_+}k_{-1}e^{(r+1)S_-}\| \\
&\quad + \|(e^{-(r+1)S_+}k_{-1}e^{S_-} - e^{-S_+}k_{-1}e^{S_-}\| \\
&\le \|e^{-(r+1)S_+}\|\|k_{-1}(e^{(r+1)S_-} - e^{S_-})\| + \|(e^{-(r+1)S_+} - e^{-S_+})k_{-1}\|\|e^{S_-}\| \\
&\le C_1\|k_{-1}(e^{(r+1)S_-} - e^{S_-})\| + C_2\|(e^{-(r+1)S_+} - e^{-S_+})k_{-1}\|.
\end{aligned}$$

The last inequality follows from the continuity of $e^{(r+1)S_+}$ in norm for r near 0 and the fact that e^{S_-} is independent of r. Continuing the estimate (using that k_{-1} is bounded), we obtain

$$\|k_r - k_0\| \le \|k_{-1}\|(C_1\|e^{(r+1)S_-} - e^{S_-}\| + C_2\|e^{-(r+1)S_+} - e^{-S_+}\|). \tag{A.5}$$

The right-hand side approaches 0 as $r \to 0$ since e^{tS_-} and e^{-tS_+} are continuous in norm at $t = 1$. This proves the lemma. □

In the decomposition

$$L^2(E|_\Sigma) = (M_r \oplus P_\nu^-) \oplus (JM_r \oplus P_\nu^+),$$

the matrix

$$Q_r = \begin{pmatrix} 1 & 0 \\ k_r & 0 \end{pmatrix}$$

is a (nonorthogonal) projection to Λ^r. The formula of Lemma 12.8 in [3] shows that

$$\begin{aligned}
\mathcal{P}^r &= Q_r Q_r^*(Q_r Q_r^* + (\mathrm{Id} - Q_r^*)(\mathrm{Id} - Q_r))^{-1} \\
&= \begin{pmatrix} (\mathrm{Id} + k_r^* k_r)^{-1} & (\mathrm{Id} + k_r^* k_r)^{-1}k_r^* \\ k_r(\mathrm{Id} + k_r^* k_r)^{-1} & k_r(\mathrm{Id} + k_r^* k_r)^{-1}k_r^* \end{pmatrix}.
\end{aligned} \tag{A.6}$$

Thus, the $\mathcal{P}^r$ are continuous in r for r near 0, completing the proof of Lemma 3.2. □

We note that the proof of Lemma A.3 can can be modified to show that

$$\lim_{r\to\infty}\Lambda^r = \Big(\lim_{r\to\infty} M_r\Big) \oplus P_\nu^- = \Big(\lim_{r\to\infty} e^{rS}L\Big) \oplus P_\nu^-.$$

In contrast to the proof just given of continuity at finite r, in this case one must be careful with the estimates over the finite-dimensional piece H_ν. The argument is straightforward and is essentially the proof given in Nicolaescu [15].

Note also that in fact we have proved the following important result.

THEOREM A.4. *The difference $\mathcal{P}^r - \pi_-$ is an operator with a smooth kernel.*

Proof. The difference $\pi_r - \pi_-$ is a smoothing operator. Using Equation A.6, the difference $\mathcal{P}^r - \pi_r$ can be represented as

$$\mathcal{P}^r - \pi_r = \begin{pmatrix} (\mathrm{Id} + k_r^* k_r)^{-1} - \mathrm{Id} & (\mathrm{Id} + k_r^* k_r)^{-1} k_r^* \\ k_r (\mathrm{Id} + k_r^* k_r)^{-1} & k_r (\mathrm{Id} + k_r^* k_r)^{-1} k_r^* \end{pmatrix}.$$

All entries in this formula are smoothing operators because k_r has a smooth kernel. □

Scott [17] proved this theorem in the nonresonant case. The proof given here basically extends his proof to cover the general case. A different proof, purely analytical, was offered by Grubb (see [10]).

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