Classification Theorem for a Class of Flat Connections and Representations of Kähler Groups

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1. Introduction

1.1

Let M be a compact Kähler manifold. For a matrix Lie group G, the representation variety $\mathcal{M}_G$ of the fundamental group $\pi_1(M)$ is defined as the quotient $\mathrm{Hom}(\pi_1(M), G)//G$. Here G acts on the set $\mathrm{Hom}(\pi_1(M), G)$ by pointwise conjugation: $(gf)(s) = gf(s)g^{-1}$, $s \in \pi_1(M)$. A study of geometric properties of $\mathcal{M}_G$ is of interest because of the relation to the problem of classifying Kähler groups (a problem posed by J.-P. Serre in the 1950s). For a simply connected nilpotent Lie group G, every element of $\mathcal{M}_G$ is uniquely determined by a d-harmonic nilpotent matrix 1-form ω on M such that $\omega \wedge \omega$ represents 0 in the corresponding de Rham cohomology group. This follows, for example, from a theorem on formality of a compact Kähler manifold [DGMS]. The main result of our paper gives, in particular, a similar description for elements of $\mathcal{M}_G$ with a simply connected solvable Lie group G. Our arguments are straightforward and based on cohomology techniques only. As a consequence of the main theorem we obtain several results on the structure of Kähler groups. We now proceed to a formulation of the results.

It is well known that $\mathcal{M}_{\mathrm{GL}_n(\mathbb{C})}$ is equivalently characterized as moduli spaces of flat bundles over M with structure group $\mathrm{GL}_n(\mathbb{C})$. In this paper we consider a family of C^∞-trivial complex flat vector bundles over M. Every bundle from this family is determined by a flat connection on the trivial bundle $M \times \mathbb{C}^n$, that is, by a matrix-valued 1-form ω on M satisfying

$$d\omega - \omega \wedge \omega = 0. \tag{1.1}$$

Moreover, we assume that the $(0,1)$-component ω_2 of ω is an upper triangular matrix form. Denote this class of connections by $\mathcal{A}_n^t$.

REMARK 1.1. Connections from $\mathcal{A}_n^t$ determine (by iterated path integration) all representations of $\pi_1(M)$ into simply connected complex solvable Lie groups.

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(Here, according to Lie's theorem, we think of every such group as a subgroup of a complex Lie group of upper triangular matrices.)

Let $T_n(\mathbb{C})$ denote the complex Lie group of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. Then the group $C^\infty(M, T_n(\mathbb{C}))$ acts by d-gauge transforms on the set $\mathcal{A}^t_n$:

$$d_g(\alpha) = g^{-1}\alpha g - g^{-1}dg, \quad g \in C^\infty(M, T_n(\mathbb{C})), \ \alpha \in \mathcal{A}^t_n. \tag{1.2}$$

Denote the corresponding quotient set by $\mathcal{B}^t_n$. (We regard $\mathcal{B}^t_n$ as the set of d-gauge equivalent classes of connections from $\mathcal{A}^t_n$.) In this paper we study the structure of $\mathcal{B}^t_n$. Also, our result gives a characterization of the subset of $\mathcal{M}_{\mathrm{GL}_n(\mathbb{C})}$ consisting of conjugate classes of representations determined by elements of $\mathcal{A}^t_n$.

Let $\mathcal{U}^n_\oplus$ be a class of flat vector bundles over M of complex rank n whose elements are direct sums of topologically trivial flat vector bundles of complex rank 1 with unitary structure group. Note that every $E \in \mathcal{U}^n_\oplus$ can be represented by a unitary diagonal cocycle $\{c_{ij}\}_{i,j\in I}$ defined on an open covering $\{U_i\}_{i\in I}$. All definitions formulated herein depend not on the choice of such a cocycle but only on its cohomology class.

A family $\{\eta_i\}_{i\in I}$ of matrix-valued p-forms satisfying

$$\eta_j = c_{ij}^{-1}\eta_i c_{ij} \quad \text{on } U_i \cap U_j \tag{1.3}$$

is, by definition, a p-form with values in the bundle $\mathrm{End}(E)$. We say that such a form is *nilpotent* if every η_i takes its values in the Lie algebra of the Lie group of upper triangular unipotent matrices. Since $\mathrm{End}(E) \in \mathcal{U}^{n^2}_\oplus$, there exists a natural flat Hermitian metric on $\mathrm{End}(E)$. As usual, one can use the metric to construct a d-Laplacian on the space of $\mathrm{End}(E)$-valued forms. In what follows, harmonic forms are determined by this Laplacian. Denote by $\mathbf{H}^1_d(\mathrm{End}(E))$ the finite-dimensional complex vector space of $\mathrm{End}(E)$-valued harmonic 1-forms, and denote by $H^2(\mathrm{End}(E))$ the de Rham cohomology group of $\mathrm{End}(E)$-valued d-closed 2-forms. Further, consider the set $\mathbf{H}^t_0(\mathrm{End}(E)) \subset \mathbf{H}^1_d(\mathrm{End}(E))$ of harmonic forms η that satisfy the following conditions:

(i) the $(0,1)$-component η_2 of η is nilpotent;
(ii) $\eta \wedge \eta$ represents 0 in $H^2(\mathrm{End}(E))$.

Observe that $\mathbf{H}^t_0(\mathrm{End}(E))$ is a complex affine subvariety of $\mathbf{H}^1_d(\mathrm{End}(E))$ defined by homogeneous quadratic equations.

Let $\mathrm{Aut}^t_f(E)$ be the group of triangular flat automorphisms of E. Elements of $\mathrm{Aut}^t_f(E)$ are, by definition, locally constant sections of $\mathrm{End}(E)$ satisfying (1.3) with $\eta_i \in T_n(\mathbb{C})$ $(i \in I)$. Clearly, $\mathrm{Aut}^t_f(E)$ is a complex solvable Lie group. It acts by conjugation on the space of $\mathrm{End}(E)$-valued forms and commutes with the Laplacian. In particular, it acts on $\mathbf{H}^t_0(\mathrm{End}(E))$. Consider the quotient set $\mathcal{S}^n_E := \mathbf{H}^t_0(\mathrm{End}(E))/\mathrm{Aut}^t_f(E)$, and denote by $\mathcal{S}^n$ the disjoint union $\bigsqcup_{E\in\mathcal{U}^n_\oplus} \mathcal{S}^n_E$. (Note that, according to Green–Lazarsfeld theorem [GL], if the dimension of the image of the Albanese mapping of $M \geq 2$ then the set $\mathcal{S}^n_E$ with the generic E consists of a single point.)

THEOREM 1.2. *There is a one-to-one correspondence between the sets $\mathcal{B}^t_n$ and $\mathcal{S}^n$.*

Using Theorem 1.2 for the case of flat connections corresponding to unipotent representations of $\pi_1(M)$, one can give alternative proofs of some results due for example to Campana (for references see [ABCKT]) and Benson and Gordon [BG]. In the following section we describe the 1–1 correspondence in more detail.

1.2

We now formulate several geometrical applications of Theorem 1.2. They describe some properties of the set $S_n(M)$ of representations of $\pi_1(M)$ into $\mathrm{GL}_n(\mathbb{C})$ generated by connections from $\mathcal{A}_n^t$.

Let T_2^u denote the Lie group of upper triangular 2×2 matrices with unitary elements on the diagonal. Further, denote by $S_2^u(M)$ a class of homomorphisms $\rho\colon \pi_1(M) \to T_2^u$ whose diagonal elements ρ_{ii} satisfy $\rho_{ii} = \exp(\tilde{\rho}_{ii})$ for some $\tilde{\rho}_{ii} \in \mathrm{Hom}(\pi_1(M), \mathbb{C})$, $i = 1, 2$. (For example, if $H^2(M, \mathbb{Z})$ is torsion-free then each element of $\mathrm{Hom}(\pi_1(M), T_2^u)$ belongs to $S_2^u(M)$.) In what follows, $f\colon M_1 \to M_2$ is a complex surjective mapping of compact Kähler manifolds and G', G'' denote the first and second (resp.) commutant groups of a group G.

THEOREM 1.3. *Assume that for any $\tau \in S_2^u(M_1)$ there is a $\tau' \in S_2^u(M_2)$ such that $\tau = \tau' \circ f_*$. Then, for any $\rho \in S_n(M_1)$, there exists $\rho' \in S_n(M_2)$ such that $\rho = \rho' \circ f_*$.*

REMARK 1.4. A result similar to Theorem 1.3 is also valid in the case of representations generated by connections from $\mathcal{A}_n^t$ with nilpotent $(0,1)$-components. In this case it suffices to assume that f induces an isomorphism of $H_1(M_1, \mathbb{R})$ and $H_1(M_2, \mathbb{R})$; see [Br]. This assumption holds, for example, if f is a smoothing of the Albanese map α_M of a compact Kähler manifold M (here M_1 and M_2 are desingularizations of M and $\alpha_M(M) \subset \mathrm{Alb}(M)$, respectively). Then the analog of Theorem 1.3 implies the following (see e.g. [ABCKT, Prop. 3.33]).

THEOREM (Campana). *The mapping f induces an isomorphism of the de Rham fundamental groups of M_1 and M_2.*

Let us now introduce the class S of compact Kähler manifolds M for which $\bigcup_{n\geq 1} S_n(M)$ separates the elements of $\pi_1(M)$.

THEOREM 1.5. *Assume that $M_1 \in S$ and f induces an isomorphism of $\pi_1(M_1)/\pi_1(M_1)''$ and $\pi_1(M_2)/\pi_1(M_2)''$. Then f_* imbeds $\pi_1(M_1)$ as a subgroup of finite index in $\pi_1(M_2)$.*

In a forthcoming paper we will demonstrate the following application of Theorem 1.2.

THEOREM. *Assume that $M \in S$ satisfies*

(i) $\pi_2(M) = 0$ *and*
(ii) $\dim_{\mathbb{C}} M \geq \frac{1}{2} \operatorname{rank}(\pi_1(M)/\pi_1(M)'')$.

Then

(a) $\dim_{\mathbb{C}} M = \frac{1}{2}\operatorname{rank}(\pi_1(M)/\pi_1(M)'')$ *and*

(b) $\pi_1(M)$ *is isomorphic to a lattice in a Lie group* G *that is a semidirect product of* $\mathbb{C}^m$ *and* $\mathbb{R}^{2k}$ *determined by a unitary representation* $\mathbb{R}^{2k} \to U_m(\mathbb{C})$.

Here $2m + 2k = \operatorname{rank}(\pi_1(M)/\pi_1(M)'')$ *and* $2m = \operatorname{rank}(\pi_1(M)/\pi_1(M)')$.

This gives, in particular, a classification of compact solvmanifolds admitting a Kähler structure.

At the end of the paper we will show that the results just formulated hold also for the class of manifolds dominated by a compact Kähler manifold.

2. Theorem 1.2: Main Steps toward the Proof

2.1

The proof of Theorem 1.2 follows from the results formulated in this section. In order to formulate the first of them, recall that any flat connection ω on a topologically trivial complex vector bundle $M \times \mathbb{C}^n$ (over a compact Kähler manifold M) determines a system of ODEs

$$df = \omega f, \quad f \in C^\infty(M, \mathrm{GL}_n(\mathbb{C})), \tag{2.1}$$

with ω satisfying (1.1) (the condition of local solvability). Conversely, for a family $\{f_i\}_{i\in I}$ of local solutions of (2.1) defined on an open covering $\{U_i\}_{i\in I}$, the flat structure on $M \times \mathbb{C}^n$ is determined by the locally constant cocycle $\{c_{ij} := f_i^{-1} f_j\}_{i,j\in I}$. Furthermore, we can rewrite (2.1) in the equivalent form

$$\partial f = \omega_1 f, \tag{2.2}$$

$$\bar\partial f = \omega_2 f, \tag{2.3}$$

where ω_1 and ω_2 are a $(1,0)$-form and a $(0,1)$-form (respectively) and $\omega = \omega_1 + \omega_2$. As follows from (1.1), the system (2.2)–(2.3) is locally solvable. Note that the local solvability of each of these equations separately is equivalent to the fulfillment of one of the corresponding conditions:

$$\partial\omega_1 - \omega_1 \wedge \omega_1 = 0, \tag{2.4}$$

$$\bar\partial\omega_2 - \omega_2 \wedge \omega_2 = 0. \tag{2.5}$$

Our first result is related to the following.

> *Complement Problem:* Given ω_2 satisfying (2.5), find ω_1 for which the system (2.2)–(2.3) is locally solvable.

THEOREM 2.1. *Suppose that* ω_2 *is a triangular* $(0,1)$*-form satisfying* (2.5). *Then there exists a triangular* $(1,0)$*-form* ω_1 *such that* $\omega = \omega_1 + \omega_2 \in \mathcal{A}_n^t$; *that is,* ω *satisfies* (1.1). *In addition, there exists a* $T_n(\mathbb{C})$*-valued* d*-gauge transform sending* ω *to a triangular* 1*-form* $\eta = \eta_1 + \eta_2$ *such that*

$$\operatorname{diag}(\eta_2) = -\overline{\eta_1}.$$

Here $\operatorname{diag}(\phi)$ *is the diagonal of* ϕ, *and* $\bar{\phi}$ *denotes the complex conjugate of* ϕ.

2.2

Let E be a flat vector bundle over M of complex rank n represented via a locally constant cocycle $\{c_{ij}\}_{i,j\in I}$ defined on an open covering $\{U_i\}_{i\in I}$. Denote by $\operatorname{End}(E)$ the vector bundle of linear endomorphisms of E. According to (1.3), the operators d and $\wedge$ are well-defined on the set of matrix-valued 1-forms with values in $\operatorname{End}(E)$. In particular, it makes sense to consider 1-forms satisfying an equation similar to (1.1). Let h be a linear C^∞-automorphism of E determined by a family $\{h_i\}_{i\in I}$, $h_i \in C^\infty(U_i, \mathrm{GL}_n(\mathbb{C}))$, satisfying

$$h_j = c_{ij}^{-1} h_i c_{ij} \quad \text{on } U_i \cap U_j.$$

Then a d-gauge transformation d_h^E defined on the set of matrix-valued 1-forms α with values in $\operatorname{End}(E)$ is given by a formula similar to (1.2),

$$d_h^E(\alpha) = h^{-1}\alpha h - h^{-1}dh.$$

Clearly, d_h^E preserves the class of 1-forms satisfying an $\operatorname{End}(E)$-valued equation (1.1). Let now $E \in \mathcal{U}_\oplus^n$ and h belong to $\operatorname{Aut}_\infty^t(E)$, the group of triangular C^∞-automorphisms of E. Then d_h^E preserves also the class of $\operatorname{End}(E)$-valued 1-forms with nilpotent $(0,1)$-components. Since E is a direct sum of topologically trivial vector bundles $M\times\mathbb{C}$, the group $\operatorname{Aut}_\infty^t(E)$ is isomorphic to $C^\infty(M, T_n(\mathbb{C}))$. In what follows we identify these two groups.

We now proceed to describe the correspondence map from Theorem 1.2. Denote by $\mathcal{E}_\psi$ the class of connections from $\mathcal{A}_n^t$ such that the diagonals of their $(0,1)$-components are equal to ψ.

PROPOSITION 2.2. (1) *For every* $\bar{\partial}$*-closed* $(0,1)$*-form* ψ *there is an invertible diagonal matrix-valued function* h_ψ *such that* $d_{h_\psi}(\mathcal{E}_\psi) = \mathcal{E}_{\tilde{\psi}}$, *where* $\tilde{\psi}$ *is the harmonic component in the Hodge decomposition of* ψ.

(2) *For every diagonal harmonic* $(0,1)$*-form* ψ *there exist a vector bundle* E_ψ *over* M *and an injective mapping* τ_ψ *of* $\mathcal{E}_\psi$ *to the set of* $\operatorname{End}(E_\psi)$*-valued* 1*-forms such that*:

(a) $E_\psi \in \mathcal{U}_\oplus^n$;
(b) $\tau_\psi \circ d_g = d_g^{E_\psi} \circ \tau_\psi$ *for every* $g \in C^\infty(M, T_n(\mathbb{C}))$; *and*
(c) $\tau_\psi(\mathcal{E}_\psi)$ *consists of forms with nilpotent* $(0,1)$*-components satisfying* (1.1).

The proof of the proposition (in Section 5) will also show that every element of $\mathcal{U}_\oplus^n$ coincides with some E_ψ.

According to Proposition 2.2, the moduli space $\mathcal{B}_n^t$ of flat connections from $\mathcal{A}_n^t$ is isomorphic to a similar moduli space of forms from $\tau_\psi(\mathcal{E}_\psi)$.

PROPOSITION 2.3. *For every* $\eta \in \tau_\psi(\mathcal{E}_\psi)$, *there is a transform* $d_g^{E_\psi}$ *with* $g \in \operatorname{Aut}_\infty^t(E_\psi)$ *such that*

$$d_g^{E_\psi}(\eta) = \tilde{\eta}_1 + \tilde{\eta}_2,$$

where $\partial\tilde{\eta}_1 = 0$ *and* $\tilde{\eta}_2$ *is a* d*-closed nilpotent antiholomorphic form*.

This result implies that $\tilde{\eta}_1$ can be decomposed into the sum $\alpha + \partial h$, where α is its harmonic component in the Hodge decomposition. Observe that $\tilde{\eta}_2$ and α belong to the space $\mathbf{H}_d^1(\mathrm{End}(E_\psi))$ of d-harmonic forms described in Section 1.1 (see also Proposition 3.7). Moreover, condition (1.1) together with the $\partial\bar{\partial}$-lemma (see Lemma 3.8) implies that $[\alpha + \tilde{\eta}_2, \alpha + \tilde{\eta}_2]$ represents 0 in the de Rham cohomology group $H^2(M, \mathrm{End}(E_\psi))$. The converse of the latter statement is also true. Namely, let α be an $\mathrm{End}(E_\psi)$-valued d-harmonic $(1,0)$-form and let θ be a d-harmonic nilpotent $\mathrm{End}(E_\psi)$-valued $(0,1)$-form.

PROPOSITION 2.4. *Let* $[\alpha + \theta, \alpha + \theta]$ *represent zero in* $H^2(M, \mathrm{End}(E_\psi))$. *Then there exists a unique* (*up to a flat additive summand*) *section* h *such that* $(\alpha + \partial h) + \theta$ *satisfies* (1.1).

Finally, to complete Theorem 1.2 we must prove the following uniqueness result.

PROPOSITION 2.5. *Let* α_1, β_1 *and* α_2, β_2 *be* $\mathrm{End}(E_\psi)$*-valued* $(1,0)$- *and* $(0,1)$-*forms, respectively. Suppose that*

(a) $\alpha_1 + \alpha_2$ *and* $\beta_1 + \beta_2$ *belong to* $\tau_\psi(\mathcal{E}_\psi)$ *and are* d*-gauge equivalent, and that*
(b) α_2, β_2 *are* d*-closed nilpotent forms.*

Then the d*-gauge equivalence is defined by a flat automorphism of* E_ψ.

In other words, $\tilde{\eta}_1 + \tilde{\eta}_2$ in Proposition 2.3 is unique up to conjugation by flat automorphisms. We now summarize the foregoing results.

The space $\mathcal{B}_n^t$ is isomorphic to the disjoint union of moduli spaces $\mathcal{E}_\psi / C^\infty(M, T_n(\mathbb{C}))$ with diagonal harmonic $(0,1)$-forms ψ. Further, the mapping τ_ψ defines an isomorphism between $\mathcal{E}_\psi / C^\infty(M, T_n(\mathbb{C}))$ and $\tau_\psi(\mathcal{E}_\psi)/\mathrm{Aut}_\infty^t(E_\psi)$. The latter, in turn, is isomorphic to $\mathcal{S}_{E_\psi}^n := \mathbf{H}_0^t(\mathrm{End}(E_\psi))/\mathrm{Aut}_f^t(E_\psi)$. This completes the description of the correspondence of Theorem 1.2.

REMARK 2.6. It was proved by Goldman and Millson [GM] and independently by Simpson [S] that the representation varieties of Kähler groups have at worst quadratic singularities at reductive representations. Theorem 1.2 shows that this "quadratic law" is also of global nature if we restrict ourselves to some naturally determined subsets of $\mathcal{M}_{\mathrm{GL}_n(\mathbb{C})}$.

3. Auxiliary Results

3.1

Let D be one of the operators d, $\bar{\partial}$, or ∂. If $g \subset \mathrm{gl}_n(\mathbb{C})$ is the Lie algebra of a Lie group $G \subset \mathrm{GL}_n(\mathbb{C})$ then we denote by $\mathcal{A}_D(g)$ the space of locally integrable D-connections in the principle bundle $M \times G$ over M defined by

$$Df = \omega f, \quad f \in C^\infty(M, G) \tag{3.1}$$

with a g-valued differential form ω. The condition of integrabilty of a connection is

$$D\omega - \omega \wedge \omega = 0.$$

Let $\mathcal{B}_D(g)$ denote the moduli space of $\mathcal{A}_D(g)$, that is, the set of D-gauge equivalent classes of connections from $\mathcal{A}_D(g)$. Further, we introduce the class $\mathcal{V}_D(G)$ of isomorphic G-topologically trivial vector bundles with D-trivial cocycles $\{c_{ij}\}$ (this means that the principle G-bundle constructed by this cocycle is topologically trivial and $Dc_{ij} = 0$ for all i, j). In particular, $\{c_{ij}\}$ is holomorphic for $D = \bar\partial$, locally constant for $D = d$, and antiholomorphic for $D = \partial$.

Then there is a bijection

$$i_D \colon \mathcal{B}_D(g) \to \mathcal{V}_D(G)$$

defined in the following way (see e.g. [O, Sec. 5, 6] for details). Let $\{U_i\}_{i\in I}$ be an open covering of M and let $f_i \in C^\infty(U_i, G)$ be a solution of (3.1) on U_i. If we set $c_{ij} = f_i^{-1} f_j$, then $\{c_{ij}\}$ is a D-trivial cocycle and so determines an element of $\mathcal{V}_D(G)$. The construction is independent of the choice of the element of an equivalence class in $\mathcal{B}_D(g)$ and thus it correctly defines the required mapping i_D. For an $\omega \in \mathcal{A}_D(g)$ we let $[\omega] \in \mathcal{B}_D(g)$ denote its D-gauge equivalence class.

Because each locally constant cocycle is holomorphic and antiholomorphic simultaneously, the identity mapping induces natural mappings

$$h \colon \mathcal{V}_d(G) \to \mathcal{V}_{\bar\partial}(G) \quad \text{and} \quad \bar h \colon \mathcal{V}_d(G) \to \mathcal{V}_\partial(G). \tag{3.2}$$

Namely, if $\mathbf{E}$ is the sheaf of locally constant sections of a vector bundle $E \in \mathcal{V}_d(G)$ then vector bundles $h(E)$ and $\bar h(E)$ are determined by sheaves $\mathbf{E} \otimes_{\mathbb{C}} \mathcal{O}_M$ and $\mathbf{E} \otimes_{\mathbb{C}} \bar{\mathcal{O}}_M$, respectively.

It is worth noting that the moduli space of isomorphic vector bundles with locally constant G-cocycles (flat bundles) is isomorphic to the quotient $\mathcal{M}_G := \mathrm{Hom}(\pi_1(M), G)/G$ of the space of representations of $\pi_1(M)$ in G, by the action of G given by conjugation (see e.g. [KN, Chap. 2, Sec. 9]).

PROPOSITION 3.1. *Let $\omega_2 \in \mathcal{A}_{\bar\partial}(\mathrm{gl}_n(\mathbb{C}))$. Then the following statements are equivalent*:

(i) *there exists a $\mathrm{gl}_n(\mathbb{C})$-valued $(1, 0)$-form ω_1 such that $\omega = \omega_1 + \omega_2$ belongs to $\mathcal{A}_d(\mathrm{gl}_n(\mathbb{C}))$*;

(ii) *there exists an element $E \in \mathcal{V}_d(\mathrm{GL}_n(\mathbb{C}))$ such that*

$$h(E) = i_{\bar\partial}([\omega_2]).$$

Proof. Let $\Pi_{0,1} \colon \mathcal{E}^1(M) \otimes \mathrm{gl}_n(\mathbb{C}) \to \mathcal{E}^{0,1}(M) \otimes \mathrm{gl}_n(\mathbb{C})$ be the projection from the space of matrix-valued 1-forms defined on M onto the space of $(0, 1)$-forms induced by the type decomposition. Clearly, $\Pi_{0,1}$ maps $\mathcal{A}_d(\mathrm{gl}_n(\mathbb{C}))$ in $\mathcal{A}_{\bar\partial}(\mathrm{gl}_n(\mathbb{C}))$ and commutes with the actions of the corresponding gauge groups. Denote by $\tilde\Pi_{0,1} \colon \mathcal{B}_d(\mathrm{gl}_n(\mathbb{C})) \to \mathcal{B}_{\bar\partial}(\mathrm{gl}_n(\mathbb{C}))$ the mapping induced by $\Pi_{0,1}$. Then the required statement follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{B}_d(\mathrm{gl}_n(\mathbb{C})) & \xrightarrow{\tilde{\Pi}_{0,1}} & \mathcal{B}_{\bar\partial}(\mathrm{gl}_n(\mathbb{C})) \\ {\scriptstyle i_d}\downarrow & & \downarrow{\scriptstyle i_{\bar\partial}} \\ \mathcal{V}_d(\mathrm{GL}_n(\mathbb{C})) & \xrightarrow{h} & \mathcal{V}_{\bar\partial}(\mathrm{GL}_n(\mathbb{C})). \end{array} \tag{3.3}$$

□

3.2

Hereafter we denote by $\mathcal{V}$ the category of vector bundles equipped with one of the following structures: C^∞, holomorphic, antiholomorphic, or flat. If $E \in \mathcal{V}$ then $\mathbf{E}$ denotes the sheaf of its local sections determining the structure of E.

Let now E, E_1, E_2 belong to $\mathcal{V}$.

DEFINITION 3.2. The element E is said to be an *extension* of E_2 by E_1 if the sequence

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0 \tag{3.4}$$

is exact. Extensions E of E_2 by E_1 and F of F_2 by F_1 are *isomorphic* in $\mathcal{V}$ if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E & \longrightarrow & E_2 & \longrightarrow & 0 \\ & & {\scriptstyle j_1}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j_2}\downarrow & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & F & \longrightarrow & F_2 & \longrightarrow & 0, \end{array} \tag{3.5}$$

where j_1, j, j_2 are isomorphisms of the corresponding $\mathcal{V}$-bundles. In the case of $j_1 = \mathrm{id}$ and $j_2 = \mathrm{id}$, these extensions are called *equivalent*.

Let E be an extension of E_2 by E_1. Then (3.4) induces the exact sequence

$$0 \longrightarrow \mathrm{Hom}(E_2, E_1) \longrightarrow \mathrm{Hom}(E_2, E) \longrightarrow \mathrm{Hom}(E_2, E_2) \longrightarrow 0$$

(here all bundles have the same structure as E_1 and E_2). This sequence, in turn, induces the exact sequence of Čech cohomology groups of the corresponding sheaves

$$0 \longrightarrow H^0(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1)) \longrightarrow H^0(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E})) \longrightarrow$$

$$H^0(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_2)) \xrightarrow{\delta} H^1(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1)) \longrightarrow \cdots.$$

Let $I \in H^0(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_2))$ be the identity section. Then it is well known that $\delta(I)$ uniquely determines the class of extensions of E_2 by E_1 equivalent to E.

PROPOSITION 3.3 [A, Prop. 2]. *The equivalence classes of extensions of E_2 by E_1 are in one-to-one correspondence with the elements of $H^1(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1))$, and the trivial extension corresponds to the trivial element.*

REMARK 3.4. It follows directly from Definition 3.2 that if $E_i \in \mathcal{V}_D(\mathrm{GL}_{k_i}(\mathbb{C}))$ $(i = 1, 2)$ then $E \in \mathcal{V}_D(G)$, where the structure group G consists of elements of the form

$$\begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix}$$

with $A_i \in \mathrm{GL}_{k_i}(\mathbb{C})$, $i = 1, 2$.

Let now E and F be isomorphic extensions of E_2 by E_1 and F_2 by F_1, respectively. Let k_i be the rank of E_i $(i = 1, 2)$, and let G be the Lie group from Remark 3.4. Consider principle bundles E_G and F_G with the structure group G corresponding to E and F. Then our next proposition follows immediately from the definitions.

PROPOSITION 3.5. *Any isomorphism $j: E \to F$ determined by* (3.5) *induces an isomorphism j_G of G-bundles E_G and F_G. Moreover, restriction of j_G to a fibre is determined as left multiplication by an element of G.*

Consider now an extension E of E_2 by E_1 in the category of flat bundles. (So structure group G of E is now defined as in Remark 3.4.) In this case, the natural mappings $h: \mathcal{V}_d(G) \to \mathcal{V}_{\bar\partial}(G)$ and $\bar h: \mathcal{V}_d(G) \to \mathcal{V}_\partial(G)$—see (3.2)—determine extensions $h(E)$ of $h(E_2)$ by $h(E_1)$ and $\bar h(E)$ of $\bar h(E_2)$ by $\bar h(E_1)$. According to Proposition 3.3 and the Dolbeault theorem, the former extension is defined by an element of the group $H^1(M, \mathrm{Hom}(\mathbf{E}_2 \otimes_{\mathbb{C}} \mathcal{O}_M, \mathbf{E}_1 \otimes_{\mathbb{C}} \mathcal{O}_M))$, and each element of this group is given by a $\bar\partial$-closed $(0,1)$-form with values in $\mathrm{Hom}(E_2, E_1)$. The latter extension is defined in the same way by a ∂-closed $(1,0)$-form with values in $\mathrm{Hom}(E_2, E_1)$.

The elements of the cohomology groups that appeared here can be described as follows. Let $\eta \in H^1(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1))$ be an element defining the extension E. Let $\Pi_{0,1}$ and $\Pi_{1,0}$ be the natural projections from the space of 1-forms onto spaces of $(0,1)$- and $(1,0)$-forms, respectively. By the same symbols we denote mappings of the corresponding cohomology groups induced by $\Pi_{0,1}$ and $\Pi_{1,0}$. Hence

$$\Pi_{0,1}(\eta) \in H^1(M, \mathrm{Hom}(\mathbf{E}_2 \otimes_{\mathbb{C}} \mathcal{O}_M, \mathbf{E}_1 \otimes_{\mathbb{C}} \mathcal{O}_M)),$$

$$\Pi_{1,0}(\eta) \in H^1(M, \mathrm{Hom}(\mathbf{E}_2 \otimes_{\mathbb{C}} \bar{\mathcal{O}}_M, \mathbf{E}_1 \otimes_{\mathbb{C}} \bar{\mathcal{O}}_M)).$$

PROPOSITION 3.6. *The classes of extensions equivalent to $h(E)$ and $\bar h(E)$ are uniquely defined by $\Pi_{0,1}(\eta)$ and $\Pi_{1,0}(\eta)$, respectively.*

Proof. In the case of $h(E)$, the result follows directly from de Rham's and Dolbeault's theorems applied to the second column of the commutative diagram

$$\begin{array}{ccc} H^0(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_2)) & \xrightarrow{\delta} & H^1(M, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1)) \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \Pi_{0,1}} \\ H^0(M, \mathrm{Hom}(\mathbf{E}_2 \otimes_{\mathbb{C}} \mathcal{O}_M, \mathbf{E}_2 \otimes_{\mathbb{C}} \mathcal{O}_M)) & \xrightarrow{\delta} & H^1(M, \mathrm{Hom}(\mathbf{E}_2 \otimes_{\mathbb{C}} \mathcal{O}_M, \mathbf{E}_1 \otimes_{\mathbb{C}} \mathcal{O}_M)). \end{array}$$

The case of $\bar h(E)$ is similar. □

3.3

In this section we collect several facts on the class $\mathcal{SB}$ of bundles with connected solvable complex Lie groups as structure groups.

(a) The class $\mathcal{SB}$ is closed under tensor products and duality; that is, E^* and $E \otimes D$ belong to $\mathcal{SB}$ together with E and D.

(b) Every element $E \in \mathcal{SB}$ can be thought of as a vector bundle with structure group $T_n(\mathbb{C})$ (for some n).

Actually, according to the Lie theorem, for any connected solvable subgroup G of $\mathrm{GL}_n(\mathbb{C})$ there exists a matrix $B \in \mathrm{GL}_n(\mathbb{C})$ such that $B^{-1}GB$ is imbedded as a subgroup in the group $T_n(\mathbb{C})$. Moreover, let E have one of the structures: holomorphic, antiholomorphic, or flat. Then the foregoing transform generates an isomorphism of E preserving this structure.

(c) Every $E \in \mathcal{SB}$ is the result of successive extensions of bundles with triangular structure groups by means of rank-1 vector bundles.

Indeed, for the action of $T_n(\mathbb{C})$ on $\mathbb{C}^n$ there exists a 1-dimensional invariant subspace such that $T_{n-1}(\mathbb{C})$ acts on the factor space. Therefore, E with structure group $T_n(\mathbb{C})$ is an extension of the bundle E_{n-1} by the bundle E_1; here E_i has structure group $T_i(\mathbb{C})$ $(i = 1 \text{ or } n-1)$.

Let

$$\{0\} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E$$

and

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = F$$

be isomorphic flags of bundles with triangular structure groups. According to Proposition 3.5, this isomorphism is defined (in corresponding local coordinates on E and F) by triangular matrices.

Let now

$$\mathrm{Gr}^* E := \bigoplus_{i=1}^{n} E_i/E_{i-1}$$

be the associated graded vector bundle with cocycle defined as the diagonal of the cocycle of E.

(d) The bundle E is isomorphic to $\mathrm{Gr}^* E$ in the category of C^∞-bundles.

By Proposition 3.3, every vector bundle E over M with structure group $T_n(\mathbb{C})$ is defined by E_1, E_{n-1}, and an element $H^1(M, \mathrm{Hom}(\mathbf{E}_{n-1}, \mathbf{E}_1))$. But the latter group is trivial in the category of C^∞-bundles, because $\mathrm{Hom}(\mathbf{E}_{n-1}, \mathbf{E}_1)$ is a fine sheaf.

As a corollary we have the following statement.

(e) Every bundle $E \in \mathcal{V}_D(T_n(\mathbb{C}))$ is $T_n(\mathbb{C})$-isomorphic to the direct sum of topologically trivial vector bundles $M \times \mathbb{C}$.

(f) The class $\bigcup_{n\geq 1} \mathcal{V}_D(T_n(\mathbb{C}))$ is closed under tensor products and duality.

3.4

In this section we recall some facts of Hodge theory.

Let E be a flat vector bundle with structure group $U_n(\mathbb{C})$ over a compact Kähler manifold M. Then the operator of differentiation d is well-defined on the set $\mathcal{E}(E)$ of E-valued forms and determines a connection on E compatible with the complex structure and the flat Hermitian metric on E. Let $Z_d^{p,q}(E)$ be the space of d-closed E-valued (p, q)-forms. As usual, one defines the cohomology groups of E by

$$H^{p,q}(E) := Z_d^{p,q}(E)/(d\mathcal{E}(E) \cap Z_d^{p,q}(E)),$$

$$\mathbf{H}^{p,q}(E) := \{\eta \in \mathcal{E}^{p,q}(E),\ \Delta_d \eta = 0\}, \qquad \mathbf{H}_d^r := \{\eta \in \mathcal{E}^r(E),\ \Delta_d \eta = 0\},$$

where Δ_d denotes the d-Laplacian on E.

Let $H^r(M, \mathbf{E})$ denote the Čech cohomology of the sheaf $\mathbf{E}$ of locally constant sections of E.

PROPOSITION 3.7 (Hodge Decomposition).

$$H^r(M, \mathbf{E}) \cong \bigoplus_{p+q=r} \mathbf{H}^{p,q}(E) \cong \bigoplus_{p+q=r} H^{p,q}(E), \quad \overline{\mathbf{H}^{p,q}(E)} \cong \mathbf{H}^{q,p}(E^*).$$

The proof follows from Kähler's identities for the connection d. See, for example, [ABCKT, p. 104], which give the identities between Laplacians

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial},$$

where Δ_∂ and $\Delta_{\bar\partial}$ are ∂- and $\bar\partial$-Laplacians on E.

These identities and the Dolbeault theorem give also the isomorphisms

$$\mathbf{H}^{p,q}(E) \cong H^{p,q}_{\bar\partial}(E) \cong H^q(M, \Omega^p_M \otimes_{\mathbb{C}} \mathbf{E}),$$

where Ω^p_M is the sheaf of germs of holomorphic p-forms on M.

Arguing as in the proof of the lemma in [GH, Chap. 1, Sec. 2] and applying the very same identities, we obtain the following.

LEMMA 3.8 ($\partial\bar\partial$-Lemma). *Let E be a flat bundle with structure group $U_n(\mathbb{C})$. Suppose that ω is a d-closed E-valued (p, q)-form that is ∂- or $\bar\partial$-exact. Then there exists an E-valued $(p-1, q-1)$-form κ such that*

$$\omega = \partial\bar\partial(\kappa).$$

3.5

In this section we collect several facts on relations between equations of type (2.1) and vector bundles $\mathrm{Hom}(E_1, E_2)$.

We begin with the equation

$$df = \omega_1 f - f\omega_2, \tag{3.6}$$

where ω_1, ω_2 satisfy (1.1). The right side can be written as $(1 \otimes \omega_1 - \omega_2^t \otimes 1) f$, where f is now thought of as n^2-vector. The mapping i_d in the following proposition is defined as in Section 3.1.

PROPOSITION 3.9. *$i_d(1 \otimes \omega_1 - \omega_2^t \otimes 1)$ is a flat vector bundle isomorphic to* $\mathrm{Hom}(i_d(\omega_2), i_d(\omega_1))$.

Proof. Let $\{U_i\}_{i\in I}$ be an open covering of M, and let $f_{ki} \in C^\infty(U_i, \mathrm{GL}_n(\mathbb{C}))$ be a solution on U_i of equation (2.1) with $\omega = \omega_k$ $(k = 1, 2)$. Then

$$\begin{aligned} d((f_{2i}^t)^{-1} \otimes f_{1i}) &= -\omega_2^t (f_{2i}^t)^{-1} \otimes f_{1i} + (f_{2i}^t)^{-1} \otimes \omega_1 f_{1i} \\ &= (1 \otimes \omega_1 - \omega_2^t \otimes 1)((f_{2i}^t)^{-1} \otimes f_{1i}). \end{aligned}$$

This means that equation (3.6) is locally solvable and defines a flat vector bundle with cocycle

$$\{((f_{2i}^t)^{-1} \otimes f_{1i})^{-1} \cdot ((f_{2j}^t)^{-1} \otimes f_{1j})\} := \{(c_{2ij}^t)^{-1} \otimes c_{1ij}\}.$$

Here $\{c_{kij} := f_{ki}^{-1} f_{kj}\}$ is a cocycle determining flat vector bundle $i_d(\omega_k)$, $k = 1, 2$. Moreover, $\{(c_{2ij}^t)^{-1}\}$ is a cocycle determining conjugate vector bundle $(i_d(\omega_2))^*$ (see [GH, Chap. 0]). This implies that $i_d(1 \otimes \omega_1 - \omega_2^t \otimes 1)$ is a flat vector bundle isomorphic to $(i_d(\omega_2))^* \otimes (i_d(\omega_1))$. But the latter is isomorphic to $\mathrm{Hom}(i_d(\omega_2), i_d(\omega_1))$. □

Let now η be a vector-valued (p, q)-form on M satisfying

$$\bar{\partial}\eta = \Pi_{0,1}(\omega) \wedge \eta, \tag{3.7}$$

where ω satisfies (1.1) and $\Pi_{0,1}$ is the natural projection from $\mathcal{E}^1(M)$ onto $\mathcal{E}^{0,1}(M)$. Let us check that η is a $\bar{\partial}$-closed $i_d(\omega)$-valued (p, q)-form. Clearly, η is a section of $i_d(\omega)$ that is C^∞-isomorphic to the vector bundle $M \times \mathbb{C}^n$ (for some n). Furthermore, in flat coordinates on $i_d(\omega)$ determined by flat connection ω, the section η is given by the family

$$\{\eta_i := f_i^{-1}\eta\}_{i \in I}.$$

Here f_i is a local solution on U_i of equation (2.1) with the form ω. From the definition of f_i it follows that

$$\bar{\partial}(f_i^{-1}\eta) = -(f_i^{-1}\Pi_{0,1}(\omega)) \wedge \eta + f_i^{-1}(\Pi_{0,1}(\omega) \wedge \eta) = 0.$$

Therefore, η is $\bar{\partial}$-closed.

Applying the same arguments in reverse order, one deduces that each $\bar{\partial}$-closed $i_d(\omega)$-valued (p, q)-form given by a family $\{\eta_i\}_{i \in I}$ defines a global form η on M, equal to $f_i\eta_i$ on U_i, satisfying (3.7). In the same way, we can also examine the equation

$$\partial\eta = \Pi_{1,0}(\omega) \wedge \eta \tag{3.8}$$

and prove that η is a ∂-closed $i_d(\omega)$-valued (p, q)-form. Here $\Pi_{1,0}\colon \mathcal{E}^1(M) \to \mathcal{E}^{1,0}(M)$ is the natural projection.

Finally, let us consider the equations

$$\bar{\partial}\eta = \Pi_{0,1}(\omega_1) \wedge \eta + (-1)^{p+q+1}\eta \wedge \Pi_{0,1}(\omega_2), \tag{3.9}$$

$$\partial\psi = \Pi_{1,0}(\omega_1) \wedge \psi + (-1)^{p+q+1}\psi \wedge \Pi_{1,0}(\omega_2), \tag{3.10}$$

with matrix (p, q)-forms η and ψ. They can be written in equivalent forms as

$$\bar{\partial}\eta = (1 \otimes \Pi_{0,1}(\omega_1) - \Pi_{0,1}(\omega_2^t) \otimes 1) \wedge \eta,$$

$$\partial\psi = (1 \otimes \Pi_{1,0}(\omega_1) - \Pi_{1,0}(\omega_2^t) \otimes 1) \wedge \psi,$$

where η and ψ are thought of as *vector* (p, q)-forms.

Together, the results proved above for such equations yield the following.

PROPOSITION 3.10. *There exists a one-to-one correspondence between solutions of equations* (3.9) (*or* (3.10)) *with* ω_i *satisfying condition* (1.1) ($i = 1, 2$) *and* $\bar{\partial}$*-closed* (∂*-closed, respectively*) (p, q)*-forms with values in* $\mathrm{Hom}(i_d(\omega_2), i_d(\omega_1))$.

4. Proof of Theorem 2.1

The proof is based on Lemmas 4.1 and 4.2. To formulate the first of these results, let T_n^u denote the subgroup of elements $A \in T_n(\mathbb{C})$ such that all of its diagonal elements belong to $U_1(\mathbb{C}) := \{z;\ |z| = 1\}$. One considers a class $\mathcal{U}_n$ of flat vector bundles F with the structure group T_n^u satisfying

$$\begin{aligned}&\bar{h}(F) \text{ is isomorphic to } \mathrm{Gr}^*\,\bar{h}(F) \text{ in the category of}\\ &\text{antiholomorphic vector bundles with structure group } T_n(\mathbb{C}).\end{aligned} \tag{4.1}$$

Let $\mathcal{U} = \bigcup_{n\geq 1}\mathcal{U}_n$. Clearly, $\mathcal{U}$ is closed under tensor products and duality.

As explained in Section 3.3(c), any bundle $F \in \mathcal{U}_n$ is a result of successive extensions of flat bundles F_i with structure group T_i^u by flat bundles F^i of complex rank 1 with structure group $U_1(\mathbb{C})$ $(i = 1, \ldots, n)$, so that $F = F_n$. From property (4.1) it follows that $\bar{h}(F_i)$ is the trivial extension of $\bar{h}(F_{i-1})$ by $\bar{h}(F^i)$. Hence, the short exact sequence of sheaves of germs of antiholomorphic p-forms $(p \geq 0)$ with values in the corresponding bundles

$$0 \longrightarrow \bar{\Omega}^p(\bar{h}(F^i)) \xrightarrow{\lambda} \bar{\Omega}^p(\bar{h}(F_i)) \xrightarrow{\kappa} \bar{\Omega}^p(\bar{h}(F_{i-1})) \longrightarrow 0 \tag{4.2}$$

is split.

For a flat vector bundle F we let $\bar{\Omega}^1_d(F)$ denote the space of F-valued d-closed antiholomorphic 1-forms. The space defines a subgroup $[\bar{\Omega}^1_d(F)]$ of $H^1(M, \mathbf{F})$. Here $\mathbf{F}$ is the sheaf of locally constant sections of F. Further, let $\Pi_{0,1}$: $H^1(M, \mathbf{F}) \to H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F})$ be the mapping induced by the projection sending a 1-form to its $(0,1)$-component.

LEMMA 4.1. *Let $F \in \mathcal{U}$. Then the following statements hold*:

(a) $\Pi_{0,1}\colon [\bar{\Omega}^1_d(F)] \to H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F})$ *is a surjection; and*
(b) *every holomorphic F-valued q-form α is d-closed.*

In addition, if α is ∂-exact then $\alpha = 0$.

Proof. We will prove the lemma by induction on the dimension i of a fibre of F.

(a) In case $i = 1$, the structure group of F_1 is $U_1(\mathbb{C})$. Then, according to the Hodge decomposition (see Section 3.4), there exists an isomorphism f: $H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}_1) \to [\bar{\Omega}^1_d(F_1)]$ such that $\Pi_{0,1} \circ f = \mathrm{id}$.

Assume now that statement (a) holds for $i - 1 \geq 1$; we will prove it for i. The definition of extensions of bundles leads to the following commutative diagram:

$$\begin{array}{ccccccc}
H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}^i) & \xrightarrow{\lambda} & H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}_i) & \xrightarrow{\kappa} & H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}_{i-1}) & \xrightarrow{\delta} & H^2(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}^i)\\
\uparrow{\scriptstyle \Pi_{0,1}} & & \uparrow{\scriptstyle \Pi_{0,1}} & & \uparrow{\scriptstyle \Pi_{0,1}} & & \\
H^1(M, \mathbf{F}^i) & \xrightarrow{\lambda} & H^1(M, \mathbf{F}_i) & \xrightarrow{\kappa} & H^1(M, \mathbf{F}_{i-1}) & \xrightarrow{\delta} & H^2(M, \mathbf{F}^i)\,.
\end{array}$$

By de Rham's and Dolbeault's theorems, each of the elements of these cohomology groups is represented by an F-valued form. Let α be a F_i-valued $\bar{\partial}$-closed $(0,1)$-form representing an element of $H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}_i) \cong H^{0,1}_{\bar{\partial}}(M, F_i)$. According

to the above diagram and the inductive hypothesis, there exists a C^∞-section g of F_{i-1} such that

$$\kappa(\alpha) + \bar\partial(g) \in \bar\Omega^1_d(F_{i-1}).$$

Because F_i is a trivial extension in the category of C^∞-bundles, we can find a C^∞-section t of F_i such that $\kappa(t) = g$. Then $\omega := \kappa(\alpha - \bar\partial t)$ is a d-closed 1-form and thus the $(1,1)$-form

$$\alpha' := d(\alpha - \bar\partial t) = \partial(\alpha - \bar\partial t)$$

can be considered as an F^i-valued one. Since $\lambda(\alpha')$ represents 0 in $H^{1,1}_\partial(M, F_i) \cong H^1(M, \bar\Omega^1(\bar h(F_i)))$, and since the mapping

$$\lambda\colon H^1(M, \bar\Omega^1(\bar h(F^i))) \to H^1(M, \bar\Omega^1(\bar h(F_i)))$$

is an injection (by (4.2)), we can deduce that

$$[\alpha'] = 0 \in H^1(M, \bar\Omega^1(F^i)).$$

Hence α' is a d-closed ∂-exact $(1,1)$-form with values in a flat vector bundle with structure group $U_1(\mathbb{C})$. Then, according to the $\partial\bar\partial$-lemma of Section 3.4, there exists a C^∞-section s of F^i such that

$$\partial\bar\partial(s) = \alpha'.$$

We now set

$$\beta := \alpha - \bar\partial t - \bar\partial(\lambda(s)).$$

Then β is a $\bar\partial$-closed $(0,1)$-form such that

$$[\beta] = [\alpha] \in H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}_i) \quad \text{and} \quad d\beta = 0.$$

Therefore, β represents an element $\tilde\beta$ of $[\bar\Omega^1_d(F_i)]$ such that $\Pi_{0,1}(\tilde\beta) = [\alpha]$. The proof of part (a) is complete.

(b) We again make use of induction on i. Let ω_i be a F_i-valued holomorphic q-form. In case $i = 1$, the Hodge identity for Laplacians (see Section 3.4) acquires the form

$$\Delta_d(\omega_1) = 2\Delta_{\bar\partial}(\omega_1) = 0,$$

and from this it follows that $d\omega_1 = 0$. In addition, if ω_1 is ∂-exact then its d-harmonicity implies $\omega_1 = 0$.

Assume now that statement (b) holds for $i - 1 \geq 1$; we will prove it for i. By the induction hypothesis we have $d(\kappa(\omega_i)) = 0$. But $d(\kappa(\omega_i)) = \kappa(d\omega_i)$ and so $d\omega_i$ can be regarded as a d-closed F^i-valued holomorphic $(q+1)$-form.

It is clear, as well, that

$$[d\omega_i] = [\partial\omega_i] \in H^{q+1,0}_\partial(M, F^i) \cong H^{q+1}(M, \bar{\mathcal{O}}_M \otimes_{\mathbb{C}} \mathbf{F}^i).$$

Now, on account of (4.2), the mapping

$$\lambda\colon H^{q+1}(M, \bar{\mathcal{O}}_M \otimes_{\mathbb{C}} \mathbf{F}^i) \to H^{q+1}(M, \bar{\mathcal{O}}_M \otimes_{\mathbb{C}} \mathbf{F}_i)$$

is an injection. On the other hand, $\lambda([\partial\omega_i]) = 0$ and hence $[\partial\omega_i] = 0$ in $H^{q+1}(M, \bar{\mathcal{O}}_M \otimes_{\mathbb{C}} \mathbf{F}^i)$. Taking into account the aforementioned identity for Laplacians in the 1-dimensional case, we can deduce that $\partial\omega_i$ is a d-harmonic F^i-valued form. Since it is ∂-exact, we have $\partial\omega_i = d\omega_i = 0$.

It remains to prove that if ω_i is, in addition, a ∂-exact form, then it equals 0. But in this case $\kappa(\omega_i)$ is a ∂-exact F_{i-1}-valued holomorphic form; consequently, $\kappa(\omega_i) = 0$ by the induction hypothesis. Hence ω_i can be regarded as a F^i-valued holomorphic form. Moreover, according to the equality

$$H^q(M, \bar{\mathcal{O}}_M \otimes_{\mathbb{C}} \mathbf{F}_i) = H^q(M, \bar{\mathcal{O}}_M \otimes_{\mathbb{C}} \mathbf{F}^i) \oplus H^q(M, \bar{\mathcal{O}}_M \otimes_{\mathbb{C}} \mathbf{F}_{i-1})$$

(see (4.2)), ω_i is ∂-exact. Therefore, ω_i is a F^i-valued ∂-exact holomorphic form and thus it equals 0, as we have already shown at the first step of the induction. □

Let us now suppose that ω_2 is a triangular $(0,1)$-form of the class $\mathcal{A}_{\bar{\partial}}(t_n)$ (see Section 3.1 for the definition of this class). Here t_n denotes the Lie algebra of $T_n(\mathbb{C})$.

LEMMA 4.2. *The following conditions are equivalent*:

(i) *for ω_2, Theorem 2.1 holds*;

(ii) *there exists a T_n^u-topologically trivial flat vector bundle $F \in \mathcal{U}$ such that*

$$h(F) = i_{\bar{\partial}}(\omega_2) \ (\in \mathcal{V}_{\bar{\partial}}(T_n(\mathbb{C}))).$$

Proof.

(i) $\Rightarrow$ (ii) According to Theorem 2.1, there exists a form $\eta \in \mathcal{A}_d(t_n)$ with the canonical decomposition $\eta = \eta_1 + \eta_2$ such that

$$[\omega_2] = [\eta_2] \in \mathcal{B}_{\bar{\partial}}(t_n) \quad \text{and} \quad \operatorname{diag}(\eta_2) = -\overline{\eta_1}.$$

Since $\operatorname{diag}(\eta) = \eta_1 - \overline{\eta_1}$ is $(\sqrt{-1}\cdot\mathbb{R})^n$-valued, the form η defines a unique element of $\mathcal{B}_d(t_n^u)$. Here t_n^u is the Lie algebra of T_n^u, which clearly consists of elements $A \in T_n(\mathbb{C})$ with $\operatorname{diag}(A) \in (\sqrt{-1}\cdot\mathbb{R})^n$. Therefore, the flat bundle $i_d(\eta)$ has structure group T_n^u (see Section 3.1).

Now we make use of the identities

$$\bar{h}(i_d(\eta)) = i_\partial(\eta_1) \in \mathcal{V}_\partial(T_n(\mathbb{C})), \qquad h(i_d(\eta)) = i_{\bar{\partial}}(\omega_2) \in \mathcal{V}_{\bar{\partial}}(T_n(\mathbb{C}))$$

(see Proposition 3.1 for details). But η_1 is a diagonal matrix form, and thus the first identity implies that $\bar{h}(i_d(\eta))$ is isomorphic to $\operatorname{Gr}^* \bar{h}(i_d(\eta))$ in the category of antiholomorphic vector bundles with structure group $T_n(\mathbb{C})$. Therefore $i_d(\eta)$ belongs to the class $\mathcal{U}$ of flat vector bundles with structure groups T_n^u and is $T_n(\mathbb{C})$-topologically trivial by the definition of the class $\mathcal{V}_D(T_n(\mathbb{C}))$. Moreover, every $T_n(\mathbb{C})$-topologically trivial vector bundle with structure group T_n^u is T_n^u-topologically trivial. Bearing in mind the second identity, we deduce now that $i_d(\eta)$ can be taken as the bundle F of statement (ii).

(ii) $\Rightarrow$ (i) Let F be the vector bundle of statement (ii). According to the results of Section 3.1, there exists a form $\theta \in \mathcal{A}_d(t_n^u)$ with the canonical decomposition $\theta = \theta_1 + \theta_2$ such that

$$i_d(\theta) = F.$$

In particular, we have $\operatorname{diag}(\theta_1) = -\operatorname{diag}(\overline{\theta_2})$. Moreover, as established in the first part of the proof, the equalities

$$i_\partial(\theta_1) = \bar{h}(i_d(\theta)) = \bar{h}(F) = \bar{h}(\mathrm{Gr}^* F) = \bar{h}(i_d(\operatorname{diag}(\theta))) = i_\partial(\operatorname{diag}(\theta_1))$$

hold in the class $\mathcal{V}_\partial(T_n(\mathbb{C}))$. This implies the existence of a ∂-gauge transform ∂_g with a triangular matrix function g such that

$$\partial_g(\theta_1) = \operatorname{diag}(\theta_1).$$

Then, for $\psi := d_g(\theta)$ we have

$$i_d(\psi) = i_d(\theta) = F,$$

and the first component ψ_1 of the canonical decomposition $\psi = \psi_1 + \psi_2$ equals $\partial_g(\theta_1)$, (i.e., is a diagonal $(1,0)$-form). Moreover,

$$i_d(\operatorname{diag}(\psi)) = i_d(\operatorname{diag}(\theta))$$

in the category of flat vector bundles with the diagonal matrix structure group. This implies the existence of a d-gauge transform d_h with a diagonal matrix function h such that

$$\bar{\partial}_h(\operatorname{diag}(\psi_2)) = -\overline{\psi_1}.$$

Putting now

$$\eta := d_h(\psi)$$

we have defined a t_n-valued 1-form such that $\operatorname{diag}(\eta_2) = -\overline{\eta_1}$. So η satisfies the conditions of Theorem 2.1.

It remains to define a triangular form ω with the second component ω_2 in its canonical decomposition satisfying $\eta = d_q(\omega)$ for some $T_n(\mathbb{C})$-valued function q. To accomplish this, we note that

$$i_{\bar\partial}(\theta_2) = h(F) = i_{\bar\partial}(\omega_2)$$

and therefore $\bar{\partial}_p(\theta_2) = \omega_2$ for some $T_n(\mathbb{C})$-valued function p. If we set $\omega := d_p(\theta)$, then ω satisfies condition (1.1) because $\theta \in \mathcal{A}_d(t_n^u)$. Moreover, $d_q(\omega) = \eta$ where $q := hgp^{-1}$. □

PROOF OF THEOREM 2.1. Let $\omega_2 \in \mathcal{A}_{\bar\partial}(t_n)$. According to Lemma 4.2, we must find a T_n^u-topologically trivial flat vector bundle $F \in \mathcal{U}$ such that

$$h(F) = i_{\bar\partial}(\omega_2) \in \mathcal{V}_{\bar\partial}(T_n(\mathbb{C})).$$

We will prove this by induction on the rank n of the holomorphic vector bundle $i_{\bar\partial}(\omega_2)$. This bundle is a result of successive extensions of holomorphic vector bundles $V_i \in \mathcal{V}_{\bar\partial}(T_i(\mathbb{C}))$ by rank-1 holomorphic vector bundles $V^i \in \mathcal{V}_{\bar\partial}(\mathbb{C}^*)$, $i = 1, \dots, n-1$ (see Section 3.3). In particular, $i_{\bar\partial}(\omega_2)$ is an extension of V_{n-1} by V^{n-1}.

We begin with the observation that every rank-1 holomorphic vector bundle $V \in \mathcal{V}_{\bar\partial}(\mathbb{C}^*)$ is determined by an equation $\bar\partial f = \kappa f$, with a 1-form κ satisfying the condition $\bar\partial\kappa = 0$. Moreover, a $\bar\partial$-gauge transform $\bar\partial_g$ in this case has the form

$$\omega \mapsto \omega - g^{-1}\bar\partial g, \quad g \in \mathbb{C}^\infty(M, \mathbb{C}^*).$$

Now we are in a position to prove the result for the 1-dimensional case. Let V and κ be as before. Since M is a compact Kähler manifold, there exists a function $r \in C^\infty(M)$ such that $\gamma = \kappa - \bar\partial r$ is a harmonic form and, in particular, is d-closed. It is clear that $\bar\partial_g(\gamma) = \kappa$, where $g = \exp(-r)$. Let us consider now the locally solvable equation

$$df = (\gamma - \bar\gamma) f.$$

It follows that $d_g(\gamma - \bar\gamma) = \sigma + \kappa$, where $\sigma = -\bar\gamma + g^{-1}\partial g$. Hence, we obtain

$$h(i_d(\gamma - \bar\gamma)) = i_{\bar\partial}(\kappa) = V.$$

But $\gamma - \bar\gamma \in \sqrt{-1} \cdot \mathbb{R}$ and therefore $i_d(\gamma - \bar\gamma) \in \mathcal{V}_d(U_1(\mathbb{C}))$. It remains to set

$$F := i_d(\gamma - \bar\gamma).$$

Let us assume that the result holds for rank $n-1$; we will prove it for n. Toward this end, let $i_{\bar\partial}(\omega_2)$ be an extension of V_{n-1} by V^{n-1}. According to the induction hypothesis, there exist bundles $F_{n-1} \in \mathcal{V}_d(t^u_{n-1}) \cap \mathcal{U}$ and $F^{n-1} \in \mathcal{V}_d(U_1(\mathbb{C}))$ such that

$$h(F_{n-1}) = V_{n-1} \quad \text{and} \quad h(F^{n-1}) = V^{n-1}.$$

From this it follows that the sheaves $\mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}_{n-1}$ and $\mathcal{O}_M \otimes_{\mathbb{C}} \mathbf{F}^{n-1}$ determine V_{n-1} and V^{n-1}, respectively (see Section 3.1). By Proposition 3.3 there exists an element δ of $H^1(M, \mathcal{O}_M \otimes_{\mathbb{C}} \mathrm{Hom}(\mathbf{F}_{n-1}, \mathbf{F}^{n-1}))$ that determines V_n. Since the flat bundle $\mathrm{Hom}(F_{n-1}, F^{n-1})$ is isomorphic to $(F_{n-1})^* \otimes F^{n-1}$ and thus belongs to $\mathcal{U}$, we can apply Lemma 4.1. By the lemma there exists an element $\beta \in [\bar\Omega^1_d(\mathrm{Hom}(F_{n-1}, F^{n-1}))] \subseteq H^1(M, \mathrm{Hom}(\mathbf{F}_{n-1}, \mathbf{F}^{n-1}))$ such that

$$\Pi_{0,1}(\beta) = \delta \quad \text{and} \quad \Pi_{1,0}(\beta) = 0.$$

Moreover, β defines an extension F_n of F_{n-1} by F^{n-1} by Proposition 3.3. From these two statements and Proposition 3.6, we conclude that

$$h(F_n) = V_n \quad \text{and} \quad \bar h(F_n) = \bar h(F_{n-1}) \oplus \bar h(F^{n-1}).$$

But $F_{n-1} \in \mathcal{U}$ by the induction hypothesis and so the latter direct sum equals

$$\bigoplus_{k=1}^{n-1} \bar h(F^k) \oplus \bar h(F_1) = \mathrm{Gr}^* \bar h(F_n).$$

Thus F_n belongs to $\mathcal{U}$. Furthermore, F_n is an extension of the bundle F_{n-1} by the bundle F^{n-1}, and by the induction hypothesis these two bundles are T^u_{n-1}- and T^u_1-topologically trivial, respectively. Hence, F_n is T^u_n-topologically trivial and so the proof is complete. □

REMARK 4.3. If in Theorem 2.1 the form ω_2 is nilpotent, then it is $\bar\partial$-gauge equivalent to an antiholomorphic nilpotent form.

5. Proof of Theorem 1.2

In order to prove the theorem we must prove propositions of Section 2.

PROOF OF PROPOSITION 2.2. (1) Let ψ be a diagonal $\bar\partial$-closed $(0,1)$-form on M. According to the Hodge decomposition,

$$\psi = \tilde{\psi} + \bar{\partial} f, \tag{5.1}$$

where $\tilde{\psi}$ is a diagonal harmonic $(0,1)$-form. Put $h_\psi := \exp(f)$. Then we have $d_{h_\psi}(\omega) \in \mathcal{E}_{\tilde{\psi}}$ for any $\omega \in \mathcal{E}_\psi$.

(2) Let ψ be a diagonal harmonic $(0,1)$-form on a compact Kähler manifold M (which, in particular, is d-closed antiholomorphic). Then we determine a flat vector bundle E_ψ over M as $E_\psi = i_d(\psi - \bar{\psi})$. As follows from arguments used in the proof of Theorem 2.1, $E_\psi \in \mathcal{U}^n_\oplus$; that is, it is a direct sum of rank-1 topologically trivial flat vector bundles with structure group $U_1(\mathbb{C})$. Moreover, each element of $\mathcal{U}^n_\oplus$ coincides with $i_d(\psi - \bar{\psi})$ for some diagonal harmonic $(0,1)$-form ψ. This proves part (a).

Let now $\omega \in \mathcal{E}_\psi$; that is, it has a triangular $(0,1)$-component ω_2 such that $\operatorname{diag}(\omega_2) = \psi$ and satisfies (1.1). Further, define the mapping τ_ψ by

$$\tau_\psi(\omega) := \omega - (\psi - \bar{\psi}). \tag{5.2}$$

The latter form can be thought of as a 1-form with values in the flat vector bundle $\operatorname{End}(E_\psi)$ whose $(0,1)$-component is nilpotent. In fact, let $\{g_i\}_{i\in I}$ be a family of invertible diagonal matrix functions defined on an open covering $\{U_i\}_{i\in I}$ and satisfying

$$dg_i = (\psi - \bar{\psi})g_i, \quad i \in I.$$

Then, in a flat coordinate system on $\operatorname{End}(E_\psi)$, the form $\tau_\psi(\omega)$ is given by the family $\{\theta_i := g_i^{-1}\tau_\psi(\omega)g_i\}_{i\in I}$. Clearly, the $(0,1)$-component of θ_i is nilpotent and hence $\tau_\psi(\omega)$ is, by definition, an $\operatorname{End}(E_\psi)$-valued form with a nilpotent $(0,1)$-component. Simple calculation—based on the identities

$$d\omega - \omega \wedge \omega = 0$$

and

$$d(\psi - \bar{\psi}) = (\psi - \bar{\psi}) \wedge (\psi - \bar{\psi}) = 0$$

and the diagonality of g_i and ψ—yields

$$d\theta_i - \theta_i \wedge \theta_i = 0, \quad i \in I.$$

This proves part (c).

Let $h \in C^\infty(M, T_n(\mathbb{C}))$. Then h determines an element from the group $\operatorname{Aut}^t_\infty(E_\psi)$ of triangular automorphisms of E_ψ given by the family $\{h_i := g_i^{-1}hg_i\}_{i\in I}$. Substituting these expressions in the definition of the d-gauge transform $d_h^{E_\psi}$ and taking into account diagonality of g_i and ψ, we obtain $\tau_\psi \circ d_h = d_h^{E_\psi} \circ \tau_\psi$. This proves part (b).

To finish the proof of the proposition, observe that the mapping τ_ψ defined on $\mathcal{E}_\psi$ by (5.2) is injective; it has the inverse defined on the set of $\operatorname{End}(E_\psi)$-valued 1-forms with nilpotent $(0,1)$-components satisfying (1.1). □

PROOF OF PROPOSITION 2.3. In proving the proposition we make use of the relation between elements of $\mathcal{E}_\psi$ with a diagonal harmonic $(0,1)$-form ψ and $\operatorname{End}(E_\psi)$-valued locally solvable equations with nilpotent $(0,1)$-components (see Proposition 2.2).

Let $\omega \in \mathcal{E}_\psi$ and let $\eta := \tau_\psi(\omega)$ be an $\mathrm{End}(E_\psi)$-valued differential 1-form with a nilpotent $(0,1)$-component satisfying the analog of (1.1). As follows from Theorem 2.1, ω can be reduced by a d-gauge transform d_g with $g \in C^\infty(M, T_n(\mathbb{C}))$ to a form $\omega' \in \mathcal{E}_\psi$ with the type decomposition $\omega'_1 + \omega'_2$ such that $\omega'_2 - \mathrm{diag}(\overline{\omega'_2}) \in \mathcal{E}_\psi$. Set now

$$\tilde{\eta}_1 := \tau_\psi(\omega'_1 + \mathrm{diag}(\omega'_2)) \quad \text{and} \quad \tilde{\eta}_2 := \tau_\psi(\omega'_2 - \mathrm{diag}(\overline{\omega'_2})).$$

Then clearly $\tau_\psi(\omega'_1 + \omega'_2) = \tilde{\eta}_1 + \tilde{\eta}_2$ (type decomposition). According to Proposition 2.2(2)(b), $d_g^{E_\psi}(\eta) = \tau_\psi(\omega') = \tilde{\eta}_1 + \tilde{\eta}_2$, where g is now thought of as an element of $\mathrm{Aut}^t_\infty(E_\psi)$. It remains to prove that $\partial\tilde{\eta}_1 = 0$ and that $\tilde{\eta}_2$ is a d-closed antiholomorphic 1-form.

By definition, the $(0,1)$-form $\tilde{\eta}_2$ satisfies the $\mathrm{End}(E_\psi)$-valued equation (1.1). This implies immediately that $\tilde{\eta}_2$ is antiholomorphic and (owing to the Hodge decomposition; see Section 3.4) d-closed. Next we prove that $\tilde{\eta}_1$ is ∂-closed. To accomplish this we observe that conditions (1.1) for forms $\omega'_1 + \omega'_2$ and $\omega'_2 - \mathrm{diag}(\overline{\omega'_2})$ include, in particular, the following identities:

$$\bar{\partial}\omega'_1 = \omega'_1 \wedge \omega'_2 + \omega'_2 \wedge \omega'_1 - \partial\omega'_2; \tag{5.3}$$

$$\bar{\partial}(-\mathrm{diag}(\overline{\omega'_2})) = (-\mathrm{diag}(\overline{\omega'_2})) \wedge \omega'_2 + \omega'_2 \wedge (-\mathrm{diag}(\overline{\omega'_2})) - \partial\omega'_2. \tag{5.4}$$

Then, subtracting the second equation from the first we obtain

$$\bar{\partial}(\omega'_1 + \mathrm{diag}(\overline{\omega'_2})) = (\omega'_1 + \mathrm{diag}(\overline{\omega'_2})) \wedge \omega'_2 + \omega'_2 \wedge (\omega'_1 + \mathrm{diag}(\overline{\omega'_2})). \tag{5.5}$$

Consider now the flat vector bundle $F := i_d(\omega'_2 - \mathrm{diag}(\overline{\omega'_2}))$. From (5.5) it follows that $\omega'_1 + \mathrm{diag}(\overline{\omega'_2})$ is an $\mathrm{End}(F)$-valued holomorphic 1-form (see Section 3.5). Since F belongs to the class $\mathcal{U}$, which is closed with respect to tensor products and duality, Lemma 4.1(b) implies in this case that $\omega'_1 + \mathrm{diag}(\overline{\omega'_2})$ is a d-closed $\mathrm{End}(F)$-valued form. But by the definition, $\mathrm{End}(F)$ is antiholomorphically isomorphic to $\mathrm{End}(\mathrm{Gr}^* F)$, which in turn coincides with $\mathrm{End}(E_\psi)$ (see the proof of Proposition 2.2). This shows that $\omega'_1 + \mathrm{diag}(\overline{\omega'_2})$ (regarded now as an $\mathrm{End}(E_\psi)$-valued 1-form) is ∂-closed. It remains to note that the latter form coincides with $\tilde{\eta}_1$. The proof of Proposition 2.3 is complete. □

Let now $\{\tilde{\eta}_1\}$ and $\{\tilde{\eta}_2\}$ be the harmonic components in the Hodge decomposition of $\mathrm{End}(E_\psi)$-valued forms $\tilde{\eta}_1$ and $\tilde{\eta}_2$, respectively. Then the $\mathrm{End}(E_\psi)$-valued condition (1.1) and the $\partial\bar{\partial}$-lemma of Section 3.4 yield

$$[\{\tilde{\eta}_i\}, \{\tilde{\eta}_i\}] = 0, \quad i = 1, 2;$$

$$[\{\tilde{\eta}_1\}, \{\tilde{\eta}_2\}] \text{ represents } 0 \text{ in } H^2(M, \mathrm{End}(E_\psi)).$$

Proof of Proposition 2.5. Let α_1, β_1 be $\mathrm{End}(E_\psi)$-valued $(1,0)$-forms, and let α_2, β_2 be $\mathrm{End}(E_\psi)$-valued d-closed nilpotent $(0,1)$-forms. Recall that E_ψ is a direct sum of rank-1 topologically trivial flat vector bundles with unitary structure group. Suppose that

$$d_g^{E_\psi}(\alpha_1+\alpha_2)=\beta_1+\beta_2 \tag{5.6}$$

for some C^∞-automorphism g of E_ψ, and that $\alpha_1+\alpha_2$ and $\beta_1+\beta_2$ belong to $\tau_\psi(\mathcal{E}_\psi)$.

We have to prove that g is flat. According to Propositions 2.2 and 2.3, there exist triangular $(0,1)$-forms θ_1 and θ_2 such that:

(i) $\tau_\psi(\theta_1-\bar\psi)=\alpha_2$ and $\tau_\psi(\theta_2-\bar\psi)=\beta_2$;
(ii) $\operatorname{diag}(\theta_i)=\psi$ for $i=1,2$; and
(iii) $\theta_i-\operatorname{diag}(\overline{\theta_i})\in\mathcal{E}_\psi$ for $i=1,2$.

If we now identify the group of C^∞-automorphisms of E_ψ with $C^\infty(M,\mathrm{GL}_n(\mathbb{C}))$ (as in the case of triangular automorphisms), then—arguing as in the proof of Proposition 2.2—we obtain $\tau_\psi\circ d_g=d_g^{E_\psi}\circ\tau_\psi$. In particular, (5.6) implies

$$\bar\partial g=\theta_1 g-\theta_2 g.$$

But this is a special case of equation (3.9). Applying Proposition 3.10, we conclude that g is a holomorphic section of flat vector bundle $V:=\operatorname{Hom}\big(i_d(\theta_2-\operatorname{diag}(\overline{\theta_2})),\ i_d(\theta_1-\operatorname{diag}(\overline{\theta_1}))\big)$. This vector bundle belongs to the class $\mathcal{U}$, and therefore g is d-closed by Lemma 4.1(b). Since by definition V is antiholomorphically isomorphic to $\operatorname{End}(E_\psi)$, the automorphism g of E_ψ is ∂-closed. Applying now the Hodge decomposition of Section 3.4, we deduce that g is locally constant—that is, flat. □

PROOF OF PROPOSITION 2.4. Let α be a holomorphic $\operatorname{End}(E_\psi)$-valued form and θ an antiholomorphic nilpotent one, and let the 2-form $[\alpha+\theta,\alpha+\theta]$ represent 0 in $H^2(M,\operatorname{End}(E_\psi))$. We have to prove that there exists a section h, unique up to an additive flat summand, such that the equation

$$df=(\alpha+\theta+\partial h)f$$

is locally solvable. To accomplish this, we first remark that $\partial\bar\partial$-lemma of Section 3.4 implies that

$$\alpha\wedge\theta+\theta\wedge\alpha=[\alpha+\theta,\alpha+\theta],$$

since the form on the right represents 0 in $H^2(M,\operatorname{End}(E_\psi))$. Applying the $\partial\bar\partial$-lemma to the left-hand side and taking into account the holomorphicity of α, we obtain

$$\bar\partial\alpha-\alpha\wedge\theta-\theta\wedge\alpha=\partial\bar\partial P \tag{5.7}$$

for some C^∞-section P of $\operatorname{End}(E_\psi)$. Since by assumption $d\theta-\theta\wedge\theta=0$, arguments similar to those of Proposition 2.2 show that there exists a triangular $(0,1)$-form η defined on M such that $\eta-\operatorname{diag}(\bar\eta)$ satisfies (1.1), $\operatorname{diag}(\eta)=\psi$, and $\tau_\psi(\eta-\operatorname{diag}(\bar\eta))=\theta$. Then, in the global C^∞-coordinates on $\operatorname{End}(E_\psi)$ (chosen as in the proof of Proposition 2.2), (5.7) can be written as

$$\bar\partial\alpha'-\eta\wedge\alpha'-\alpha'\wedge\eta=\partial\beta+\operatorname{diag}(\bar\eta)\wedge\beta+\beta\wedge\operatorname{diag}(\bar\eta)$$

(see Section 3.5). Here $\alpha := g_i^{-1}\alpha' g_i$ on U_i, $\bar\partial P := g_i^{-1}\beta g_i$ on U_i, and $\{g_i\}_{i\in I}$ is a family of invertible diagonal matrix functions satisfying $dg_i = (\psi - \bar\psi)g_i$ on U_i.

Consider now the flat vector bundle $F := i_d(\eta - \mathrm{diag}(\bar\eta))$ of the class $\mathcal{U}$. If we think of α as an $\mathrm{End}(F)$-valued $(1,0)$-form (F is C^∞-trivial), then the left-hand side of the previous expression determines its $\bar\partial$-differential. But the right-hand side shows that $\bar\partial\alpha$ is a ∂-exact $\mathrm{End}(F)$-valued form. The proof of the theorem will be complete if we find a C^∞-section h of $\mathrm{End}(F)$ such that $\alpha + \partial h$ is a holomorphic $\mathrm{End}(F)$-valued form. Actually, let $\{f_i\}_{i\in I}$ be a family of triangular invertible C^∞-functions determined on the open covering $\{U_i\}_{i\in I}$ (the same covering as for $\{g_i\}_{i\in I}$ previously) and satisfying $df_i = (\eta - \mathrm{diag}(\bar\eta))f_i$, $i \in I$. Then the holomorphicity of $\alpha + \partial h$ is equivalent to the equation

$$\bar\partial(\alpha' + \gamma) = (\alpha' + \gamma)\wedge\eta + \eta\wedge(\alpha' + \gamma),$$

where $\gamma = f_i\partial h f_i^{-1}$ on U_i. The latter equation, in turn, determines the $\mathrm{End}(E_\psi)$-valued equation

$$\bar\partial(\alpha + \tilde\gamma) = (\alpha + \tilde\gamma)\wedge\theta + \theta\wedge(\alpha + \tilde\gamma). \tag{5.8}$$

Here $\tilde\gamma = g_i^{-1}\gamma g_i$ on U_i.

Clearly, $\partial(g_i^{-1}f_i) = 0$ and therefore

$$\tilde\gamma = \partial(g_i^{-1}f_i h f_i^{-1} g_i)$$

on U_i. But $\{g_i^{-1}f_i h f_i^{-1}g_i\}_{i\in I}$ determines a section $\tilde h$ of $\mathrm{End}(E_\psi)$, so $\tilde\gamma = \partial\tilde h$. Equation (5.8) is one of the conditions of local solvability contained in (1.1). Observe that (1.1) in our case is equivalent to the fulfillment of (5.8) together with the identity

$$(\alpha + \partial\tilde h)\wedge(\alpha + \partial\tilde h) = 0, \tag{5.9}$$

since $\partial(\alpha + \partial\tilde h) = 0$ by assumptions of the proposition.

To check this identity, we first note that $\mathrm{End}(E_\psi)$ is antiholomorphically isomorphic to $\mathrm{End}(F)$. This isomorphism is given locally by conjugations by matrix functions $f_i^{-1}g_i$ ($i \in I$) and so it commutes with the operator $\wedge$. Therefore, it suffices to prove an identity similar to (5.9) for $\alpha + \partial h$. Here α is thought of as an $\mathrm{End}(F)$-valued section (image of α by the previous isomorphism). Furthermore, since $\alpha\wedge\alpha = 0$ we have

$$(\alpha + \partial h)\wedge(\alpha + \partial h) = \partial(h\alpha - \alpha h + h\partial h). \tag{5.10}$$

This implies that the $\mathrm{End}(F)$-valued holomorphic 1-form $\alpha + \partial h$ is ∂-exact. Applying Lemma 4.1(b) to this form, one concludes that the identity (5.9) holds. The uniqueness part of the proposition follows from the fact that there is a unique (up to a flat additive summand) section h such that $\alpha + \partial h$ is $\mathrm{End}(F)$-valued holomorphic (see Lemma 4.1(b)).

Thus it remains to find the section h such that $\alpha + \partial h$ is a holomorphic $\mathrm{End}(F)$-valued 1-form. We do this by a procedure reducing the n-dimensional statement to the $(n-1)$-dimensional one; here n is the dimension of a fibre of $\mathrm{End}(F)$.

We begin with the following remark. Since $\mathrm{End}(F) \in \mathcal{U}$ it can be regarded as an extension of a rank-1 flat vector bundle F_1 with unitary structure group by a flat vector bundle $F_{n-1} \in \mathcal{U}$. In other words, the following sequence of flat vector bundles

$$0 \longrightarrow F_{n-1} \xrightarrow{i} \mathrm{End}(F) \xrightarrow{j} F_1 \longrightarrow 0$$

is exact. We can analogously represent F_{n-1} as an extension of a rank-1 flat vector bundle with unitary structure group by a flat vector bundle $F_{n-2} \in \mathcal{U}$, and so on. In particular, F_0 is a vector bundle over M with null-dimensional fibre. In the next part of the proof we let the same letters i, j denote the corresponding mappings induced by i, j on the space of differential forms.

Let us consider now the F_1-valued ∂-exact $(1,1)$-form $j(\bar\partial\alpha) = \bar\partial(j(\alpha))$. Since $\partial\alpha = 0$, we have $\bar\partial\alpha = d\alpha$ and hence $j(\bar\partial\alpha)$ is a d-exact F_1-valued 1-form. The $\partial\bar\partial$-lemma implies then that

$$\bar\partial(j(\alpha)) = \bar\partial\partial(g)$$

for $g \in C^\infty(F_1)$. Because $\mathrm{End}(F)$ is a trivial extension of F_1 by F_{n-1} in the category of C^∞-bundles, there exists an $\mathrm{End}(F)$-valued C^∞-section k_1 such that $j(k_1) = g$.

If we put now $\alpha_1 := \alpha - \partial k_1$, then

$$\partial\alpha_1 = \partial\alpha = 0 \quad \text{and} \quad \bar\partial(j(\alpha_1)) = j(\bar\partial\alpha_1) = j(\bar\partial\alpha - \bar\partial\partial k_1) = 0.$$

It follows from the second identity that $\bar\partial\alpha_1$ can be regarded as an F_{n-1}-valued form. Since $\mathrm{End}(F) = F_1 \oplus F_{n-1}$ in the class of antiholomorphic vector bundles, the mapping

$$i\colon H^1(M, \bar\Omega^1(F_{n-1})) \to H^1(M, \bar\Omega^1(\mathrm{End}(F)))$$

is an injection. Furthermore, the ∂-exactness of $\bar\partial\alpha$ implies that

$$i([\bar\partial\alpha_1]) = 0 \in H^1(M, \bar\Omega^1(\mathrm{End}(F)))$$

and so $[\bar\partial\alpha_1] = 0 \in H^1(M, \bar\Omega^1(F_{n-1}))$. From this it follows that $\bar\partial\alpha_1$ is an F_{n-1}-valued ∂-exact form.

Starting with the F_{n-1}-valued form $\bar\partial\alpha_1$ and proceeding in the same way, we can now find a C^∞-section k_2 such that, for

$$\alpha_2 := \alpha_1 - \partial k_2 = \alpha - \partial k_1 - \partial k_2,$$

$\bar\partial\alpha_2$ is an F_{n-2}-valued ∂-exact $(1,1)$-form. Continuing in this fashion, we obtain after n steps the form $\alpha_n := \alpha_{n-1} - \partial k_{n-1}$ such that $\bar\partial\alpha_n$ is an F_0-valued ∂-exact $(1,0)$-form; that is, $\bar\partial\alpha_n = 0$. If we now set

$$h := -\sum_{i=1}^{n} k_i,$$

then $\alpha + \partial h$ equals the holomorphic $\mathrm{End}(F)$-valued 1-form α_n. □

REMARK 5.1. If the form α of Proposition 2.4 is, in addition, triangular, then the section h can also be chosen as triangular.

In fact, let t_n be the Lie algebra of the Lie group $T_n(\mathbb{C})$ of upper triangular matrices. The vector space t_n is invariant with respect to the linear operators $(A^t)^{-1} \otimes A\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with $A \in T_n(\mathbb{C})$. Hence there exists a subbundle $T \in \mathcal{U}$ of the bundle $\mathrm{End}(F)$ in the proof of Proposition 2.4 with a fibre isomorphic to t_n. In fact, the latter bundle is defined by a cocycle of the form

$$\{(c_{ij}^t)^{-1} \otimes c_{ij};\ c_{ij} \in T_n(\mathbb{C})\}$$

(see Section 3.5). Since the form α is t_n-valued and $T \in \mathcal{U}$, it follows that α is a T-valued $(1,0)$-form. We can now apply the arguments of the proof of Theorem 2.4 to α but with T instead of $\mathrm{End}(F)$. In this way, we obtain the required section h but in this case with values in $T_n(\mathbb{C})$.

6. Proof of Theorems 1.3 and 1.5

Proof of Theorem 1.3. Let $V_2(M)$ be a class of flat vector bundles over M whose elements are constructed by homomorphism from $S_2^u(M)$. According to the assumptions, for any $E \in V_2(M_1)$ there exists an $F \in V_2(M_2)$ such that $f^*F \cong E$. Moreover, every such bundle E is, by definition, determined by $\mathrm{Gr}^* E = E_1 \oplus E_2$ and an element of $H^1(M_1, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1))$. Here E_1, E_2 are topologically trivial rank-1 flat vector bundles with unitary structure group. Then the conditions of the theorem imply the following.

Statement. *For every topologically trivial rank-*1 *flat vector bundle V_1 over M_1 with unitary structure group, there exists a topologically trivial flat vector bundle V_2 over M_2 with unitary structure group such that $f^*V_2 = V_1$ and $f^*(H^1(M_2, \mathbf{V}_2)) = H^1(M_1, \mathbf{V}_1)$.*

Let now $\rho\colon \pi_1(M_1) \to \mathrm{GL}_n(\mathbb{C})$ be a homomorphism of the class $S_n(M_1)$. Then, according to Theorem 1.2, we have that ρ is uniquely defined by the $\mathrm{End}(E')$-valued harmonic $(1,0)$-form α and a harmonic nilpotent $(0,1)$-form η satisfying $[\alpha+\eta, \alpha+\eta]$ represents 0 in $H^2(M_1, \mathrm{End}(E'))$. Here E' is a direct sum of topologically trivial rank-1 flat vector bundles with unitary structure group. Furthermore, from the Statement it follows that there exist a flat vector bundle F' over M_2 that is isomorphic to a direct sum of topologically trivial rank-1 flat vector bundles with unitary structure group and an $\mathrm{End}(F')$-valued harmonic $(1,0)$-form α' and a harmonic nilpotent $(0,1)$-form η' such that

$$f^*\,\mathrm{End}(F') = \mathrm{End}(E'), \quad f^*(\alpha') = \alpha, \quad f^*(\eta') = \eta.$$

In addition, assume that $[\alpha', \eta']$ represents 0 in $H^2(M_2, \mathrm{End}(F'))$. The foregoing conditions imply also that $[\alpha', \alpha'] = [\eta', \eta'] = 0$ and hence the triple $(\mathrm{End}(F'), \alpha', \eta')$ determines a representation $\rho' \in S_n(M_2)$. Then the uniqueness part of Theorem 1.2 (see Proposition 2.4) yields $\rho = \rho' \circ f_*$.

Thus it remains to prove that $[\alpha', \eta']$ represents 0 in $H^2(M_2, \mathrm{End}(F'))$. Note that $f^*([\alpha', \eta']) = [\alpha, \eta]$ represents 0 in $H^2(M_1, \mathrm{End}(E'))$. The required statement is then a consequence of the following general result.

Let $f: N \to M$ be a surjective mapping of compact Kähler manifolds, and let E be a flat vector bundle over M with unitary structure group.

PROPOSITION 6.1. *Let $\alpha \in \mathcal{E}^{1,1}(E)$ be a d-closed E-valued form. If $f^*(\alpha) \in \mathcal{E}^{1,1}(f^*E)$ is d-exact, then α is also d-exact.*

Proof. Consider the flat vector bundle f^*E over N and the d-exact form $f^*(\alpha) \in \mathcal{E}^{1,1}(f^*E)$. Because this bundle has unitary structure group and N is a compact Kähler manifold, there exists an $h \in C^\infty(f^*E)$ such that $f^*(\alpha) = \bar\partial\partial h$. Let

$$N \xrightarrow{p_1} Y \xrightarrow{p_2} M$$

be the Stein factorization of f (here the fibres of p_1 are connected and p_2 is a finite analytic covering). For a point $x \in M$, consider an open neighborhood U_x of x such that $E|_{U_x}$ is the trivial flat vector bundle. Then f^*E is trivial over $f^{-1}(U_x)$ and, for any fibre V of f over a point of U_x, the restriction $\alpha_V := f^*(\alpha)|_V = 0$. This implies that $h|_V$ is locally constant. (To prove this fact in the case of singular V, one must pull back h to its desingularization.) Then there exists a section h' of p_2^*E such that $p_2^*\alpha = \bar\partial\partial h'$ on nonsingular part of Y.

Consider now the average of h' over points of regular fibres of p_2,

$$h''(y) := \frac{1}{\#\{p_2^{-1}(y)\}} \sum_{z \in p_2^{-1}(y)} h'(z), \quad y \in M.$$

Clearly h'' is a bounded section of E, smooth at regular values of p_2, and so $\alpha = \bar\partial\partial h''$ outside of a proper analytic subset of M. Moreover, according to assumptions of the proposition, α is locally $\bar\partial\partial$-exact. Further, boundedness of h'' together with regularity of the operator $\partial\bar\partial$ implies that h'' can be extended to M as a C^∞-section of E satisfying $\alpha = \bar\partial\partial h''$. This shows that α is d-exact. □

The proof of Theorem 1.3 is complete. □

PROOF OF THEOREM 1.5. Let $\tau: \pi_1(M_1) \to T_2^u$ be a representation of the class $S_2^u(M_1)$. Clearly $\mathrm{Ker}(\tau)$ contains $\pi_1(M_1)''$ and so τ determines a homomorphism $\tau_1: \pi_1(M_1)/\pi_1(M_1)'' \to T_2^u$. Furthermore, according to the assumptions of the theorem, there exists a homomorphism $\tau_2: \pi_1(M_2)/\pi_1(M_2)'' \to T_2^u$ such that $\tau_1 = \tau_2 \circ f_*$ whose diagonal elements have the logarithm. Obviously, we can extend τ_2 to a homomorphism $\tau': \pi_1(M_2) \to T_2^u$ of the class $S_2^u(M_2)$ satisfying $\tau = \tau' \circ f_*$. Thus, the conditions of Theorem 1.3 are fulfilled. According to this theorem, for any representation $\rho: \pi_1(M_1) \to \mathrm{GL}_n(\mathbb{C})$ of the class $S_n(M_1)$ there exists a representation $\rho': \pi_1(M_2) \to \mathrm{GL}_n(\mathbb{C})$ such that $\rho = \rho' \circ f_*$. The latter, in particular, shows that $\mathrm{Ker}\, f_*$ belongs to the kernel of every matrix representation of the class $S_n(M_1)$, $n \geq 1$. But by the assumption of the theorem, $\pi_1(M_1)$ belongs to the class S. Therefore $\mathrm{Ker}\, f_* = \{e\}$ and f_* is an injective homomorphism. Finally, from the Stein factorization of f, one obtains that $f_*(\pi_1(M_1))$ is a subgroup of a finite index in $\pi_1(M_2)$. □

7. Concluding Remarks

All results of this paper hold true also for the class of manifolds dominated by a compact Kähler. We recall the following definition.

DEFINITION 7.1. A manifold M is said to be *dominated* by a compact Kähler manifold N if there exists a complex surjective mapping $f: N \to M$.

Let M be a manifold dominated by a compact Kähler manifold $N: N \xrightarrow{f} M$, and let E be a flat vector bundle over M with unitary structure group. The proof of the following is similar to that of Proposition 6.1.

PROPOSITION 7.2. (a) *Let $\alpha \in \mathcal{E}^{0,1}(E)$ be an E-valued $\bar{\partial}$-closed $(0,1)$-form. Then there exists a C^∞-section h of E such that $\alpha - \bar{\partial}h$ is d-closed.*

(b) *Let the E-valued $(1,1)$-form β satisfy $d\beta = 0$ and $\beta = \partial\gamma$ for some E-valued $(0,1)$-form γ. Then there exists an E-valued function g such that $\beta = \partial\bar{\partial}g$.*

Using this result and applying the very same arguments, one can prove the validity of the results of this paper for the class of manifolds dominated by compact Kähler ones.

In [BO] we describe the de Rham 1-cohomology $H^1_{\mathrm{DR}}(M, G)$ of a compact Kähler manifold M with values in a solvable complex linear algebraic group G of a special class. The result obtained is similar to Theorem 1.2.

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