Gluck Surgery along a 2-Sphere in a 4-Manifold is Realized by Surgery along a Projective Plane

Atsuko Katanaga, Osamu Saeki, Masakazu Teragaito, & Yuichi Yamada

0. Introduction

One of the well-known methods to construct a new 4-manifold from an old one is the Gluck surgery along an embedded 2-sphere with trivial normal bundle, which is defined as follows (see [G]). Let M be a smooth 4-manifold and K a smoothly embedded 2-sphere in M. We suppose that the tubular neighborhood $N(K)$ of K in M is diffeomorphic to $S^2 \times D^2$. Let τ be the self-diffeomorphism of $S^2 \times S^1 = \partial(S^2 \times D^2)$ defined by $\tau(z, \alpha) = (\alpha z, \alpha)$, where we identify S^1 with the unit circle of $\mathbf{C}$ and S^2 with the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then consider the 4-manifold obtained from $M - \operatorname{Int} N(K)$ by regluing $S^2 \times D^2$ along the boundary using τ. We say that the resulting 4-manifold, denoted by $\Sigma(K)$, is obtained from M by the *Gluck surgery* along K (see [G] or [Kir2, p. 16]).

When the ambient 4-manifold is the 4-sphere S^4, we call a smoothly embedded 2-sphere K in S^4 a 2-*knot*. In this case, the resulting 4-manifold $\Sigma(K)$ is always a homotopy 4-sphere. It has been known that, for certain 2-knots K, $\Sigma(K)$ is again diffeomorphic to S^4 (see e.g. [Gom1; Gor; HMY; Mo; Pl]). It has not been known if the Gluck surgery along a 2-knot K in S^4 produces a 4-manifold $\Sigma(K)$ not diffeomorphic to S^4 for some K (see [Kir1, 4.11, 4.24, 4.45] and [Gom2]). On the other hand, for 2-spheres embedded in 4-manifolds M not necessarily diffeomorphic to S^4, Akbulut [Ak1; Ak2] constructed an example of an embedded 2-sphere K in such an M such that $\Sigma(K)$ is homeomorphic but is not diffeomorphic to M. For Gluck surgeries, see also [Ak3; AK; AR; Gom1; Gom2].

Price [P] considered a similar construction using embedded projective planes in S^4. Let P be a smoothly embedded projective plane in S^4. In the following, we fix an orientation for S^4. Then it is known that the tubular neighborhood $N(P)$ of P is always diffeomorphic to the nonorientable D^2-bundle over $\mathbf{R}P^2$ with Euler number ± 2 (see [M1; M2]), which we denote by N_e with $e = \pm 2$ the Euler number. Note that ∂N_e is diffeomorphic to the quaternion space Q, whose fundamental

Received December 18, 1998. Revision received June 22, 1999.

The second author has been partially supported by the Anglo-Japanese Scientific Exchange Programme, run by JSPS and the Royal Society. The fourth author was partially supported by JSPS Research Fellowship for Young Scientists.

Michigan Math. J. 46 (1999).

group is isomorphic to the quaternion group of order 8. Then consider the closed orientable 4-manifold $\Pi(P)_\varphi$ obtained from $S^4 - \operatorname{Int} N(P)$ by regluing N_e along the boundary using a self-diffeomorphism φ of Q. Then we say that $\Pi(P)_\varphi$ is obtained from S^4 by a *Price surgery* along P with respect to φ. In fact, Price [P] showed that there are exactly six isotopy classes of orientation-preserving self-diffeomorphisms of Q—thus we have essentially six choices for φ—and that exactly four of them produce homotopy 4-spheres by a Price surgery. Furthermore, he has also shown that there are at most two diffeomorphism types among the four homotopy 4-spheres thus constructed, one of which is the standard 4-sphere. In the following, $\Pi(P)$ will denote the unique homotopy 4-sphere obtained by the Price surgery along P with respect to a nontrivial self-diffeomorphism of Q, which may not be diffeomorphic to the 4-sphere.

Obviously we can generalize this definition of Price surgeries to those along projective planes embedded in arbitrary 4-manifolds with normal Euler number ± 2.

Let P_0 be a standardly embedded projective plane in S^4 whose normal Euler number is either 2 or -2 (see e.g. [La1; La2; PR; Y1]). One of our main results of the present paper is the following theorem concerning the relationship between Gluck surgeries and Price surgeries.

Theorem 0.1. *Let K be a 2-knot in S^4. Then the homotopy 4-sphere $\Sigma(K)$ obtained by the Gluck surgery along K is diffeomorphic to the homotopy 4-sphere $\Pi(P_0 \sharp K)$ obtained by the Price surgery along the projective plane $P_0 \sharp K$, where $\sharp$ denotes the connected sum.*

In fact, this theorem is a direct consequence of a more general result as follows. In the following, for a projective plane P smoothly embedded in S^4, we denote by $N(P)$ and $E(P)$ its tubular neighborhood in S^4 and $S^4 - \operatorname{Int} N(P)$, respectively; we call $E(P)$ the *exterior* of P.

Theorem 0.2. *Let K and K' be an arbitrary pair of 2-knots in S^4. Then there exist four self-diffeomorphisms φ_j $(j = 1, 2, 3, 4)$ of Q such that the closed oriented 4-manifold $E(P_0 \sharp K) \cup_{\varphi_j} -E(P_0 \sharp K')$, obtained by gluing $E(P_0 \sharp K)$ and $-E(P_0 \sharp K')$ along their boundaries using φ_j, is orientation-preservingly diffeomorphic to S^4, $\Sigma(K)$, $\Sigma(K'!)$, and $\Sigma(K) \sharp \Sigma(K'!)$ for $j = 1, 2, 3,$ and 4 (respectively), where $-E(P_0 \sharp K')$ denotes $E(P_0 \sharp K')$ with the reversed orientation and $K'!$ denotes the mirror image of K'.*

In this theorem, if K' is unknotted then $E(P_0) = E(P_0 \sharp K')$ is diffeomorphic to $N_{\mp 2}$ (see [La1; La2; M2; P; PR; Y1]) and $\Sigma(K'!)$ is diffeomorphic to S^4. Thus, Theorem 0.1 follows from Theorem 0.2. Note that, in Theorem 0.2, the fact that $E(P_0 \sharp K) \cup_{\varphi_1} -E(P_0 \sharp K')$ is diffeomorphic to S^4 for some φ_1 has already been obtained by the fourth author [Y3] when K and K' are 2-twist spun 2-knots (see [Z]).

Using our Theorem 0.1, we will show that the Gluck surgery along a smoothly embedded 2-sphere K in an arbitrary 4-manifold M is always realized by a Price surgery along the connected sum $P_0 \sharp K$ of K and a standardly embedded projective plane P_0 contained in a 4-disk in M.

The paper is organized as follows. In Section 1, we study the decomposition $S^4 = N(P_0) \cup E(P_0)$ and show that, for every pair of 2-knots K and K' in S^4, the 4-sphere S^4 decomposes as $E(P_0 \sharp K) \cup -E(P_0 \sharp K')$. In Section 2, we review the result of Price [P] concerning the mapping class group $\mathcal{M}(Q)$ of the quaternion space Q. Recall that Q admits a structure of a Seifert fibered space over S^2 with three singular fibers (see [Y1]). We will identify $\mathcal{M}(Q)$ with the symmetric group on three letters, where to a self-diffeomorphism φ of Q corresponds the bijection on the set of the singular fibers associated with a fiber-preserving diffeomorphism isotopic to φ. In Section 3, we will prove Theorem 0.2. In Section 4, we show that every Gluck surgery in an arbitrary 4-manifold is realized by a Price surgery. In the Appendix, we will introduce a method to describe the homotopy 4-sphere $\Sigma(K)$ obtained by the Gluck surgery along a 2-knot K in S^4 by using a framed link in S^3. This result will be used in the proof of Theorem 0.2 in Section 3. In fact, the result itself seems to be folklore; however, we have included it because (to the authors' knowledge) there has been nothing explicitly written in the literature.

Throughout the paper, all manifolds and maps are of class C^∞ unless otherwise indicated. The symbol "$\cong$" denotes a (orientation-preserving) diffeomorphism between (oriented) manifolds or an appropriate isomorphism between algebraic objects.

Acknowledgment. The authors would like to thank Katsuyuki Yoshikawa and Seiichi Kamada for their helpful comments on embedded projective planes in S^4. They would also like to express their sincere gratitude to the referee for helpful comments and suggestions. The second author would like to express his thanks to the people at the University of Liverpool for their hospitality during the preparation of the manuscript.

1. Decompositions of the 4-Sphere

In this section, we study decompositions of the 4-sphere S^4 into the union of the exteriors of two embedded projective planes in S^4.

Let P_+ (resp. P_-) denote the standardly embedded projective plane in S^4 whose normal Euler number is equal to 2 (resp. -2) (see e.g. [La1; La2; PR; Y1]). First we review the decomposition of S^4 into the union of a tubular neighborhood $N(P_+)$ ($\cong N_2$) of P_+ in S^4 and its exterior $E(P_+)$ (see [Y1]). It is well known that $E(P_+)$ is diffeomorphic to $-N_2$ ($\cong N_{-2}$) [La1; La2; M2; P; PR; Y1], where $-N_2$ denotes N_2 with the reversed orientation. Hence we have the decomposition $S^4 \cong -N_2 \cup_\partial N_2$, where $\cup_\partial$ means that we glue $-N_2$ and N_2 along their boundaries.

In [Y1], the fourth author gave a handlebody decomposition of N_2 and described it by a framed link. Here we describe $E(P_+) \cong -N_2$ by the framed link as in Figure 1. We denote by $-N_2 = H^0 \cup H^1 \cup H^2$ the handlebody decomposition corresponding to the left-hand side framed link of Figure 1, where H^r denotes a handle of index r.

Using the handlebody decomposition of $-N_2$ and the decomposition $S^4 \cong -N_2 \cup_\partial N_2$, we obtain the decomposition of S^4 as follows:

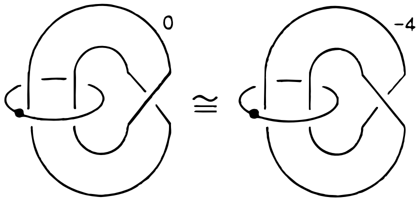


Figure 1

$$S^4 \cong (H_-^0 \cup H_-^1 \cup H_-^2) \cup (H_+^2 \cup H_+^1 \cup H_+^0), \tag{1}$$

where $H_\pm^r$ denotes the r-handle of $\pm N_2$. In the following, the $(4-r)$-handle dual to the r-handle $H_\pm^r$ will be denoted by $(H_\pm^r)^\perp$. The attaching circle and the framing of $(H_+^2)^\perp$, which is attached to $H_-^0 \cup H_-^1 \cup H_-^2$, is studied in [Y1, Sec. 5].

In the theory of framed links, it is usual to omit drawing 3- and 4-handles (see [LP]). In this sense, the decomposition of S^4 in (1) gives a nontrivial handlebody decomposition that is described by the framed link as in Figure 2 (see also [Y1, Fig. 6]).

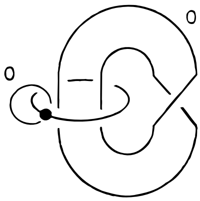


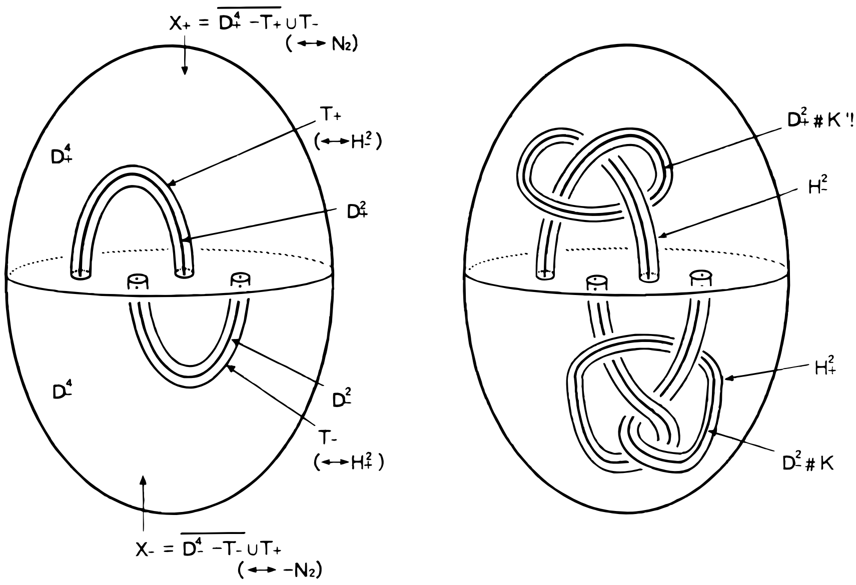
Figure 2

Examining the framed link representation, we see easily that $H_-^0 \cup H_-^1 \cup (H_+^2)^\perp$ is diffeomorphic to the 4-disk D^4. In the following, we identify $H_-^0 \cup H_-^1 \cup (H_+^2)^\perp$ with D^4. Then we can summarize the decomposition of S^4 as follows:

$$\begin{aligned} S^4 &\cong -N_2 \cup_\partial N_2 && (2)\\ &= (H_-^0 \cup H_-^1 \cup H_-^2) \cup (H_+^2 \cup H_+^1 \cup H_+^0) && (3)\\ &= (H_-^0 \cup H_-^1 \cup (H_+^2)^\perp) \cup ((H_-^2)^\perp \cup H_+^1 \cup H_+^0) && (4)\\ &= D^4 \cup_\partial -D^4. && (5)\end{aligned}$$

Intuitively, we can regard the decomposition $S^4 \cong -N_2 \cup_\partial N_2$ shown in Figure 3(1) as follows (cf. [Y1, Fig. 3]). Let D_+^4 (resp. D_-^4) denote the upper (resp. lower) hemisphere of S^4, and let $k = k_+ \cup k_-$ be the torus link of type $(4, 2)$ in $\partial D_+^4 = \partial D_-^4 = D_+^4 \cap D_-^4$, where $k_\pm$ are the components of k. Since k_+ (resp. k_-) is an unknotted circle in ∂D_+^4 (resp. in ∂D_-^4), it bounds a 2-disk D_+^2 (resp. D_-^2)

properly embedded in D^4_+ (resp. in D^4_-) such that (D^4_+, D^2_+) (resp. (D^4_-, D^2_-)) is a standard disk pair. Let $T_\pm$ be a small tubular neighborhood of $D^2_\pm$ in $D^4_\pm$ and let $X_\pm$ denote the union of the closure of $D^4_\pm - T_\pm$ in $D^4_\pm$ and $T_\mp$. Then the decomposition $(S^4; -N_2, N_2)$ is diffeomorphic to the decomposition $(D^4_+ \cup D^4_-; X_-, X_+)$. Then, by replacing the 2-disks $D^2_\pm$ in $D^4_\pm$ with knotted 2-disks, we can construct new decompositions of S^4. The idea of such a decomposition is seen in Figure 3(2).



(1) (2)

Figure 3

In the foregoing handlebody decomposition $\pm N_2 = H^0_\pm \cup H^1_\pm \cup H^2_\pm$, the union of the 0-handle and the 1-handle is diffeomorphic to $S^1 \times D^3$, which is diffeomorphic to the exterior of an unknotted 2-sphere in S^4. In other words, $H^0_\pm \cup H^1_\pm$ corresponds to the closure of $D^4_\pm - T_\pm$ in $D^4_\pm$. Furthermore, $H^2_\pm$ corresponds to $T_\mp$.

For given 2-knots K and K' in S^4, let us consider the operation of replacing D^2_+ and D^2_- with $D^2_+ \sharp K'!$ and $D^2_- \sharp K$, respectively. This corresponds to replacing $H^0_- \cup H^1_-$ with $E(K)$ and $H^1_+ \cup H^0_+$ with $E(K'!)$, respectively, in the decomposition (3), where $K'!$ is the mirror image of K'. We can regard the small tubular neighborhoods of $D^2_+ \sharp K'!$ and $D^2_- \sharp K$ in D^4_+ and D^4_- (respectively) as 2-handles, and we denote them using the same notation $H^2_\mp$ as before. Then, since we have $E(K) \cup H^2_- \cong E(P_+ \sharp K)$ and $H^2_+ \cup E(K'!) \cong -E(P_+ \sharp K')$, it follows that

$$S^4 \cong (E(K) \cup H^2_-) \cup (H^2_+ \cup E(K'!)) \tag{6}$$

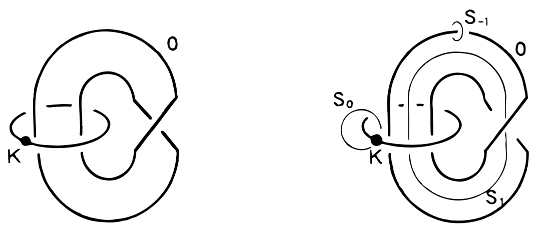
$$\cong E(P_+ \sharp K) \cup_\partial -E(P_+ \sharp K'). \tag{7}$$

Thus we have the following lemma.

LEMMA 1.1. *For every pair of 2-knots K and K' in S^4, we can decompose the standard 4-sphere S^4 into the union of $E(P_+ \sharp K)$ and $-E(P_+ \sharp K')$.*

REMARK 1.2. This result has already been obtained by the fourth author [Y3] in the case where K and K' are the 2-twist spun 2-knots (see [Z]).

Note that—using the notation to be introduced in the Appendix—we can represent $E(P_+ \sharp K)$ by the framed link as in Figure 4(1).



(1) (2)

Figure 4

2. Mapping Class Group of the Quaternion Space

Let Q denote the quaternion space, which is identified with ∂N_2. We denote by $\mathcal{M}(Q)$ the mapping class group of Q, which is (by definition) the group of isotopy classes of orientation-preserving self-diffeomorphisms of Q. In [P], Price investigated the self-diffeomorphisms of the quaternion space Q and showed that $\mathcal{M}(Q)$ is isomorphic to $\mathcal{S}_3$, the symmetric group on three letters. In this section, we study the self-diffeomorphisms of Q using a Seifert fibered structure of Q and define four self-diffeomorphisms f_{ij} $(i, j = 0, 1)$ of Q.

Recall that Q admits a Seifert fibered structure whose Seifert invariants in the sense of [Or, Section 5.2] are given by $\{-1; (o_1, 0); (2, 1), (2, 1), (2, 1)\}$. Let K be a 2-knot in S^4. In the framed link representation of $E(P_+ \sharp K)$, the three singular fibers S_{-1}, S_0, S_1 correspond to the circles in $\partial E(P_+ \sharp K) = -Q$, as in Figure 4(2). Here S_{-1} is a co-core of the 2-handle H^2_-, S_0 is a meridional circle of $D^2_- \sharp K$, and the third one is S_1. For an element σ of $\mathcal{S}_3$, the symmetric group on the three letters $\{-1, 0, 1\}$, there exists a self-diffeomorphism f_σ of Q which preserves the Seifert fibered structure and which satisfies $f_\sigma(S_i) = S_{\sigma(i)}$. By using a result of Price [P] together with a calculation of the automorphisms of $\pi_1(Q)$ induced by f_σ, it is not difficult to show that every self-diffeomorphism of Q is isotopic to f_σ for a unique $\sigma \in \mathcal{S}_3$.

Now let K' be another 2-knot in S^4. Then we can construct closed oriented 4-manifolds by gluing $E(P_+ \sharp K)$ and $-E(P_+ \sharp K')$ along their boundaries. By the previous paragraph, we have at most six diffeomorphism types for the resulting 4-manifolds. Here we are interested only in homotopy 4-spheres. By an argument using the Mayer–Vietoris sequence, we see easily that if the gluing map sends the singular fiber S_{-1} onto itself then the resulting 4-manifold has nontrivial first homology group. Thus we consider the following four self-diffeomorphisms that send S_{-1} onto an S_k ($k \neq -1$) as gluing maps. Let f_{ij} ($i, j = 0, 1$) denote the self-diffeomorphism of Q which preserves the Seifert fibered structure and which satisfies $f_{ij}(S_{-1}) = S_i$ and $f_{ij}^{-1}(S_{-1}) = S_j$. In other words, $f_{ij} = f_\sigma$ for

$$\sigma = \begin{pmatrix} -1 & j & \bar{j} \\ i & -1 & \bar{i} \end{pmatrix} \in \mathcal{S}_3,$$

where $\{-1, i, \bar{i}\} = \{-1, j, \bar{j}\} = \{-1, 0, 1\}$.

3. Proof of Theorem 0.2

Let K and K' be 2-knots in S^4. In this section, we study the homotopy 4-spheres obtained by gluing $E(P_0 \sharp K)$ and $-E(P_0 \sharp K')$ along their boundaries.

Proof of Theorem 0.2. We may assume that $P_0 = P_+$. Let us begin with the decomposition of S^4 obtained in Section 1 as follows:

$$S^4 \cong E(P_+ \sharp K) \cup_\partial -E(P_+ \sharp K') \tag{8}$$

$$\cong (E(K) \cup H_-^2) \cup (H_+^2 \cup E(K'!)) \tag{9}$$

$$= (E(K) \cup (H_+^2)^\perp) \cup ((H_-^2)^\perp \cup E(K'!)). \tag{10}$$

Note that, in (10), $E(K) \cup (H_+^2)^\perp$ and $(H_-^2)^\perp \cup E(K'!)$ are diffeomorphic to D^4 and $-D^4$, respectively. Let us denote by g the gluing map $\partial(-E(P_+ \sharp K')) \to \partial E(P_+ \sharp K)$ in the decomposition (8).

Let us verify that the gluing map g is isotopic to f_{00}. As we have seen in Section 1 (see Figure 2), the co-core S_{-1} of H_+^2 lying on $\partial(H_+^2 \cup E(K'!))$ corresponds by g to S_0 lying on $\partial(E(K) \cup H_-^2)$. Thus we have $g(S_{-1}) = S_0$. By a similar argument, we see that $g^{-1}(S_{-1}) = S_0$. Thus g is isotopic to f_{00} and hence we have the first diffeomorphism of the theorem with $\varphi_1 = f_{00}$.

Next let us consider the following:

$$E(P_+ \sharp K) \cup_{f_{ij}} -E(P_+ \sharp K')$$

$$\cong (E(K) \cup H_-^2) \cup_{f_{ij}} (H_+^2 \cup E(K'!)) \tag{11}$$

$$= (E(K) \cup_{S_i} (H_+^2)^\perp) \cup ((H_-^2)^\perp \cup_{S_j} E(K'!)), \tag{12}$$

where $E(K) \cup_{S_i} (H_+^2)^\perp$ (resp. $(H_-^2)^\perp \cup_{S_j} E(K'!)$) denotes the compact 4-manifold obtained from $E(K)$ (resp. from $E(K'!)$) by attaching the 2-handle $(H_+^2)^\perp$ (resp. $(H_-^2)^\perp$) along the circle S_i (resp. S_j). By an argument similar to the foregoing, we see that if $i = 0$ then $E(K) \cup_{S_i} (H_+^2)^\perp$ is diffeomorphic to D^4 and that if $j = 0$ then $(H_-^2)^\perp \cup_{S_j} E(K'!)$ is diffeomorphic to $-D^4$.

Let us show that if $i = 1$ then $E(K) \cup_{S_i} (H^2_+)^\perp$ is diffeomorphic to $\Sigma(K)^\circ$, where $\Sigma(K)$ is the homotopy 4-sphere obtained by the Gluck surgery along K (see Section 0) and $\Sigma(K)^\circ = \Sigma(K) - \operatorname{Int} D^4$.

We shall prove this claim by using the framed link theory together with a method to be explained in the Appendix. The 4-manifold $E(K)$ is represented by an unknotted circle with a dot together with the symbol K, and the attaching circle of $(H^2_+)^\perp$ coincides with S_1 as in Figure 4(2). Let us determine the framing n of the 2-handle $(H^2_+)^\perp$. First note that $\partial(E(K) \cup H^2_- \cup_{S_1} (H^2_+)^\perp) \cong \partial E(K'!) \cong S^2 \times S^1$. By Lemma A.3, we can represent this boundary by the framed link as in Figure 5. Because the linking matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & n & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

of the framed link is a presentation matrix of $H_1(S^2 \times S^1; \mathbf{Z}) \cong \mathbf{Z}$, its determinant $\det A = 4 - 4n$ must vanish. Thus we have $n = +1$. Hence $E(K) \cup_{S_1} (H^2_+)^\perp$ is described by the framed link $M(K, 1)$, as in Figure A2 (in the Appendix). Therefore it is diffeomorphic to $\Sigma(K)^\circ$, by Lemma A.2.

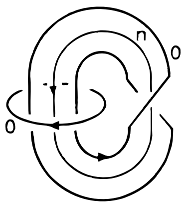


Figure 5

By the same argument, we can show that $(H^2_-)^\perp \cup_{S_1} E(K'!) \cong \Sigma(K'!)^\circ$. Therefore, by (12), we see that

$$E(P_+ \sharp K) \cup_{f_{01}} -E(P_+ \sharp K') \cong D^4 \cup \Sigma(K'!)^\circ,$$

$$E(P_+ \sharp K) \cup_{f_{10}} -E(P_+ \sharp K') \cong \Sigma(K)^\circ \cup -D^4,$$

$$E(P_+ \sharp K) \cup_{f_{11}} -E(P_+ \sharp K') \cong \Sigma(K)^\circ \cup \Sigma(K'!)^\circ.$$

Thus we have the conclusion of Theorem 0.2 with $\varphi_2 = f_{10}$, $\varphi_3 = f_{01}$, and $\varphi_4 = f_{11}$. This completes the proof. □

As a direct consequence of Theorem 0.2, we have the following.

COROLLARY 3.1. *Let K be a 2-knot in S^4. Then there exist self-diffeomorphisms φ and ψ of Q such that*

$$E(P_0 \sharp K) \cup_\varphi -E(P_0 \sharp K) \cong S^4,$$

$$E(P_0 \sharp K) \cup_\psi -E(P_0 \sharp K) \cong \Sigma(K).$$

In other words, S^4 and $\Sigma(K)$ can be decomposed as twisted doubles of the exterior of $P_0 \sharp K$.

For twisted double decompositions of the 4-sphere, see [La1; Y1; Y2].

We now discuss further related results and problems concerning Theorems 0.1 and 0.2.

REMARK 3.2. Concerning surgeries along embedded tori in S^4, a result similar to Theorem 0.1 has been obtained by Iwase [I, Prop. 3.5].

COROLLARY 3.3. *Let K be a 2-knot in S^4 such that the homotopy 4-sphere $\Sigma(K)$ obtained by the Gluck surgery along K is diffeomorphic to S^4. Then all the homotopy 4-spheres obtained by Price surgeries along $P_0 \sharp K$ are also diffeomorphic to S^4.*

COROLLARY 3.4. *Let K and K' be 2-knots in S^4. If $E(P_0 \sharp K)$ is diffeomorphic to $E(P_0 \sharp K')$, then $\Sigma(K')$ is diffeomorphic to $\Sigma(K)$.*

Proof. By applying Theorem 0.2 to K and the trivial 2-knot, we see that the homotopy 4-sphere obtained by gluing $E(P_0 \sharp K)$ and $-E(P_0)$ along their boundaries is diffeomorphic to S^4 or $\Sigma(K)$. Similarly, the homotopy 4-sphere obtained by gluing $E(P_0 \sharp K')$ and $-E(P_0)$ is diffeomorphic to S^4 or $\Sigma(K')$. Thus, by our assumption, we have $\{S^4, \Sigma(K)\} = \{S^4, \Sigma(K')\}$ as sets of diffeomorphism classes of homotopy 4-spheres. If $\Sigma(K) \cong S^4$, then the number of elements of the set is equal to 1 and hence we have $S^4 \cong \Sigma(K')$. If $\Sigma(K) \ncong S^4$, then the number of elements of the set is equal to 2. Thus $\Sigma(K') \ncong S^4$ and, by the foregoing equality, we have $\Sigma(K') \cong \Sigma(K)$. This completes the proof. □

Viro showed that there exists a nontrivial 2-knot K in S^4 such that $P_0 \sharp K$ is isotopic to P_0 ([V]; see also [PR]). By Theorem 0.1 (or Corollary 3.4), for such a 2-knot K, $\Sigma(K)$ is diffeomorphic to the 4-sphere.

As a generalization of Viro's construction, let us consider a 2-knot K in S^4 and consider the local move as depicted in Figure 6. A disk and an annulus, which are parts of K, are properly embedded in a 4-ball $D^4 \cong D^3 \times [-1, 1]$ in S^4, and we change those parts of K in the 4-ball (or, more precisely, in $D^3 \times \{0\}$), as in Figure 6, without changing the other parts. Then it is not difficult to show that if a 2-knot K is changed to K' by a finite number of local changes of this type, then the pair K and K' satisfies the assumption of Corollary 3.4. Note that this type of operation, which is a slight generalization of Viro's example, does create many pairs of *distinct* 2-knots (K, K') such that $K \sharp P_\pm$ is isotopic to $K' \sharp P_\pm$.

Thus we have the following result, which purely concerns Gluck surgeries along 2-knots.

COROLLARY 3.5. *Let K and K' be 2-knots in S^4. If K is transformed to K' by a finite iteration of local changes of the type just described, then $\Sigma(K')$ is diffeomorphic to $\Sigma(K)$.*

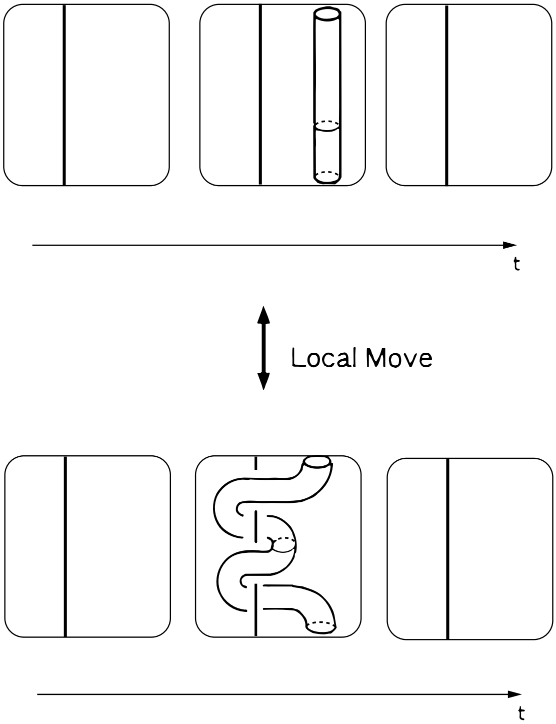


Figure 6

Note that a more general version of this corollary has been obtained in [HMY] by using a different method.

It has been known that, for certain 2-knots in S^4, the results of the Gluck surgeries along them are diffeomorphic to S^4 (see e.g. [Gom1; Gor; HMY; Mo; Pl]). Note that we do not know if the Gluck surgery along every 2-knot gives a homotopy 4-sphere diffeomorphic to the standard 4-sphere (see [Kir1, 4.11, 4.24, 4.45]). The answer to this question will be affirmative if the following problem is negatively solved.

PROBLEM 3.6. Does there exist a smoothly embedded projective plane P in S^4 such that the Price surgery along P gives a homotopy 4-sphere that is not diffeomorphic to S^4?

The following corollary is a direct consequence of our Theorem 0.1 and [AR, Thm. 4.6].

COROLLARY 3.7. *Let K be a 2-knot in S^4 and let $\Pi(P_0 \sharp K)$ be the homotopy 4-sphere obtained by the Price surgery along $P_0 \sharp K$. Then $\Pi(P_0 \sharp K) \sharp \mathbf{C}P^2$ is always diffeomorphic to $\mathbf{C}P^2$.*

We do not know if this corollary holds for homotopy 4-spheres obtained by Price surgeries along arbitrary projective planes in S^4.

REMARK 3.8. Here we note that all known examples of projective planes smoothly embedded in S^4 are isotopic to the connected sum of a standardly embedded projective plane and a 2-knot. In [Ka1; Ka2; Kin; PR, Sec. V], some examples of such knotted projective planes whose exteriors have fundamental groups not isomorphic to $\mathbf{Z}_2$ have been constructed and studied. In fact, the *Kinoshita conjecture* posits that every smoothly embedded projective plane in S^4 is isotopic to such a connected sum. (Although this conjecture has not appeared in the literature, it has been known to knot theorists in Japan for many years; see e.g. [Yo].) If this conjecture is true, then our Theorem 0.1 would imply that the homotopy 4-spheres produced by Price surgeries along embedded projective planes in S^4 are nothing but the homotopy 4-spheres produced by Gluck surgeries along embedded 2-spheres in S^4.

REMARK 3.9. It has been known that there exist inequivalent 2-knots in S^4 with diffeomorphic exteriors (see [CS]). We do not know if there exist inequivalent projective planes in S^4 with diffeomorphic exteriors. Note that, by [P], there are at most two such projective planes with a fixed exterior. For example, for the 2-knots K and K' of [CS] with the properties just described, we do not know if $P_0 \sharp K$ and $P_0 \sharp K'$ also have the same property.

REMARK 3.10. In [Y3] it was shown that, for certain 2-knots K and K' in S^4, there exists a self-diffeomorphism φ of Q such that $E(P_+ \sharp K) \cup_\varphi -E(P_+ \sharp K')$ is diffeomorphic to S^4. Furthermore, in [KS] it was shown that, if P and P' are topologically locally flatly embedded projective planes in S^4 with the same normal Euler number and such that either P or P' is the connected sum of P_0 and a locally flat topological 2-knot in S^4, then there exists a self-homeomorphism φ of Q such that $E(P) \cup_\varphi -E(P')$ is homeomorphic to S^4.

REMARK 3.11. Theorem 0.2 (or Lemma 1.1) shows that one can construct infinitely many mutually nonisotopic smooth embeddings of the quaternion space Q into S^4. Conversely, let $f : Q \to S^4$ be an arbitrary smooth embedding. In [KS] it was shown that the closure of each connected component of $S^4 - f(Q)$ is homeomorphic to the exterior of some topologically locally flatly embedded projective plane in S^4.

4. Gluck Surgery in an Arbitrary 4-Manifold

Let M be a connected smooth 4-manifold and K a smoothly embedded 2-sphere in M with trivial normal bundle. Furthermore, let P_0 be a smoothly embedded projective plane in M such that (i) it is contained in a 4-disk D^4 in M, (ii) it is standard as an embedding into D^4, and (iii) it has normal Euler number ± 2. Note that P_0 is uniquely determined up to isotopy if M is nonorientable and that there are exactly two isotopy classes corresponding to the normal Euler numbers ± 2

if M is orientable. We denote by $P_0 \sharp K$ the connected sum of P_0 with K in M. Note that $\partial N(P_0 \sharp K)$ is diffeomorphic to Q, where $N(P_0 \sharp K)$ is a tubular neighborhood of $P_0 \sharp K$ in M.

Our main result of this section is the following.

Theorem 4.1. *Let M, K, and P_0 be as before. Then there exists a self-diffeomorphism φ of Q such that the 4-manifold $\Pi(P_0 \sharp K)_\varphi$ obtained by the Price surgery along $P_0 \sharp K$ with respect to φ is diffeomorphic to the 4-manifold $\Sigma(K)$ obtained by the Gluck surgery along K.*

Proof. We may assume that $P_0 = P_+$. Let D_0^4 be a smoothly embedded 4-disk in M such that the following conditions hold:

(1) the 2-sphere K and ∂D_0^4 intersect transversely along a circle; and
(2) the pair $(D_0^4, D_0^4 \cap K)$ is a standard disk pair.

Let M_1 denote the closure of $M - D_0^4$ and let T_1 be a tubular neighborhood of $K \cap M_1$ in M_1. Note that the closure $E'(K)$ of $M_1 - T_1$ in M_1 is diffeomorphic to the exterior of K in M and that T_1 is considered to be a 2-handle attached to $E'(K)$.

Let D' be a properly embedded 2-disk in D_0^4 which does not intersect $K \cap \partial D_0^4$, such that (D_0^4, D') is a standard disk pair and the link $\partial D' \cup (K \cap \partial D_0^4)$ is as in the left-hand side of Figure 1, where the dotted circle corresponds to $K \cap \partial D_0^4$. Let T' denote a small tubular neighborhood of D' in D_0^4. Note that T' is also considered to be a 2-handle attached to M_1 (see Figure 7) and that $E(P_+ \sharp K) \cong E'(K) \cup T'$. We denote by E_0 the closure of $D_0^4 - T'$ in D_0^4. Then the operation of a Price surgery along $P_+ \sharp K$ is to cut off $E_0 \cup T_1$ from M and then reglue it using a self-diffeomorphism of Q.

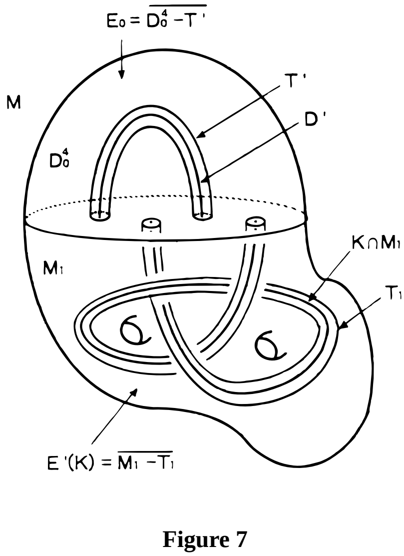


Figure 7

Then (by an argument similar to that in the proof of Theorem 0.2) we see that, by a self-diffeomorphism φ of $Q \cong \partial(E_0 \cup T_1)$ corresponding to f_{10}, the 2-handle T_1 is attached to $E'(K)$ along a circle in $\partial D_0^4 \cap E'(K)$ that corresponds to the circle with framing n in Figure 5. By an argument similar to the proof of Theorem 0.2 and Lemma A.2, we see that the resulting 4-manifold $E'(K) \cup T_1$ is diffeomorphic to $\Sigma(K)^\circ$ and that the union of E_0 and T' is diffeomorphic to D^4. Hence the result of the Price surgery along $P_+ \sharp K$ with respect to φ is diffeomorphic to $\Sigma(K)$. This completes the proof. □

Remark 4.2. Akbulut [Ak1; Ak2] constructed an example of an embedded 2-sphere K in a 4-manifold M such that $\Sigma(K)$ is homeomorphic but is not diffeomorphic to M. Such an example, together with our Theorem 4.1, gives an example of a smoothly embedded projective plane in a 4-manifold such that a Price surgery gives an exotic 4-manifold.

Appendix: Framed Link Representation of $\Sigma(K)$

In this section we introduce a method to describe a 2-knot exterior by a framed link in S^3, and we also describe the homotopy 4-sphere $\Sigma(K)$ obtained by the Gluck surgery along a 2-knot K in S^4 by using such framed links.

Let (D^4, D_0^2) be a standard disk pair; that is, D_0^2 is an unknotted 2-disk properly embedded in the 4-disk D^4. Let K be a 2-knot in S^4. We consider K to be embedded in the interior of D^4 and let $D_0^2 \sharp K$ denote the connected sum of D_0^2 and K in the interior of D^4. Let $N(D_0^2 \sharp K)$ be a tubular neighborhood of $D_0^2 \sharp K$ in D^4. Note that the closure $E'(K)$ of $D^4 - N(D_0^2 \sharp K)$ in D^4 is diffeomorphic to the exterior $E(K)$ of K in S^4.

Definition A.1. We denote the compact 4-manifold $E'(K)$ by the framed link consisting of an unknotted circle in S^3 with a dot and with symbol K attached to it (see Figure A1). In the framed link representation, the exterior of the unknotted circle in S^3 coincides with $\partial D^4 \cap E'(K)$.

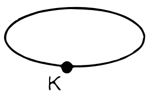
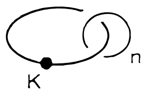


Figure A1

When the 2-knot K is unknotted, $E'(K)$ is diffeomorphic to $S^1 \times D^3$ and the framed link representation of $E'(K)$ as in the preceding definition (but without the symbol K) coincides with the usual framed link representation of a 1-handle attached to a 0-handle (see e.g. [Kir2, I, Sec. 2]). In other words, Definition A.1 is a generalization of the framed link representation of a 1-handle.

Our main result of this appendix is the following.

LEMMA A.2. *For an integer n and a 2-knot K in S^4, consider the compact 4-manifold $M(K,n)$ represented by the framed link as in Figure A2. In other words, $M(K,n)$ is obtained from $E'(K)$ by attaching a 2-handle along the undotted circle in $\partial E'(K)$ as in Figure A2 with framing n. Then $M(K,n)$ is diffeomorphic to D^4 if n is even and to $\Sigma(K)^\circ$ if n is odd, where $\Sigma(K)$ denotes the homotopy 4-sphere obtained by the Gluck surgery along K and $\Sigma(K)^\circ = \Sigma(K) - \mathrm{Int}\, D^4$.*



M(K,n)

Figure A2

This lemma may already be a "folklore" fact. However, we include a proof here for completeness.

Proof of Lemma A.2. First we consider K to be embedded in S^4. Take a small 4-disk D_0^4 in S^4 such that the following conditions hold:

(1) the 2-knot K and ∂D_0^4 intersect transversely along a circle; and
(2) the pair $(D_0^4, D_0^4 \cap K)$ is a standard disk pair.

Let the closure of $S^4 - D_0^4$ be denoted by D_1^4. Then let $T_i \cong D^2 \times D^2$ $(i = 0,1)$ be a tubular neighborhood of $D_i^4 \cap K$ in D_i^4 such that $T_0 \cap \partial D_0^4 = T_1 \cap \partial D_1^4 = T_0 \cap T_1$ is a tubular neighborhood of $K \cap \partial D_0^4$ in ∂D_0^4. Note that $T_0 \cup T_1 \cong S^2 \times D^2$ is a tubular neighborhood of K in S^4 and that the closure of $D_1^4 - T_1$ is diffeomorphic to $E'(K)$.

Consider $\tau_n = \tau^n$, where τ is the self-diffeomorphism of $S^2 \times S^1 = \hat{\mathbf{C}} \times S^1$ as in Section 0. We identify the boundary of $T_0 \cup T_1$ with $S^2 \times S^1$ so that $T_0 \cap \partial(T_0 \cup T_1)$ (resp. $T_1 \cap \partial(T_0 \cup T_1)$) corresponds to $D_+^2 \times S^1$ (resp. $D_-^2 \times S^1$), where D_+^2 (resp. D_-^2) is the unit disk in $\mathbf{C}$ (resp. the complement of the open unit disk in $\mathbf{C}$ together with $\{\infty\}$).

Then consider the surgery operation of cutting off $T_0 \cup T_1$ from $S^4 = D_0^4 \cup D_1^4$ and regluing $T_0 \cup T_1 \cong S^2 \times D^2$ by using τ_n. By [G], the resulting 4-manifold M is diffeomorphic to S^4 if n is even and to $\Sigma(K)$ if n is odd. Since the diffeomorphism τ_n preserves the decomposition $S^2 \times S^1 = (D_+^2 \times S^1) \cup (D_-^2 \times S^1)$, this 4-manifold decomposes as $M_0 \cup M_1$, where M_i $(i = 0,1)$ is the 4-manifold obtained by the surgery operation of cutting off T_i from D_i^4 and regluing $T_i \cong D^2 \times D^2$ by using τ_n restricted to $(T_i \cap \partial(T_0 \cup T_1))$.

Since $(D_0^4, D_0^4 \cap K)$ is a standard disk pair, it is easy to show that M_0 is always diffeomorphic to the 4-disk D^4. On the other hand, the closure of $D_1^4 - T_1$ is diffeomorphic to $E'(K)$ and $T_1 \cong D^2 \times D^2$ can be regarded as a 2-handle attached to $E'(K)$. The attaching circle of T_1 is isotopic to the undotted circle shown in Figure A2 in $\partial E'(K)$ and the framing is equal to n, since we use τ_n for the attaching map. Thus we have shown that M_1 is diffeomorphic to $M(K,n)$.

Summarizing these observations, we have that $M(K, n) \cup D^4$ is diffeomorphic to S^4 if n is even and to $\Sigma(K)$ if n is odd. Thus we have the conclusion, which completes the proof of Lemma A.2. $\square$

If a compact 4-manifold M is represented by a framed link that has dotted circles with a symbol K, then one can obtain a usual framed link representation of the boundary 3-manifold ∂M by the following lemma.

LEMMA A.3. *Suppose that a framed link L has an unknotted circle d with a dot and with a K. Let L' denote the framed link obtained from L by removing the dot and the symbol K from the component d and replacing them with a* 0 *as a framing number. Then the boundary* 3*-manifolds of the* 4*-manifolds represented by L and L' are diffeomorphic to each other.*

Proof. We use the same notation as in the paragraph just before Definition A.1. Let $\nu\colon D^2 \times D^2 \to N(D_0^2 \sharp K)$ be a diffeomorphism with $\nu(D^2 \times \{0\}) = D_0^2 \sharp K$ such that $N(d) = \nu(\partial D^2 \times D^2)$ is a tubular neighborhood of d in ∂D^4, where 0 is the center of the disk D^2. Note that the framing number corresponding to the diffeomorphism $\nu|_{\partial D^2 \times D^2}$ is equal to 0, since d and the parallel circle $d' = \nu(\partial D^2 \times \{p\})$ bound disjoint disks in D^4, where p is a point on ∂D^2.

Let W denote the 3-manifold represented by the framed link $L - d$, which coincides with that represented by $L' - d$. Then we see that the boundary of the 4-manifold represented by L is diffeomorphic to $(\overline{W - N(d)}) \cup \nu(D^2 \times \partial D^2)$. Since the framing number corresponding to the diffeomorphism $\nu|_{\partial D^2 \times D^2}$ is equal to 0, we see that the boundary 3-manifold is also represented by the framed link L'. This completes the proof. $\square$

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A. Katanaga
Institute of Mathematics
University of Tsukuba
Ibaraki 305-8571
Japan

fuji@math.tsukuba.ac.jp

O. Saeki
Department of Mathematics
Hiroshima University
Higashi-Hiroshima 739-8526
Japan

saeki@math.sci.hiroshima-u.ac.jp

M. Teragaito
Department of Mathematics Education
Hiroshima University
Higashi-Hiroshima 739-8524
Japan

mteraga@dean.sed.hiroshima-u.ac.jp

Y. Yamada
Department of Systems Engineering
University of Electro-Communications
Tokyo 182-8585
Japan

yyyamada@e-one.uec.ac.jp