On the Length of Lemniscates

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For a monic polynomial p of degree d, we write $E(p) := \{z : |p(z)| = 1\}$. A conjecture of Erdős, Herzog and Piranian [4], repeated by Erdős in [5, Prob. 4.10] and elsewhere, is that the length $|E(p)|$ is maximal when $p(z) := z^d + 1$. It is easy to see that, in this conjectured extremal case, $|E(p)| = 2d + O(1)$ when $d \to \infty$.

The first upper estimate $|E(p)| \le 74d^2$ is due to Pommerenke [10]. Recently, Borwein [2] gave an estimate that is linear in d, namely

$$|E(p)| \le 8\pi ed \approx 68.32d.$$

Here we improve Borwein's result.

Let α_0 be the least upper bound of perimeters of the convex hulls of compact connected sets of logarithmic capacity 1. The precise value of α_0 is not known, but Pommerenke [8] proved the estimate $\alpha_0 < 9.173$. The conjectured value is $\alpha_0 = 3^{3/2}2^{2/3} \approx 8.24$.

Theorem 1. *For monic polynomials p of degree d, $|E(p)| \le \alpha_0 d < 9.173d$.*

A similar problem for rational functions turns out to be much easier, and can be solved completely by means of Lemma 1.

Theorem 2. *Let f be a rational function of degree d. Then the spherical length of the preimage under f of any circle C is at most d times the length of a great circle.*

This is best possible, as shown by the example of $f(z) = z^d$ and $C = \mathbf{R}$.

Remarks. Borwein notices that his method would give the estimate $4\pi d \approx 12.57d$ if one knew one of the following facts: (a) the precise estimate of the size of the exceptional set in Cartan's lemma (Lemma 3 here); or (b) for extremal polynomials, the set $E(p)$ is connected. It turns out that (b) is true (this is our Lemma 3), and in addition we can improve from 4π to 9.173 by using more precise arguments than those of Borwein.

The main property of the level sets $E(p)$ is the following.

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LEMMA 1. *For every rational function f of degree d, the f-preimage of any line or circle has no more than $2d$ intersections with any line or circle C, except finitely many C.*

Proof. The group of fractional-linear transformations acts transitively on the set of all circles on the Riemann sphere, and a composition of a rational function with a fractional-linear transformation is a rational function of the same degree.

Thus it is enough to prove that, for a rational function f of degree d, the set $F := \{z : f(z) \in \mathbf{R}\}$ has at most $2d$ points of intersection with the real line $\mathbf{R}$, unless $\mathbf{R} \subset F$. Let z_0 be such a point of intersection. Then z_0 is a zero of the rational function $f_1(z) = f(z) - \overline{f(\bar{z})}$. But f_1 evidently has degree at most $2d$ and thus cannot have more than $2d$ zeros, unless $f_1 \equiv 0$. □

The length of sets described in Lemma 1 can be estimated using the following lemma, in which we denote by π_x and π_y the orthogonal projections onto a pair of perpendicular coordinate axes.

LEMMA 2. *If an analytic curve Γ intersects each vertical and horizontal line at most n times, then $|\Gamma| \leq n(|\pi_x(\Gamma)| + |\pi_y(\Gamma)|)$.*

Proof. We break the curve Γ into finitely many pieces l_j such that every l_j intersects each vertical or horizontal line at most once. Then we have

$$|l_j| \leq |\pi_x(l_j)| + |\pi_y(l_j)|.$$

We obtain this by approximating l_j by broken lines whose segments are parallel to the coordinate axes. Adding these inequalities for all pieces and using the fact that both projection maps are at most n-to-1 on Γ, we obtain the result. □

COROLLARY. *Every connected subset l of $E(p)$ has the property*

$$|l| \leq 2d(|\pi_x(l)| + |\pi_y(l)|) \leq 4d\,\mathrm{diam}(l).$$

LEMMA 3 (H. Cartan, see e.g. [7, p. 19]). *For a monic polynomial p of degree d, the set $\{z : |p(z)| < M\}$ is contained in the union of discs the sum of whose radii is $2eM^{1/d}$.*

Pommerenke [9, Satz 3] improved the constant $2e$ in this lemma to 2.59, but we will not use this result.

Now we can prove the existence of extremal polynomials for our problem.

LEMMA 4. *The length $|E(p)|$ is a continuous function of the coefficients of p. For every positive integer d there exists a monic polynomial p_d with the property $|E(p_d)| \geq |E(p)|$ for every monic polynomial p of degree d.*

Proof. Every monic polynomial of degree d can be written as

$$p(z) = \prod_{j=1}^{d}(z - z_j).$$

We consider vectors $Z = (z_1, \ldots, z_d)$ in $\mathbf{C}^d$ and denote by p_Z the monic polynomial with the zero-set Z.

First we show that $|E(p)| \to 0$ as $\operatorname{diam} Z \to \infty$, $p = p_Z$. Let M be a number such that $M > (4e)^d$. If the diameter of the set Z is greater than $4Md$, then we can split Z into two parts, $Z = Z_1 \cup Z_2$, such that $\operatorname{dist}(Z_1, Z_2) > 4M$.

Indeed, Let D be the union of closed discs of radii $2M$ centered at the points $z_1, \ldots, z_d$. If D is connected, then $\operatorname{diam} D \le 4Md$, contradicting our assumption. Thus D is disconnected—that is, $D = D_1 \cup D_2$, where D_1 and D_2 are disjoint compact sets. We set $Z_i = Z \cap D_i$ for $i = 1, 2$, which proves our assertion.

Consider two polynomials

$$p_k(z) := \prod_{w \in Z_k} (z - w), \quad k = 1, 2,$$

so that $p = p_1 p_2$. By Lemma 3, the union L of two sets

$$L_k := \{ z : |p_k(z)| < M^{-1} \}, \quad k = 1, 2,$$

can be covered by discs the sum of whose radii is $4eM^{-1/d} < 1$. Thus, the sum of the lengths of the projections of L satisfies

$$|\pi_x(L)| + |\pi_y(L)| \le 16eM^{-1/d}. \tag{1}$$

On the other hand, each component of a set L_k contains a zero of p_k and has diameter less than 2, so that $\operatorname{dist}(L_1, L_2) > 4M - 4 > 2M$ since $M > 4e$.

Next we show that $E(p) \subset L_1 \cup L_2$. Indeed, suppose that $z \in E(p)$. Assume without loss of generality that $\operatorname{dist}(z, L_1) \le \operatorname{dist}(z, L_2)$. Then $\operatorname{dist}(z, L_2) > M$ and thus $|p_2(z)| > M$, so that

$$|p_1(z)| = |p(z)|/|p_2(z)| < M^{-1},$$

and this implies that $z \in L_1$.

We conclude that $|\pi_x(E(p))| + |\pi_y(E(p))| \le 16eM^{-1/d}$, which tends to 0 as $M \to \infty$. Now an application of the corollary following Lemma 2 concludes the proof of our assertion that $|E(p)| \to 0$ as $\operatorname{diam} Z \to \infty$.

Now we show that $|E(p_Z)|$ is a continuous function of the vector

$$Z = (z_1, \ldots, z_d) \in \mathbf{C}^d.$$

Consider the multivalued algebraic function

$$q(Z, w) = (d/dw)(p^{-1}(w)).$$

The coefficients of the algebraic equation defining this function q are polynomials of Z and w, and $q(Z, w) \neq 0$ in $\mathbf{C}^d \times \mathbf{C}$ because this is a derivative of an inverse function. So all branches of q are continuous with respect to w and Z at every point where these branches are finite (see e.g. [6, Thm. 12.2.1]). Denoting by $\mathbf{T}$ the unit circle, we have

$$|E(p)| = \int_{\mathbf{T}} Q(Z, w)|dw|, \quad \text{where} \;\; Q(Z, w) = \sum |q(Z, w)|;$$

the summation is over all values of the multivalued function q. To show that this integral is a continuous function of the parameter Z, we will verify that the family of functions $w \mapsto \sum |q(Z, w)|$, $\mathbf{T} \to \mathbf{R}$ has a uniform integrability property.

Let K be an arc of the unit circle of length $\delta < \pi/6$. Then this arc is contained in a disc $D(w, r)$ of radius $r = \delta/2$, centered at the middle point w of the arc K. By Lemma 3 applied to $p - w$, the full preimage $p^{-1}D(w, r)$ can be covered by discs the sum of whose radii is at most $2er^{1/d}$. So the sum of the vertical and horizontal projections of $p^{-1}D(w, r)$ is at most $8er^{1/d}$. Finally, by the corollary after Lemma 2, the length of the part of $E(p)$ that is mapped to K is at most $\varepsilon := 16der^{1/d} = 16de(\delta/2)^{1/d}$. Thus

$$\int_K Q(Z, w)|dw| < \varepsilon, \tag{2}$$

where $\varepsilon \to 0$ as $\delta \to 0$ uniformly with respect to Z.

Suppose now that $Z_0 \in \mathbf{C}^d$. Consider the points $w_1, \ldots, w_k$ on the unit circle $\mathbf{T}$ such that $Q(Z_0, w_j) = \infty$. Then $k \le d - 1$, because a polynomial p of degree d can have no more than $d - 1$ critical points. Given that $\varepsilon > 0$, we choose open arcs K_j such that $w_j \in K_j \subset \mathbf{T}$ $(1 \le j \le k)$ and that (2) is satisfied with $K = \bigcup_j K_j$ whenever $Z \in \mathbf{C}^d$. Now we have $Q(Z, w) \to Q(Z_0, w)$ as $Z \to Z_0$ uniformly with respect to w in $\mathbf{T} \backslash K$, so that

$$\left| \int_{\mathbf{T}} Q(Z, w)|dw| - \int_{\mathbf{T}} Q(Z_0, w)|dw| \right| \le 3\varepsilon$$

when Z in $\mathbf{C}^d$ is close enough to Z_0.

We have proved that $Z \mapsto |E(p_Z)|$ is a continuous function in $\mathbf{C}^d$ and that $|E(p_Z)| \to 0$ as $Z \to \infty$. It follows that a maximum of $|E(p)|$ exists. To show that $|E(p)|$ is a continuous function of the coefficients, we again refer to the well-known fact [6, Thm. 12.2.1] that the zeros of a monic polynomial are continuous functions of its coefficients. □

In what follows we will call *extremal* any polynomial p that maximizes $|E(p)|$ in the set of all monic polynomials of degree d.

LEMMA 5. *There exists an extremal polynomial p such that all critical points of p are contained in $E(p)$.*

REMARKS. From this lemma it follows that the polynomial $z^2 + 1$ is extremal for $d = 2$. The level set $\{z : |z^2 + 1| = 1\}$ is known as Bernoulli's lemniscate (it is also one of Cassini's ovals), and its length is expressed by the elliptic integral

$$4 \int_0^1 \frac{1}{\sqrt{1 - x^4}}\, dx \approx 7.4163.$$

This curve, as well as the integral played an important role in the history of mathematics (see e.g. [11]).

Proof of Lemma 5. Let p be a polynomial and a a critical value of p such that a does not lie on the unit circle $\mathbf{T}$. Let U be an open disc centered at a such that

U does not contain other critical values. Let $\Phi\colon \mathbf{C} \to \mathbf{C}$ be a smooth function whose support is contained in U and such that $\Phi(a) = 1$. If $\lambda \in \mathbf{C}$ and λ satisfies $|\lambda| < \varepsilon := (\max_U |\operatorname{grad} \Phi|)^{-1}$, then the map $\phi_\lambda\colon \mathbf{C} \to \mathbf{C}$, $\phi_\lambda(z) = z + \lambda\Phi(z)$, is a smooth quasiconformal homeomorphism of $\mathbf{C}$. Hence we have a family of quasiconformal homeomorphisms depending analytically on λ for $|\lambda| < \varepsilon$.

The composition $q_\lambda := \phi_\lambda \circ p$ is a family of quasiregular maps of the plane into itself. We denote by μ_f the Beltrami coefficient of a quasiregular map f; that is, $\mu_f := f_{\bar z}/f_z$, where $f_z := \partial f/\partial z$ and $f_{\bar z} := \partial f/\partial \bar z$. By the chain rule (see e.g. [1, p. 9]),

$$\mu_{q_\lambda} = \left(\frac{|p'|}{p'}\right)^2 \mu_{\phi_\lambda} \circ p, \tag{3}$$

so that μ_{q_λ} depends analytically on λ for $|\lambda| < \varepsilon$. According to the existence and analytic dependence on parameter theorems for the Beltrami equation (see e.g. [3, Chap. I, Thms. 7.4 and 7.6]), there exists a family of quasiconformal homeomorphisms $\psi_\lambda\colon \mathbf{C} \to \mathbf{C}$, that (a) satisfies the Beltrami equations

$$\mu_{\psi_\lambda} = \mu_{q_\lambda},$$

(b) is normalized by $\psi_\lambda = z + o(1)$, $z \to \infty$, and (c) depends analytically on λ for $|\lambda| < \varepsilon$ for every fixed z.

It follows that

$$p_\lambda := q_\lambda \circ \psi_\lambda^{-1} = \phi_\lambda \circ p \circ \psi_\lambda^{-1}$$

are entire functions. Since they are all d-to-1, they are polynomials of degree d, and the normalization of ψ implies that these polynomials are monic. These polynomials p_λ may be considered as obtained from p by shifting one critical value from a to $a + \lambda$, while all other critical values remain unchanged. The functions $\lambda \mapsto p_\lambda(z)$ are continuous (in fact, analytic) for every z. Thus the coefficients of p_λ are continuous functions of λ. It follows by Lemma 4 that $|E(p_\lambda)|$ is a continuous function of λ.

Now we assume that p is an extremal polynomial and that a critical value a of p does not belong to the unit circle $\mathbf{T}$. Then we can choose the disc U in the preceding construction such that U does not intersect the unit circle. As ϕ_λ is conformal outside U, we conclude from (3) that q_λ and thus ψ_λ are conformal away from $p^{-1}(U)$. This implies that ψ_λ is conformal in the neighborhood of $E(p)$, and we have

$$|E(p_\lambda)| = |\psi_\lambda(E(p))| = \int_{E(p)} \left|\frac{d\psi_\lambda}{dz}\right| |dz|.$$

Since ϕ_λ depends analytically on λ, so does $d\psi_\lambda/dz$; thus, for every fixed z, $|d\psi_\lambda/dz|$ is a subharmonic function of λ for $|\lambda| < \varepsilon$. It follows that $|E(p_\lambda)|$ is subharmonic for $|\lambda| < \varepsilon$. Because we assumed that p is extremal, this subharmonic function has a maximum at the point 0, so it is constant.

Now we consider all critical values $a_1, \dots, a_n$ of p that do not belong to the unit circle $\mathbf{T}$, and we connect each a_j with $\mathbf{T}$ by a curve γ_j such that all these curves are disjoint and do not intersect $\mathbf{T}$ except at one endpoint. Performing the deformation described previously, we move all critical values a_j, one at a time, along γ_j to the unit circle; as a result, we obtain a monic polynomial p^* of degree d all of whose critical values belong to $\mathbf{T}$. This is equivalent to the property that all critical points of p^* belong to $E(p^*)$. We have $|E(p^*)| = |E(p)|$, because $|E(p)|$ remains constant as a critical value a_j is moved along γ_j. Thus p^* is also extremal. $\square$

LEMMA 6. *There exists an extremal polynomial p for which the set $E(p)$ is connected.*

Proof. Put $D = \{z \in \bar{\mathbf{C}} : |P(z)| > 1\}$ and $\Delta = \{z \in \bar{\mathbf{C}} : |z| > 1\}$. Let p be an extremal polynomial constructed as in Lemma 5. Then $p\colon D \to \Delta$ is a ramified covering of degree d having exactly one critical point of index $d-1$, namely, the point ∞. By the Riemann–Hurwitz formula, D is simply connected, so $E(p)$ is connected. $\square$

REMARKS. By moving those critical values whose moduli are greater than 1 toward infinity rather than to the unit circle, and using the arguments from the proof of Lemma 4, one can show that an extremal polynomial cannot have critical values with absolute value greater than 1. In fact, it follows that, for all extremal polynomials p, the level sets $E(p)$ are connected. We will not use this additional information in the proof of Theorem 1.

LEMMA 7 (Pommerenke [8, Satz 5]). *Let E be a connected compact set of logarithmic capacity* 1. *Then the perimeter of the convex hull of E is at most*

$$\pi\left(\sqrt{10} - 3\sqrt{2} + 4\right) < 9.173.$$

Proof of Theorem 1. Let p be an extremal polynomial with connected set $E(p)$. Such a p exists by Lemma 6. Applying Lemma 7, we conclude that the perimeter of the convex hull of E is at most 9.173.

Now the integral-geometric formula [12] for the length of a curve gives

$$|E| = \frac{1}{2}\int_0^{\pi}\int_{-\infty}^{\infty} N_E(\theta, x)\,dx\,d\theta,$$

where $N_E(\theta, x)$ is the number of intersections of E with the line

$$\{z : \Re(ze^{-i\theta}) = x\}.$$

A connected compact set E intersects exactly those lines that the boundary of its convex hull intersects. But the boundary of the convex hull intersects almost every line either 0 or 2 times, while a set $E(p)$ intersects each line at most $2\deg p$ times. Thus $|E| < 9.173d$. This proves our assertion. $\square$

Proof of Theorem 2. Following Borwein, we use the Poincaré integral-geometric formula [12]. Assuming that the great circles have length 2π, we denote by $l(E)$ the spherical length of E, by dx the spherical area element, and by $v(E, x)$ the number of intersections of E with the great circle, one of whose centers is x. The Poincaré formula is

$$l(E) = \frac{1}{4}\int v(E, x)\,dx.$$

Now, if $E(f)$ is the preimage of a circle under a rational function f of degree d, then by Lemma 1 we have that $E(f)$ intersects every great circle at most $2d$ times, so that the spherical length $l(E(f))$ is at most $2\pi d$. □

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