Weak Singularity Spectra of the Patterson Measure for Geometrically Finite Kleinian Groups with Parabolic Elements

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1. Introduction and Statement of Results

In this paper we give a multifractal description of the Patterson measure μ supported on the limit set $L(G)$ of a geometrically finite Kleinian group G with parabolic elements. More precisely, we estimate the *weak singularity spectra* of μ, which means that for $\theta > 0$ we determine the Hausdorff dimensions of the following sets:

$$\mathcal{I}^{\theta}(\mu) := \left\{ \xi \in L(G) : \liminf_{r \to 0} \frac{\log \mu(B(\xi, r))}{\log r} \leq \theta \right\},$$

$$\mathcal{I}_{\theta}(\mu) := \left\{ \xi \in L(G) : \liminf_{r \to 0} \frac{\log \mu(B(\xi, r))}{\log r} \geq \theta \right\},$$

$$\mathcal{S}^{\theta}(\mu) := \left\{ \xi \in L(G) : \limsup_{r \to 0} \frac{\log \mu(B(\xi, r))}{\log r} \leq \theta \right\},$$

$$\mathcal{S}_{\theta}(\mu) := \left\{ \xi \in L(G) : \limsup_{r \to 0} \frac{\log \mu(B(\xi, r))}{\log r} \geq \theta \right\},$$

where $B(\xi, r)$ denotes the Euclidean ball of radius r centered at ξ.

This "weak multifractal analysis" of the Patterson measure will be based on a further investigation of the Hausdorff dimension $\dim_H(\mathcal{J}_\sigma(G))$ of the associated *σ-Jarník limit sets* $\mathcal{J}_\sigma(G) \subset L(G)$, which represent the natural generalization of the well-approximable real numbers to the theory of Kleinian groups ($\mathcal{J}_\sigma(G)$ is defined at the end of this section).

In [12] we derived a complete description of $\mathcal{J}_\sigma(G)$ in terms of the dimension with respect to μ. As a consequence, we were able to determine $\dim_H(\mathcal{J}_\sigma(G))$ for those cases in which $\dim_H(L(G))$ does not exceed the maximal rank of the parabolic fixed points of G. The first aim of this paper will be to show how to modify the construction in [12] in order to deal with the remaining cases. That is, based on the construction in [12], we compute $\dim_H(\mathcal{J}_\sigma(G))$ for *all* geometrically finite Kleinian groups with parabolic elements. We then discuss how these estimates

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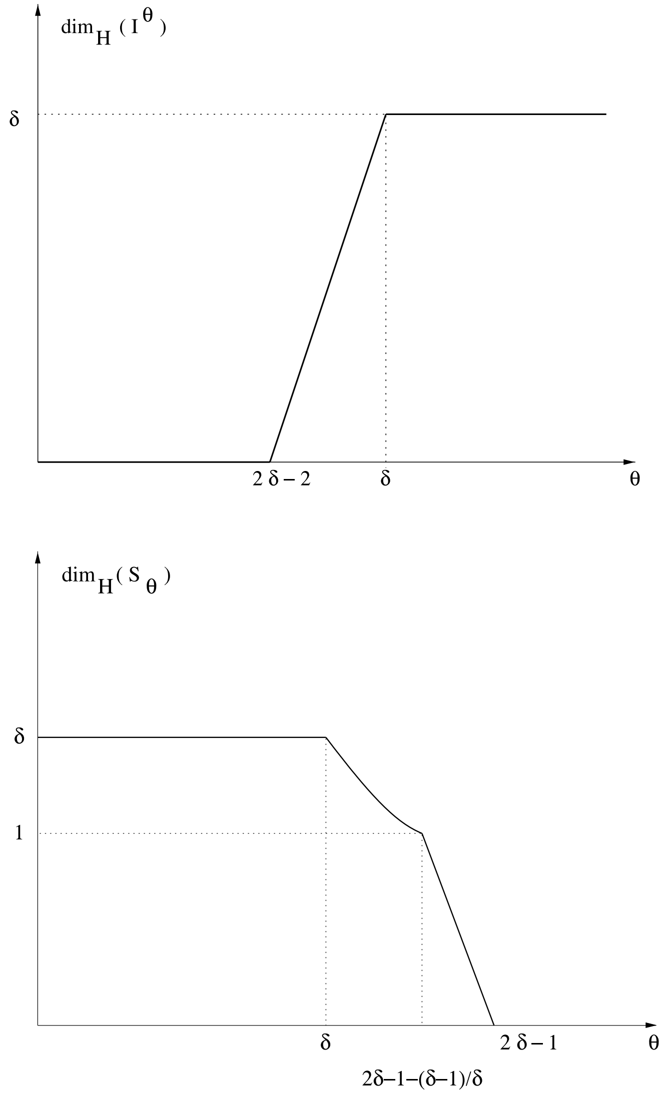

Figure 1 The interesting spectra in case (C)

give access to the aforementioned multifractal aspects of the Patterson measure. The results obtained clearly suggest that the most natural way to view the derived Hausdorff dimension of $\mathcal{J}_\sigma(G)$ is exactly this interpretation in terms of the theory of multifractals.

In order to state these main results more explicitly, we recall that $\dim_H(L(G))$ has been proven to be equal to the *exponent of convergence* $\delta = \delta(G)$ of G, which is given by the exponent of convergence of the Dirichlet series

$$\sum_{g\in G} \exp(-sd(0, g0)),$$

where d denotes the hyperbolic distance in the Poincaré ball $\mathbb{D}^3$ (note that, for simplicity, we restrict the discussion to hyperbolic 3-space) . In fact, a combination of the global measure formula (see Section 2) and the Khintchine law for geodesics (cf. [17]) immediately shows that, for μ-almost all $\xi \in L(G)$, we have

$$\delta = \liminf_{r\to 0} \frac{\log \mu(B(\xi, r))}{\log r} = \limsup_{r\to 0} \frac{\log \mu(B(\xi, r))}{\log r}.$$

Also, recall that a parabolic element of a Kleinian group acting on $\mathbb{D}^3$ may be either of rank 1 or of rank 2, depending on whether the stabilizer of the associated parabolic fixed point is isomorphic to a finite extension of $\mathbb{Z}$ or $\mathbb{Z}^2$. If $k_{\min}$ and $k_{\max}$ denote the minimal and maximal possible ranks among the parabolic elements occurring in G, then it is well known [1] that $\delta > k_{\max}/2$. Combining this fact with the global measure formula for μ then shows that the investigation of the weak singularity spectra of μ has to be split into the following five cases:

(A) $k_{\max} = 1$ and $\delta < 1$;
(B) $k_{\max} = 1$ and $\delta > 1$;
(C) $1 = k_{\min} < k_{\max} = 2$;
(D) $k_{\min} = k_{\max} = 2$;
(E) $k_{\max} = 1$ and $\delta = 1$.

Note that (A) includes as a special case all finitely generated "second-kind" Fuchsian groups with parabolic elements. Also, we shall see that, from the multifractal point of view, (B) and (C) provide the most interesting cases (see also Figure 1).

The following theorem is the main result in this paper.

THEOREM 1. *Let G be a geometrically finite Kleinian group with parabolic elements and with exponent of convergence δ. The weak singularity spectra of the associated Patterson measure μ are then determined as follows.*

(A) *If $k_{\max} = 1$ and $\delta < 1$, then*

$$\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & \text{for } 0 < \theta \le 2\delta - 1, \\ \frac{\delta}{1-\delta}\theta - \frac{\delta(2\delta-1)}{1-\delta} & \text{for } 2\delta - 1 < \theta \le \delta, \\ \delta & \text{for } \theta > \delta; \end{cases}$$

$$\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & \text{for } 0 < \theta \le \delta, \\ 0 & \text{for } \theta > \delta. \end{cases}$$

(B) *If $k_{\max} = 1$ and $\delta > 1$, then*

$$\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & \text{for } 0 < \theta < \delta, \\ \delta & \text{for } \theta \ge \delta; \end{cases}$$

$$\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & \textit{for } 0 < \theta \le \delta, \\ (1-\delta) - \frac{(1-\delta)(2\delta-1)}{\theta-1} & \textit{for } \delta \le \theta \le (2\delta-1) - \frac{\delta-1}{\delta}, \\ \frac{\delta}{1-\delta}\theta - \frac{\delta(2\delta-1)}{1-\delta} & \textit{for } (2\delta-1) - \frac{\delta-1}{\delta} \le \theta \le 2\delta-1, \\ 0 & \textit{for } 2\delta - 1 \le \theta. \end{cases}$$

(C) *If* $1 = k_{\min} < k_{\max} = 2$, *then*

$$\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & \textit{for } 0 < \theta \le 2\delta-2, \\ \frac{\delta}{2-\delta}\theta - \frac{\delta(2\delta-2)}{2-\delta} & \textit{for } 2\delta-2 < \theta \le \delta, \\ \delta & \textit{for } \theta > \delta; \end{cases}$$

$$\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & \textit{for } 0 < \theta \le \delta, \\ (1-\delta) - \frac{(1-\delta)(2\delta-1)}{\theta-1} & \textit{for } \delta \le \theta \le (2\delta-1) - \frac{\delta-1}{\delta}, \\ \frac{\delta}{1-\delta}\theta - \frac{\delta(2\delta-1)}{1-\delta} & \textit{for } (2\delta-1) - \frac{\delta-1}{\delta} \le \theta \le 2\delta-1, \\ 0 & \textit{for } 2\delta - 1 \le \theta. \end{cases}$$

(D) *If* $k_{\min} = k_{\max} = 2$, *then*

$$\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & \textit{for } 0 < \theta \le 2\delta-2, \\ \frac{\delta}{2-\delta}\theta - \frac{\delta(2\delta-2)}{2-\delta} & \textit{for } 2\delta-2 < \theta \le \delta, \\ \delta & \textit{for } \theta > \delta; \end{cases}$$

$$\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & \textit{for } 0 < \theta \le \delta, \\ 0 & \textit{for } \theta > \delta. \end{cases}$$

Furthermore, in each of the cases (A)–(D) *we have that*

$$\dim_H(\mathcal{I}_\theta(\mu)) = \begin{cases} \delta & \textit{for } 0 < \theta \le \delta, \\ 0 & \textit{for } \theta > \delta; \end{cases}$$

$$\dim_H(\mathcal{S}^\theta(\mu)) = \begin{cases} 0 & \textit{for } 0 < \theta < \delta, \\ \delta & \textit{for } \theta \ge \delta. \end{cases}$$

(E) *If* $k_{\max} = 1$ *and* $\delta = 1$, *then the weak singularity spectra of* μ *are trivial. In this case, for all* $\xi \in L(G)$ *we have that*

$$\lim_{r \to 0} \frac{\log \mu(B(\xi, r))}{\log r} = \delta.$$

Note that, in (B) and (C), the lower lim sup spectrum exhibits a *phase transition* at $\theta = \theta^*$, where $\theta^* := 2\delta - 1 - (\delta - 1)/\delta$. The significance of this value is that $\dim_H(\mathcal{S}_{\theta^*}(\mu)) = 1$ (see Figure 1).

As we have already mentioned, one of the main ingredients in the proof of this theorem will be the explicit calculation of the Hausdorff dimension of the σ-Jarník limit sets $\mathcal{J}_\sigma(G)$. In order to state this result more precisely, recall from [12] the actual definition of $\mathcal{J}_\sigma(G)$. For $t > 0$ let ξ_t denote the unique point on the ray between the origin $0 \in \mathbb{D}^3$ and $\xi \in L(G)$ whose hyperbolic distance from 0 is equal to t. Let Δ denote the *ray excursion function*, which is given by $\Delta(\xi_t) := d(\xi_t, G(0))$. For $\sigma > 0$, the *σ-Jarník limit set* $\mathcal{J}_\sigma(G)$ is defined by

$$\mathcal{J}_\sigma(G) := \left\{ \xi \in L(G) : \limsup_{t\to\infty} \frac{\Delta(\xi_t)}{t} \geq \frac{\sigma}{1+\sigma} \right\}$$

(see Section 2 for an alternative definition). We obtain the following theorem.

THEOREM 2. *Let G be a geometrically finite Kleinian group with parabolic elements and with limit set of Hausdorff dimension δ. For $\sigma > 0$, the Hausdorff dimension of the σ-Jarník limit set $\mathcal{J}_\sigma(G)$ is determined as follows.*

(1) *For $\delta \leq k_{\max}$, we have* $\dim_H(\mathcal{J}_\sigma(G)) = \frac{\delta}{1+\sigma}$.
(2) *For $\delta > k_{\max}$,*

$$\dim_H(\mathcal{J}_\sigma(G)) = \begin{cases} \frac{\delta}{1+\sigma} & \textit{for } \frac{\delta}{1+\sigma} \leq 1, \\ \frac{\delta+\sigma}{1+2\sigma} & \textit{for } \frac{\delta}{1+\sigma} \geq 1. \end{cases}$$

REMARKS. (1) Our estimates of the weak singularity spectra of μ, combined with the fact that μ has a flat Rényi dimension spectrum equal to the exponent of convergence δ (cf. [14]), reveal that μ can not be analyzed by means of the usual multifractal theory, whose goal it is to relate the dimension spectrum and the Rényi spectrum by means of a Legendre transform. Furthermore, a complete multifractal analysis for those cases in which the Patterson measure is equivalent to the 1-dimensional Lebesgue measure has not yet been fully developed. Such an analysis relates closely to the multifractal analysis of the continued fraction map, and here we refer to [11] for some very interesting preliminary results.

(2) Recall that Jarník [6] (and later also Besicovitch [2]) showed that the Hausdorff dimension of the set of *well-approximable irrational numbers* is given by

$$\dim_H\left(\left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < (q^{-2})^{1+\sigma} \text{ for infinitely many reduced } \frac{p}{q} \right\}\right) = \frac{1}{1+\sigma}.$$

Some first generalizations of this result to certain special cases of geometrically finite Kleinian groups with parabolic elements were derived in [7; 12]. Now, as explained previously, in this paper we also deal with the remaining cases and hence complete this generalization to geometrically finite Kleinian groups with parabolic elements. We mention that, for Kleinian groups of this type, the Hausdorff dimension of the set of well-approximable limit points has been computed also in [5] and that, for finitely generated second-kind Fuchsian groups, the results concerning the weak singularity spectra of the Patterson measure have been stated in [14].

(3) After submitting this paper for publication, we completed work on analogous results for the entropy-dimensional conformal measure on Julia sets of parabolic rational maps ([15]; see also [16]). In particular, combining the work in this paper with the results in [15], we obtain a new chapter in the "Julia–Klein dictionary" initiated by Sullivan in [20].

2. Preliminaries

In this section we recall a few well-known results that are fundamental for what follows. Let G denote a geometrically finite Kleinian group with parabolic elements. For simplicity we assume that G acts on hyperbolic 3-space $\mathbb{D}^3$.

THE JARNÍK SET AS A LIM SUP SET. Let P be a complete set of inequivalent parabolic fixed points of G. For $p \in P$, let $\mathcal{T}_p \subset G$ denote a set of coset representatives of G_p chosen such that $|g(0)| \le |h(0)|$ for each $g \in \mathcal{T}_p$ and for all $h \in G_{g(p)} := gG_pg^{-1}$. Also, for $p_0 \in P$ such that the rank $k(p_0)$ of p_0 is equal to $k_{\max}$, let $\mathcal{T}_{\max} := \mathcal{T}_{p_0}$ and define $\mathcal{T} := \bigcup_{p\in P} \mathcal{T}_p$. It is well known [17] that to each $g \in \mathcal{T}_p$ we may associate a *standard horoball* $H_{g(p)}(r_g) \subset \mathbb{D}^3$ at $g(p)$ with Euclidean radius $r_g \asymp (1 - |g(0)|)$, so that the set $\{ H_{g(p)}(r_g) : g \in \mathcal{T}_p,\ p \in P \}$ comprises a G-equivariant set of pairwise disjoint horoballs. If $\mathcal{H}_\xi(r)$ denotes the radial projection of a horoball $H_\xi(r)$ at ξ of radius r projected from the origin in $\mathbb{D}^3$ onto the boundary S^2 of hyperbolic 3-space, then we can express $\mathcal{J}_\sigma(G)$ in terms of the set of standard horoballs as follows. Let $\mathcal{F}$ denote the set of functions $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to 0} \log\phi(x)/\log x = 0$. Using basic hyperbolic geometry, we see that $\xi \in \mathcal{J}_\sigma(G)$ if and only if $\xi \in \bigcup_{\phi\in\mathcal{F}} \limsup\{ \mathcal{H}_{g(p)}(\phi(r_g) r_g^{1+\sigma}) : g \in \mathcal{T}_p,\ p \in P \}$; that is, we have

$$\mathcal{J}_\sigma(G) = \bigcup_{\phi\in\mathcal{F}} \bigcap_{n\in\mathbb{N}} \bigcup_{p\in P} \bigcup_{\substack{g\in\mathcal{T}_p \\ r_g \le 1/n}} \mathcal{H}_{g(p)}(\phi(r_g) r_g^{1+\sigma}).$$

THE PATTERSON MEASURE. For the construction and basic properties of the Patterson measure μ we refer to [8; 9; 10; 18]. Recall that μ is a nonatomic probability measure supported on the limit set $L(G)$, for which we derived in [17] (see also [19]) a formula concerning its scaling properties. This formula describes in a uniform way the decay of μ around arbitrary limit points. In order to restate this formula, we require the following notation. For $\xi \in L(G)$ and positive t define $k(\xi_t)$ to be equal to $k(p)$ if $\xi_t \in H_{g(p)}(r_g)$ for some $p \in P$ and $g \in \mathcal{T}_p$, and let $k(\xi_t)$ be equal to δ otherwise.

THE GLOBAL MEASURE FORMULA. For $\xi \in L(G)$ and positive t, we have

$$\mu(B(\xi, e^{-t})) \asymp e^{-t\delta} e^{-\Delta(\xi_t)(\delta - k(\xi_t))}.$$

3. Hausdorff Dimensions of Jarník Limit Sets

In this section we give the proof of Theorem 2; that is, for arbitrary geometrically finite Kleinian groups with parabolic elements we calculate the Hausdorff dimension of the Jarník limit set $\mathcal{J}_\sigma(G)$. As we mentioned in Section 1, part (1) of Theorem 2 has already been dealt with in [12], and we refer to that paper for the proof in this case. Hence, we now concentrate on the proof of part (2) of Theorem 2 and so assume that $\delta > k_{\max}$ or (equivalently) that $k_{\max} = 1$ and $\delta > 1$.

Our arguments are based on certain parts of the construction in [12]. In particular, we shall recall two lemmata derived in [12] and restate a construction of a probability measure supported on a Cantor-like subset contained in $\mathcal{J}_\sigma(G)$.

For simplicity, we adapt the following notation from [12]. For $n \in \mathbb{N}$, $\tau > 0$, and $g \in \mathcal{T}_{\max}$, define

$$A_n(\tau) := \{ h \in \mathcal{T}_{\max} : \tau^{n+1} \leq r_h < \tau^n \},$$

$$Q_n(g, \sigma, \tau) := \{ h \in A_n(\tau) : \mathcal{H}_{h(p)}(r_h) \subset \mathcal{H}_{g(p)}(r_g^{1+\sigma}) \}.$$

The following result was obtained as Proposition 1 in [12].

LEMMA 1. *There exist $\rho, k_0, k_1, k_2 > 0$ and an increasing function $\iota\colon \mathbb{N} \to \mathbb{R}^+$ such that, if $g \in A_n(\rho)$ for some $n > k_0$, then for $m > \iota(n)$ we have that*

$$k_1 \rho^{\delta(n-m)+\sigma n(2\delta - k_{\max})} \leq \operatorname{card}(Q_m(g, \sigma, \rho)) \leq k_2 \rho^{\delta(n-m)+\sigma n(2\delta - k_{\max})}.$$

For the rest of this section let us fix the number ρ derived in this lemma.

Following the construction in [12], let $\{n_k\}_{k\in\mathbb{N}}$ denote a sequence of positive integers such that $n_0 > \max\{k_0; 2\sigma^{-1}\}$, $n_k > \iota(n_{k-1})$ for all k, and

$$\lim_{m\to\infty} \left(\frac{1}{n_m} \sum_{j=0}^{m-1} n_j \right) = 0.$$

For $k \in \mathbb{N}$, let

$$N_k := \min_{g \in A_{n_{k-1}}(\rho)} \operatorname{card} Q_{n_k}(g, \sigma, \rho).$$

Further, simply by deleting certain elements if necessary, we reduce for an arbitrary $h \in A_{n_{k-1}}(\rho)$ the number of elements in $Q_{n_k}(h, \sigma, \rho)$ to N_k. The resulting set is denoted by $\tilde{Q}_{n_k}(h)$. Hence, by definition, for all $k \in \mathbb{N}$ and $h \in A_{n_{k-1}}(\rho)$ we have $N_k = \operatorname{card} \tilde{Q}_{n_k}(h)$. For a fixed $g_0 \in A_{n_0}(\rho)$, we have the following definitions.

(1) $I_0^\sigma := \{\mathcal{H}_{g_0(p_0)}(r_{g_0}^{1+\sigma})\}$.
(2) If I_{k-1}^σ is defined for k in $\mathbb{N}$, then let

$$I_k^\sigma := \{ \mathcal{H}_{h(p_0)}(r_h^{1+\sigma}) : h \in \tilde{Q}_{n_k}(g) \text{ for some } g \in A_{n_{k-1}}(\rho) \text{ such that } \mathcal{H}_{g(p_0)}(r_g^{1+\sigma}) \in I_{k-1}^\sigma \}.$$

By construction, each element in I_{k-1}^σ contains exactly N_k elements of I_k^σ. This allows us to define the Cantor-like set $I_\sigma := \bigcap_{k\geq 0} \bigcup_{I \in I_k^\sigma} I$. Next, in order to construct a probability measure on I_σ, renormalize the Patterson measure on every I_k^σ. That is, for each $k \in \mathbb{N}$, define a probability measure $\nu_\sigma^{(k)}$ on I_k^σ such that, for Borel sets $E \subset S^2$, we have

$$\nu_\sigma^{(k)}(E) = \sum_{I \in I_k^\sigma} \frac{1}{N_1 \cdots N_k} \frac{\mu(E \cap I)}{\mu(I)}.$$

Using Helly's theorem, we obtain a probability measure ν_σ on I_σ as the weak limit of the sequence of measures $\{\nu_\sigma^{(k)}\}_{k\in\mathbb{N}}$. Note that $\nu_\sigma^{(k)}(I) = \nu_\sigma(I)$ for each $k \in \mathbb{N}$ and $I \in I_k^\sigma$.

Now, recall the following lemma obtained as Lemma 6 in [12].

LEMMA 2. *There exists a* $k_3 > 0$ *such that, if* $\xi \in I_\sigma$ *and* r *has the property* $\rho^{n_k} \leq r < \rho^{n_{k-1}}$ *for some* $k \in \mathbb{N}$, *then* $B(\xi, r)$ *intersects at most one element in* I_{k-1}^σ *and*

$$\operatorname{card}\{ I \in I_k^\sigma : I \cap B(\xi, r) \neq \emptyset \} \leq k_3 \rho^{-\delta n_k} \mu(B(\xi, r)).$$

The following lemma marks the difference between the construction here and the construction in [12]. The reader is asked to compare this result with Lemma 3 in [12].

LEMMA 3. *There exists a* $k_4 > 0$ *such that, for each* $\varepsilon > 0$, *there exists a* $r_0(\varepsilon) > 0$ *with the property that, for all* $\xi \in I_\sigma$ *and for all* $0 < r < r_0(\varepsilon)$ *such that* $\rho^{n_k} \leq r < \rho^{n_{k-1}}$ *for some* $k \in \mathbb{N}$, *we have*

$$\nu_\sigma(B(\xi, r)) \leq k_4 \mu(B(\xi, r)) \rho^{-n_{k-1}(\sigma(2\delta - k_{\max}) + \varepsilon)}.$$

Proof. Let ξ and r be given as stated in the lemma. By construction of the measure ν_σ and using Lemma 2, we have that

$$\begin{aligned} \nu_\sigma(B(\xi, r)) &\leq \prod_{j=0}^{k} N_j^{-1} \operatorname{card}\{ I \in I_k^\sigma : I \cap B(\xi, r) \neq \emptyset \} \\ &\leq k_3 \rho^{-n_k \delta} \mu(B(\xi, r)) \prod_{j=0}^{k} N_j^{-1}. \end{aligned}$$

Using Lemma 1, it follows that

$$\begin{aligned} \nu_\sigma(B(\xi, r)) &\leq k_3 k_1^{-1} \rho^{-n_k \delta} \mu(B(\xi, r)) k_1^{-(k-1)} \rho^{\delta(n_k - n_{k-1})} \rho^{-\sigma n_{k-1}(2\delta - k_{\max})} \\ &\quad \cdot \rho^{\delta(n_{k-1} - n_0)} \rho^{-\sigma(2\delta - k_{\max}) \sum_{j=0}^{k-2} n_j} \\ &= k_3 k_1^{-1} \mu(B(\xi, r)) \\ &\quad \cdot \rho^{n_{k-1}(-\sigma(2\delta - k_{\max}) - n_{k-1}^{-1}(n_0 + \sigma(2\delta - k_{\max}) \sum_{j=0}^{k-2} n_j + (k-1)(\log k_1)(\log \rho)^{-1}))}. \end{aligned}$$

By construction of the sequence (n_k), for each $\varepsilon > 0$ and for sufficiently large k we have that

$$n_0 + \sigma(2\delta - k_{\max}) \sum_{j=0}^{k-2} (n_j + (k-1)(\log k_1)(\log \rho)^{-1}) < \varepsilon n_{k-1}.$$

Using this fact in the foregoing estimate, the statement of the lemma follows. $\square$

PROPOSITION 1. *For each* $\varepsilon > 0$ *there exists* $r_1(\varepsilon) > 0$ *such that, for each* $\xi \in I_\sigma$ *and* $0 < r < r_1(\varepsilon)$, *we have*

$$\nu_\sigma(B(\xi, r)) \ll \begin{cases} r^{\delta/(1+\sigma) - \varepsilon} & \text{for } \sigma \geq \delta - 1, \\ r^{(\delta+\sigma)/(1+2\sigma) - \varepsilon} & \text{for } \sigma \leq \delta - 1. \end{cases}$$

Proof. Let $\xi \in I_\sigma$ be given. Without loss of generality, assume that $\rho^{n_k} \leq r < \rho^{n_{k-1}}$ and that $\xi \in \mathcal{H}_{g(p)}(r_g^{1+\sigma})$ for some $g \in \mathcal{T}_p$ with $r_g \asymp \rho^{n_{k-1}}$. We then have the following.

(1) “Before the visit to $H_{g(p)}(r_g^{1+\sigma})$”:
If $(\rho^{n_{k-1}})^{1+\sigma} \le r < \rho^{n_{k-1}}$, then $\nu_\sigma(B(\xi, r)) \ll r^{\delta/(1+\sigma)-\varepsilon/(1+\sigma)}$.

Proof: Since by construction $\nu_\sigma(\mathcal{H}_{g(p)}(r_g)) \asymp \nu_\sigma(\mathcal{H}_{g(p)}(r_g^{1+\sigma}))$, it follows that

$$\begin{aligned}\nu_\sigma(B(\xi, r)) &\ll \nu_\sigma(\mathcal{H}_{g(p)}(r_g^{1+\sigma}))\\ &\ll \mu(\mathcal{H}_{g(p)}(r_g^{1+\sigma}))\rho^{-n_{k-1}(\sigma(2\delta-1)+\varepsilon)}\\ &\asymp r_g^\delta r_g^{\sigma(2\delta-1)}\rho^{-n_{k-1}(\sigma(2\delta-1)+\varepsilon)}\\ &\asymp r_g^\delta r_g^{-\varepsilon}\\ &\asymp (r_g^{1+\sigma})^{\delta/(1+\sigma)} r_g^{-\varepsilon}\\ &\ll r^{\delta/(1+\sigma)} r^{-\varepsilon/(1+\sigma)}.\end{aligned}$$ ■

(2) “After having met $H_{g(p)}(r_g^{1+\sigma})$ and while still visiting $H_{g(p)}(r_g)$”:
If $(\rho^{n_{k-1}})^{1+2\sigma} \le r < (\rho^{n_{k-1}})^{1+\sigma}$, then

$$\nu_\sigma(B(\xi, r)) \ll \begin{cases} r^{\delta/(1+\sigma)} r^{-\varepsilon/(1+\sigma)} & \text{for } \sigma \ge \delta - 1,\\ r^{(\delta+\sigma)/(1+2\sigma)} r^{-\varepsilon/(1+\sigma)} & \text{for } \sigma \le \delta - 1.\end{cases}$$

Proof: Let $r = \rho^{n_{k-1}(1+\sigma+\tau)}$ for some $0 < \tau \le \sigma$. In this case, the fluctuation of the Patterson measure is maximal for $\Delta(\xi_t) = -(\sigma - \tau)n_{k-1}\log\rho$ (note that $r \asymp e^{-t}$, where $t = -n_{k-1}(1 + \sigma + \tau)\log\rho$). This gives that

$$\begin{aligned}\nu_\sigma(B(\xi, r)) &\ll \mu(B(\xi, r))\rho^{-n_{k-1}(\sigma(2\delta-1)+\varepsilon)}\\ &\ll r^\delta \rho^{n_{k-1}(\sigma-\tau)(\delta-1)}\rho^{-n_{k-1}(\sigma(2\delta-1)+\varepsilon)}\\ &\ll \rho^{n_{k-1}(1+\sigma+\tau)\delta}\rho^{n_{k-1}(\sigma-\tau)(\delta-1)}\rho^{-n_{k-1}(\sigma(2\delta-1)+\varepsilon)}\\ &= (\rho^{n_{k-1}})^{\delta+\tau}\rho^{-n_{k-1}\varepsilon}\\ &= (\rho^{n_{k-1}(1+\sigma+\tau)})^{(\delta+\tau)/(1+\sigma+\tau)}(\rho^{n_{k-1}(1+\sigma+\tau)})^{-\varepsilon/(1+\varepsilon+\tau)}\\ &\ll \begin{cases} r^{\delta/(1+\sigma)} r^{-\varepsilon/(1+\sigma)} & \text{for } \sigma \ge \delta - 1,\\ r^{(\delta+\sigma)/(1+2\sigma)} r^{-\varepsilon/(1+\sigma)} & \text{for } \sigma \le \delta - 1.\end{cases}\end{aligned}$$ ■

(3) “After the visit to $H_{g(p)}(r_g)$”:
If $\rho^{n_k} \le r < (\rho^{n_{k-1}})^{1+2\sigma}$, then $\nu_\sigma(B(\xi, r)) \ll r^{(\delta+\sigma)/(1+2\sigma)-\varepsilon/(1+2\sigma)}$.

Proof: Since $\delta > 1$, the global measure formula implies that $\mu(B(\xi, r)) \ll r^\delta$. Using this and Lemma 3, it follows that

$$\begin{aligned}\nu_\sigma(B(\xi, r)) &\ll \mu(B(\xi, r))\rho^{-n_{k-1}(\sigma(2\delta-1)+\varepsilon)}\\ &\ll r^\delta r^{-(\sigma(2\delta-1)+\varepsilon)/(1+2\sigma)}\\ &\ll r^{(\delta+\sigma-\varepsilon)/(1+2\sigma)}\\ &\ll r^{(\delta+\sigma)/(1+2\sigma)} r^{-\varepsilon/(1+2\sigma)}.\end{aligned}$$ ■

The statement of the proposition now follows by summing up the previous three considerations. □

Proof of Theorem 2. Recall that we have just obtained that, for each $\sigma > 0$, there exists a set $I_\sigma \subset \mathcal{J}_\sigma(G)$ that supports a probability measure ν_σ with the property that, for each $\varepsilon > 0$ and for each $\xi \in I_\sigma$, there exists $r_1(\varepsilon)$ such that, for all $0 < r < r_1(\varepsilon)$, we have

$$\nu_\sigma(B(\xi, r)) \ll \begin{cases} r^{\delta/(1+\sigma)-\varepsilon'} & \text{for } \sigma \geq \delta - 1, \\ r^{(\delta+\sigma)/(1+2\sigma)-\varepsilon'} & \text{for } \sigma \leq \delta - 1. \end{cases}$$

Hence, by the mass distribution principle (see e.g. [4]), for each $\sigma > 0$ we have

$$\dim_H(\mathcal{J}_\sigma(G)) \geq \begin{cases} \frac{\delta}{1+\sigma} & \text{for } \sigma \geq \delta - 1, \\ \frac{\delta+\sigma}{1+2\sigma} & \text{for } \sigma \leq \delta - 1. \end{cases}$$

In order to obtain the upper bounds for the Hausdorff dimension of $\mathcal{J}_\sigma(G)$, note first that for this it is sufficient to give upper bounds for the Hausdorff dimension of the set

$$\mathcal{W}_\sigma(G) := \bigcap_{n \in \mathbb{N}} \bigcup_{p \in P} \bigcup_{\substack{g \in \mathcal{T}_p \\ r_g \leq 1/n}} \mathcal{H}_{g(p)}(r_g^{1+\sigma}).$$

Now, $\{\mathcal{H}_{g(p)}(r_g^{1+\sigma}) : p \in P,\ g \in \mathcal{T}_p\}$ provides a "natural cover" of $\mathcal{W}_\sigma(G)$. Using this and the fact that $\sum_{p \in P} \sum_{g \in \mathcal{T}_p} (r_g^{1+\sigma})^{\delta/(1+\sigma)+\varepsilon}$ converges for any $\varepsilon > 0$, it follows that

$$\dim_H(\mathcal{W}_\sigma(G)) \leq \frac{\delta}{1+\sigma}.$$

On the other hand, note that the intersection of $L(G)$ with the shadow of any σ-reduced horoball $\mathcal{H}_{g(p)}(r_g^{1+\sigma})$ is contained in some "tie-shaped region" (see Figure 2). This observation, together with the fact that the largest horoball one can possibly meet after having traveled through $H_{g(p)}(r_g^{1+\sigma})$ is of the size $r_g^{1+2\sigma}$, yields that we may cover $L(G) \cap \mathcal{H}_{g(p)}(r_g^{1+\sigma})$ by $r_g^{-\sigma}$ Euclidean balls of size $r_g^{1+2\sigma}$ (these balls are not necessarily shadows of horoballs!). Clearly, this gives us an alternative way of covering $\mathcal{W}_\sigma(G)$ (see Figure 2), and for this cover we obtain that

$$\sum_{p \in P} \sum_{g \in \mathcal{T}_p} r_g^{-\sigma} r_g^{(1+2\sigma)s} \begin{cases} \text{converges} & \text{for } s > (\delta+\sigma)/(1+2\sigma), \\ \text{diverges} & \text{for } s \leq (\delta+\sigma)/(1+2\sigma). \end{cases}$$

In particular, this implies that

$$\dim_H(\mathcal{W}_\sigma(G)) \leq \frac{\delta+\sigma}{1+2\sigma}.$$

Combining the two upper bounds for the Hausdorff dimension, an elementary calculation now gives that

$$\dim_H(\mathcal{W}_\sigma(G)) \leq \begin{cases} \frac{\delta}{1+\sigma} & \text{for } \sigma \geq \delta - 1, \\ \frac{\delta+\sigma}{1+2\sigma} & \text{for } \sigma \leq \delta - 1, \end{cases}$$

which proves the theorem. □

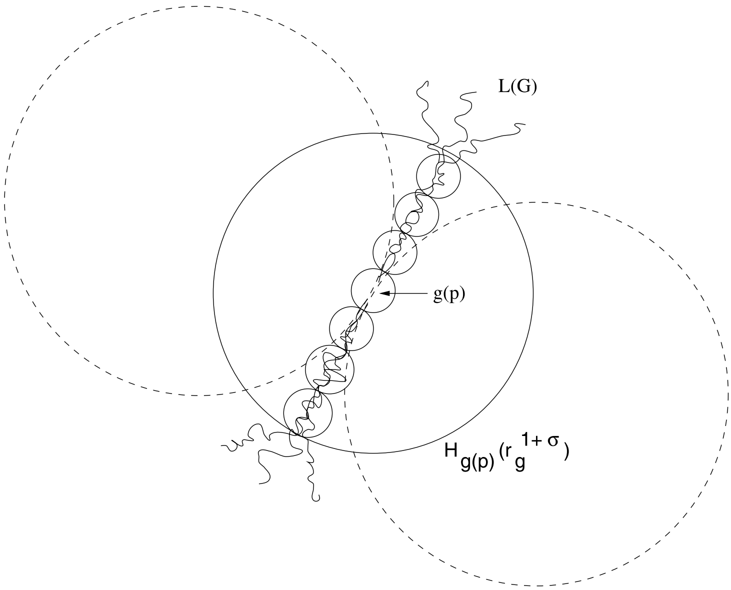

Figure 2 The "alternative cover"

4. Weak Singularity Spectra of the Patterson Measure

In this section we give the estimates that lead to the weak singularity spectra of the Patterson measure μ. The results are a consequence of a combination of Theorem 2 and the global measure formula for μ.

Proof of Theorem 1. Because the arguments are similar for cases (A)–(D), we restrict our discussion to the cases (B) and (C), which are hardest to derive and where the outcome is most interesting.

We consider case (B) first and so assume that G has exclusively rank-1 parabolic fixed points and that $\delta > 1$. In this case, the global measure formula for μ gives the existence of a universal constant $c > 1$ such that, for each $\xi \in L(G)$ and every $t > 0$, we have

$$\delta + (\delta - 1)\frac{\Delta(\xi_t)}{t} - \frac{\log c}{t} \le \frac{\log \mu(B(\xi, e^{-t}))}{\log e^{-t}} \le \delta + (\delta - 1)\frac{\Delta(\xi_t)}{t} + \frac{\log c}{t}.$$

From this we immediately see that

$$\xi \in \mathcal{J}_\sigma(G) \iff \limsup_{t\to\infty} \frac{\log \mu(B(\xi, e^{-t}))}{\log e^{-t}} \ge \delta + (\delta - 1)\frac{\sigma}{1+\sigma}.$$

Hence, if we let $\theta = \delta + (\delta - 1)\frac{\sigma}{1+\sigma}$ then, with an elementary calculation, we see that Theorem 2 implies

$$\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & \text{for } 0 < \theta \le \delta, \\ (1-\delta) - \frac{(1-\delta)(2\delta-1)}{\theta-1} & \text{for } \delta \le \theta \le (2\delta-1) - \frac{\delta-1}{\delta}, \\ \frac{\delta}{1-\delta}\theta - \frac{\delta(2\delta-1)}{1-\delta} & \text{for } (2\delta-1) - \frac{\delta-1}{\delta} \le \theta \le 2\delta-1, \\ 0 & \text{for } 2\delta-1 \le \theta. \end{cases}$$

In order to determine the remaining spectra in case (B), note that the global measure formula gives in particular that, for all $\xi \in L(G)$ and $t > 0$, we have $\mu(B(\xi, e^{-t})) \ll e^{-\delta t}$. Note also that, for $\delta < \theta \le 2\delta - 1$, the inequality $\mu(B(\xi, e^{-t})) \ll e^{-\theta t}$ holds t*-eventually* (i.e., uniformly for large values of t) only if $\xi \in G(P)$; for $\theta > 2\delta - 1$, we even have that this inequality never holds. Using these observations, we derive that

$$\dim_H(\mathcal{I}_\theta(\mu)) = \begin{cases} \delta & \text{for } 0 \le \theta \le \delta, \\ 0 & \text{for } \theta > \delta. \end{cases}$$

In a similar way, we see that for $\theta \ge \delta$ the inequality $\mu(B(\xi, e^{-t})) \gg e^{-\theta t}$ is satisfied for each radial limit point $\xi \in L_r(G) := L(G) \setminus G(P)$ for some sequence $\{t_n\}$ tending to infinity (where $\{t_n\}$ depends on ξ). For $\theta < \delta$, no such sequence $\{t_n\}$ exists for any $\xi \in L(G)$. Hence, we have

$$\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & \text{for } 0 \le \theta < \delta, \\ \delta & \text{for } \theta \ge \delta. \end{cases}$$

Finally, we see that for $\theta \ge \delta$ the inequality $\mu(B(\xi, e^{-t})) \gg e^{-\theta t}$ is t-eventually satisfied for any ξ in the *uniformly radial limit set* $L_{\mathrm{ur}}(G) := \{\xi \in L(G) : \Delta(\xi_t) \ll 1 \text{ for all } t > 0\}$. (This is also referred to as the set of badly approximable limit points; cf. [13].) For $\theta < \delta$, this inequality is never t-eventually satisfied for any $\xi \in L(G)$. Using the fact that $\dim_H(L_{\mathrm{ur}}(G)) = \delta$, which we obtained in [13] (for an alternative proof see [3]), it follows that

$$\dim_H(\mathcal{S}^\theta(\mu)) = \begin{cases} 0 & \text{for } 0 < \theta < \delta, \\ \delta & \text{for } \theta \ge \delta. \end{cases}$$

This gives the weak singularity spectra of μ in the case (B).

Next, we consider the case (C); that is, we assume $1 = k_{\min} < k_{\max} = 2$. Note that, since in general $\delta > k_{\max}/2$, we have that $\delta > 1$. In this case the global measure formula for μ gives the existence of a universal constant $c > 1$ such that, for each $\xi \in L(G)$ and every $t > 0$, we have

$$c^{-1} e^{-t(\delta + (\delta-1)\Delta(\xi_t)/t)} \le \mu(B(\xi, e^{-t})) \le c e^{-t(\delta - (2-\delta)\Delta(\xi_t)/t)}.$$

Obviously, the scaling of μ in rank-1 cusps differs from the scaling of μ in rank-2 cusps. Because of this, we now decompose $\mathcal{J}_\sigma(G)$ as follows. For $l = 1, 2$, let Δ_l denote the ray excursion function detecting only excursions into rank-l cusps. That is, for $\xi \in L(G)$ and $t > 0$ we let $\Delta_l(\xi_t) = d(\xi_t, G(0))$ if ξ_t is contained in a standard horoball associated with some rank-l parabolic fixed point; otherwise, we let Δ_l be equal to some fixed constant. Then the (σ, l)*-Jarník limit set* $\mathcal{J}_{\sigma,l}(G)$ is defined by

$$\mathcal{J}_{\sigma,l}(G) := \left\{ \xi \in L(G) : \limsup_{t\to\infty} \frac{\Delta_l(\xi_t)}{t} \geq \frac{\sigma}{1+\sigma} \right\}.$$

Clearly, we have that $\mathcal{J}_\sigma(G) = \mathcal{J}_{\sigma,1}(G) \cup \mathcal{J}_{\sigma,2}(G)$, which is of course not a disjoint union. The Hausdorff dimensions of $\mathcal{J}_{\sigma,1}(G)$ and $\mathcal{J}_{\sigma,2}(G)$ are derived following essentially the same line of arguments given in the previous section and in [12]. Here it is important to note that, as an immediate consequence of the decoupling lemma in [17] we have that, for $l = 1, 2$, the exponent of convergence of the Dirichlet series $\sum_{p\in P, rk(p)=l} \sum_{g\in \mathcal{T}_p} r_g^s$ remains equal to δ (this gives the upper bounds of $\dim_H(\mathcal{J}_{\sigma,l}(G))$). Hence, for computing $\dim_H(\mathcal{J}_{\sigma,2}(G))$, we may proceed as in the case "$k_{\min} = 2$" and so obtain that

$$\dim_H(\mathcal{J}_{\sigma,2}(G)) = \frac{\delta}{1+\sigma}.$$

For computing $\dim_H(\mathcal{J}_{\sigma,1}(G))$, we proceed as we did in the case "$k_{\max} = 1$ and $\delta > 1$", which gives that

$$\dim_H(\mathcal{J}_{\sigma,1}(G)) = \begin{cases} \frac{\delta}{1+\sigma} & \text{for } \sigma \geq \delta - 1, \\ \frac{\delta+\sigma}{1+2\sigma} & \text{for } \sigma \leq \delta - 1. \end{cases}$$

Using these observations, we can now determine the weak singularity spectra as follows. Namely, exactly as in case (B), we have that

$$\xi \in \mathcal{J}_{\sigma,1}(G) \iff \limsup_{t\to\infty} \frac{\log \mu(B(\xi, e^{-t}))}{\log e^{-t}} \geq \delta + (\delta - 1)\frac{\sigma}{1+\sigma}$$

and hence

$$\dim_H(\mathcal{S}_\theta(\mu)) = \begin{cases} \delta & \text{for } 0 < \theta \leq \delta, \\ (1-\delta) - \frac{(1-\delta)(2\delta-1)}{\theta-1} & \text{for } \delta \leq \theta \leq (2\delta-1) - \frac{\delta-1}{\delta}, \\ \frac{\delta}{1-\delta}\theta - \frac{\delta(2\delta-1)}{1-\delta} & \text{for } (2\delta-1) - \frac{\delta-1}{\delta} \leq \theta \leq 2\delta - 1, \\ 0 & \text{for } 2\delta - 1 \leq \theta. \end{cases}$$

Also, using once again the global measure, we have that

$$\xi \in \mathcal{J}_{\sigma,2}(G) \iff \liminf_{t\to\infty} \frac{\log \mu(B(\xi, e^{-t}))}{\log e^{-t}} \leq \delta - (2-\delta)\frac{\sigma}{1+\sigma},$$

which gives, by substituting $\theta = \delta - (2-\delta)\frac{\sigma}{1+\sigma}$, that

$$\dim_H(\mathcal{I}^\theta(\mu)) = \begin{cases} 0 & \text{for } 0 < \theta \leq 2\delta - 2, \\ \frac{\delta}{2-\delta}\theta - \frac{\delta(2\delta-2)}{2-\delta} & \text{for } 2\delta - 2 < \theta \leq \delta, \\ \delta & \text{for } \theta > \delta. \end{cases}$$

For the remaining parts of the spectra, note that, for $0 < \theta \leq \delta$, we have that $\mu(B(\xi, e^{-t})) \ll e^{-\theta t}$ is t-eventually satisfied for at least all $\xi \in L_{\mathrm{ur}}(G)$. For $\delta < \theta \leq 2\delta - 1$, this inequality holds t-eventually only for rank-1 parabolic fixed points, and for $\theta > 2\delta - 1$ we have that this inequality never holds t-eventually. Hence, it follows that

$$\dim_H(\mathcal{I}_\theta(\mu)) = \begin{cases} \delta & \text{for } 0 \leq \theta \leq \delta, \\ 0 & \text{for } \theta > \delta. \end{cases}$$

The computation of $\dim_H(\mathcal{S}^\theta(\mu))$ is essentially the same as in case (B). Here, the only difference is that, for $2\delta - 2 \leq \theta < \delta$, the inequality $\mu(B(\xi, e^{-t})) \gg e^{-\theta t}$ is satisfied t-eventually by all rank-2 parabolic fixed points (which are, of course, the only elements in $L(G)$ with this behavior). It follows that

$$\dim_H(\mathcal{S}^\theta(\mu)) = \begin{cases} 0 & \text{for } 0 < \theta < \delta, \\ \delta & \text{for } \theta \geq \delta. \end{cases} \qquad \square$$

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