Light-Cone Expansion of the Dirac Sea to First Order in the External Potential

Felix Finster

1. Introduction

In relativistic quantum mechanics, the problem of the unphysical negative-energy solutions of the Dirac equation is solved by the conception that all negative-energy states are occupied in the vacuum forming the so-called Dirac sea. In [1], the Dirac sea was constructed for the Dirac equation with general interaction in terms of a formal power series in the external potential. In the present paper, we turn our attention to a single Feynman diagram of this perturbation expansion. More precisely, we will analyze the contribution to first order in the potential and derive explicit formulas for the Dirac sea in position space. Since this analysis does not require a detailed knowledge of the perturbation expansion for the Dirac sea, we can make this paper self-consistent by giving a brief introduction to the mathematical problem.

In the vacuum, the Dirac sea is characterized by the integral over the lower mass shell

$$P(x,y) = \int \frac{d^4p}{(2\pi)^4} (\not{p} + m)\, \delta(p^2 - m^2)\, \Theta(-p^0)\, e^{-ip(x-y)} \tag{1.1}$$

(Θ is the Heavyside function, $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ otherwise); $P(x,y)$ is a tempered distribution that solves the free Dirac equation $(i\not{\partial}_x - m)P(x,y) = 0$. In the case with interaction, the Dirac sea is accordingly described by a tempered distribution $\tilde{P}(x,y)$ being a solution of the Dirac equation

$$(i\not{\partial}_x + \mathcal{B}(x) - m)\tilde{P}(x,y) = 0, \tag{1.2}$$

where $\mathcal{B}$ is composed of the classical bosonic potentials. We assume $\mathcal{B}$ to be a 4×4 matrix potential satisfying the condition $\gamma^0 \mathcal{B}(x)^\dagger \gamma^0 = \mathcal{B}(x)$ ("$\dagger$" denotes the transposed, complex conjugated matrix). We can thus decompose it in the form

$$\mathcal{B} = e\not{A} + e\gamma^5 \not{B} + \Phi + i\gamma^5 \Xi + \sigma_{jk} H^{jk} \tag{1.3}$$

with the electromagnetic potential A_j, an axial potential B_j, scalar and pseudoscalar potentials Φ and Ξ, and a bilinear potential H^{jk} (see e.g. [7] for a discussion of these potentials). In Appendix B, it is shown how the results can be extended to an external gravitational field.

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The Dirac equation (1.2) can be solved by a perturbation expansion. To first order in $\mathcal{B}$, one has

$$\tilde{P}(x, y) = P(x, y) + \Delta P(x, y) + \mathcal{O}(\mathcal{B}^2),$$

where ΔP satisfies the inhomogeneous Dirac equation

$$(i\partial\!\!\!/_x - m)\Delta P(x, y) = -\mathcal{B}(x) P(x, y). \tag{1.4}$$

The factor $(i\partial\!\!\!/_x - m)$ can be inverted with a Green's function: We choose as Green's function the sum of the retarded and advanced Green's functions,

$$s(x, y) = \frac{1}{2} \lim_{0<\varepsilon\to 0} \sum_{\pm} \int \frac{d^4p}{(2\pi)^4} \frac{p\!\!\!/ + m}{p^2 - m^2 \pm i\varepsilon p^0} e^{-ip(x-y)}. \tag{1.5}$$

According to its definition, s satisfies the equation

$$(i\partial\!\!\!/_x - m) s(x, y) = \delta^4(x - y). \tag{1.6}$$

As a consequence, the integral

$$\Delta P(x, y) := -\int d^4z (s(x, z)\mathcal{B}(z) P(z, y) + P(x, z)\mathcal{B}(z) s(z, y)) \tag{1.7}$$

is a solution of (1.4).

Clearly, (1.7) is not the only solution of the inhomogeneous Dirac equation (1.4). For example, we could have worked with the advanced or retarded Green's function instead of (1.5) or could have omitted the second summand in (1.7). The special form of our solution follows from the causality principle for the Dirac sea, which was introduced and discussed in [1]. We will not repeat these considerations here, and simply take (1.7) as an ad hoc formula for the perturbation of the Dirac sea. The reader who feels uncomfortable with this procedure is either referred to [1] or can, in a simplified argument, explain the special form (1.7) from the "Hermiticity condition"

$$\Delta P(x, y)^\dagger = \gamma^0 \Delta P(y, x) \gamma^0,$$

which seems quite natural to impose.

In the language of Feynman diagrams, (1.7) is a first-order tree diagram. In comparison to diagrams of higher order or to loop diagrams, this is a very simple diagram, and it might seem unnecessary to study the diagram further. Unfortunately, (1.7) gives no information on what ΔP explicitly looks like in position space. We are especially interested in the behavior of $\Delta P(x, y)$ in a neighborhood of the light cone $(y - x)^2 \equiv (y - x)_j (y - x)^j = 0$.

DEFINITION 1.1. A tempered distribution $A(x, y)$ is of the order $\mathcal{O}((y - x)^{2p})$, $p \in \mathbf{Z}$, if the product

$$(y - x)^{-2p} A(x, y)$$

is a regular distribution (i.e., a locally integrable function). It has the *light-cone expansion*

$$A(x, y) = \sum_{j=g}^{\infty} A^{[j]}(x, y) \tag{1.8}$$

if the distributions $A^{[j]}(x, y)$ are of the order $\mathcal{O}((y-x)^{2j})$ and if A is approximated by the partial sums in the way that

$$A(x, y) - \sum_{j=g}^{p} A^{[j]}(x, y) \quad \text{is of the order } \mathcal{O}((y-x)^{2p+2}) \tag{1.9}$$

for all $p \geq g$.

The first summand $A^{[g]}(x, y)$ gives the leading order of $A(x, y)$ on the light cone. If A is singular on the light cone, g will be negative. Notice that the $A^{[j]}$ are determined only up to contributions of higher order $\mathcal{O}((y-x)^{2j+2})$, but this will not lead to any problems in the following.

We point out that we do not study the convergence of the sum (1.8); we only make a statement on the approximation of A by the finite partial sums. The reason why questions of convergence are excluded is that the distributions $A^{[j]}$ will typically involve partial derivatives of order $2j$ of the potential $\mathcal{B}$, and we can thus expect convergence only if $\mathcal{B}$ is analytic (for nonanalytic functions, the partial derivatives may increase arbitrarily fast in the order of the derivative, which makes convergence impossible). Analyticity of the potential, however, is too strong a condition for physical applications; we can assume only that $\mathcal{B}$ is smooth (the reason why analytic functions are too restrictive is that they are completely determined from their behavior in a small open set, which contradicts causality). Thus, the infinite sum in (1.8) is merely a convenient notation for the approximation by the partial sums (1.9). Despite this formal character of the sum, the light-cone expansion completely describes the behavior of $A(x, y)$ near the light cone. This situation can be seen in analogy to writing down the Taylor expansion for a smooth, nonanalytic function. Although the Taylor series does not converge in general, the Taylor polynomials give local approximations of the function.

Our aim is to derive explicit formulas for the light-cone expansion of $\Delta P(x, y)$.

2. Discussion of the Method

Before performing the light-cone expansion, we briefly discuss the basic problem and describe the possible methods for calculating $\Delta P(x, y)$.

At first sight, our problem seems quite complicated because of the Dirac matrices in s, in P, and in the potential (1.3). Actually, this is not the difficult point; we can reduce to a scalar problem by pulling all Dirac matrices out of the integral (1.7) as follows. We have

$$\begin{aligned} P(x, y) &= (i\partial\!\!\!/_x + m) T_{m^2}(x, y) = (-i\partial\!\!\!/_y + m) T_{m^2}(x, y), \\ s(x, y) &= (i\partial\!\!\!/_x + m) S_{m^2}(x, y) = (-i\partial\!\!\!/_y + m) S_{m^2}(x, y), \end{aligned} \tag{2.1}$$

where T_{m^2} and S_{m^2} denote the negative-energy eigenspace and the Green's function of the Klein–Gordon operator, respectively:

$$T_{m^2}(x,y) = \int \frac{d^4p}{(2\pi)^4}\, \delta(p^2-m^2)\,\Theta(-p^0)\, e^{-ip(x-y)}, \tag{2.2}$$

$$S_{m^2}(x,y)\frac{1}{2} \lim_{0<\varepsilon\to 0} \sum_{\pm} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2-m^2 \pm i\varepsilon p^0} e^{-ip(x-y)}. \tag{2.3}$$

Using the short notation $(\gamma^a)_{a=1,\dots,16}$ for the bases $\mathbf{1}, i\gamma^5, \gamma^j, \gamma^5\gamma^j, \sigma^{jk}$ of the Dirac matrices, we can thus rewrite (1.7) in the form

$$\Delta P(x,y) = \sum_{a=1}^{16} (i\partial\!\!\!/_x + m)\gamma^a(-i\partial\!\!\!/_y + m)\,\Delta T_{m^2}[\mathcal{B}_a](x,y) \tag{2.4}$$

with

$$\begin{aligned}&\Delta T_{m^2}[V](x,y)\\ &\qquad = -\int d^4z\,(S_{m^2}(x,z)V(z)T_{m^2}(z,y) + T_{m^2}(x,z)V(z)S_{m^2}(z,y)).\end{aligned} \tag{2.5}$$

The scalar distribution $\Delta T_{m^2}[V](x,y)$ is a solution of the inhomogeneous Klein–Gordon equation

$$(-\Box_x - m^2)\,\Delta T_{m^2}(x,y) = -V(x)\,T_{m^2}(x,y), \tag{2.6}$$

as is immediately verified. Once we have derived the light-cone expansion for $\Delta T_{m^2}(x,y)$, the corresponding formula for $\Delta P(x,y)$ is obtained by calculating the partial derivatives and using the commutation rules of the Dirac matrices in (2.4), which will be a (lengthy but) straightforward computation.

We conclude that the main problem is to calculate the solution (2.5) of the Klein–Gordon equation (2.6). The simplest method is to analyze the partial differential equation (2.6). This hyperbolic equation is closely related to the wave equation, and the behavior near the light cone can be studied like the wave propagation of singularities (this method is sometimes called “integration along characteristics”; see e.g. [5]). In order to give an idea of the technique, we look at the simplified equation

$$(-\Box - m^2)\,f(x) = g(x) \tag{2.7}$$

and choose light-cone coordinates $(u=\frac{1}{2}(t+r),\ v=\frac{1}{2}(t-r),\ \vartheta,\ \varphi)$ around the origin (r, ϑ, φ are polar coordinates in $\mathbb{R}^3$). Then the $\Box$-operator takes the form

$$\Box = \partial_u\partial_v - \frac{1}{r}(\partial_u - \partial_v) - \frac{1}{r^2}\Delta_{S^2},$$

where $\Delta_{S^2} = \partial_\vartheta^2 + \cot\vartheta\,\partial_\vartheta + \sin^{-2}\vartheta\,\partial_\varphi^2$ is the spherical Laplace operator. The important point is that the $\Box$-operator contains only first derivatives in both u and v. This allows us to express the normal derivative of f on the light cone as a line integral over f and its tangential derivatives. Thus, we rewrite (2.7) on the upper light cone $u=t=r$, $v=0$, in the form

$$\partial_u(u\,\partial_v f(u,0,\vartheta,\varphi)) = \left(\partial_u - um^2 + \frac{1}{u}\Delta_{S^2}\right) f(u,0,\vartheta,\varphi) - u\,g(u,0,\vartheta,\varphi).$$

This equation can be integrated along the light cone as

$$\begin{aligned}
&u_1\,\partial_v f(u_1,0,\vartheta,\varphi) \\
&\qquad = \int_0^{u_1} \partial_u (u\,\partial_v f(u,0,\vartheta,\varphi))\,du \\
&\qquad = \int_0^{u_1} \left(\left(\partial_u - um^2 + \frac{1}{u}\Delta_{S^2} \right) f(u,0,\vartheta,\varphi) - ug(u,0,\vartheta,\varphi) \right) du \\
&\qquad = f(u_1,0,\vartheta,\varphi) - f(0,0,\vartheta,\varphi) \\
&\qquad\quad + \int_0^{u_1} \left(\left(-um^2 + \frac{1}{u}\Delta_{S^2} \right) f(u,0,\vartheta,\varphi) - ug(u,0,\vartheta,\varphi) \right) du. \qquad (2.8)
\end{aligned}$$

By iterating this method, it is possible to calculate the higher derivatives in a similar way. We conclude that knowing f on the light cone determines all its derivatives on the light cone. This makes it possible to perform the light-cone expansion. We remark that complications arise when f has singularities on the light cone. The main disadvantage of this method is that the special form of the solution (2.5) does not enter. This means, in our example, that additional input is needed to completely determine f on the light cone.

Because of these problems, it is preferable to use a different method and to directly evaluate the integral (2.5). One substitutes explicit formulas for the distributions S and T in position space and studies the asymptotic behavior of the integral for $(y-x)^2 \to 0$. This method is presented in detail in [4]. Because it is carried out purely in position space, it gives a good intuition for the behavior of ΔP near the light cone. Unfortunately, this method is rather lengthy. Furthermore, the calculation of the operator products in (2.5) and of the partial derivatives in (2.4) lead to subtle analytical difficulties.

In this paper, we use a combination of calculations in position and in momentum space, which gives a shorter and more systematic approach. It has the disadvantage that working with infinite sums in momentum space is more abstract than studying the behavior of distributions in position space. Therefore the reader may find it instructive to compare the technique of this paper with the calculations in [4].

3. The Formal Light-Cone Expansion of ΔT_{m^2}

In this section, we will perform the light-cone expansion for $\Delta T_{m^2}(x,y)$ on a formal level. The analytic justification for the expansion is postponed to the next section. We assume that $m \neq 0$ and set $a = m^2$.

Since we want to derive formulas in position space, it is useful first to consider explicitly what T_{m^2} looks like. Calculating the Fourier transform of the lower mass shell (2.2) yields an expression containing the Bessel functions J_1, Y_1, and K_1. The most convenient form for our purpose is to work with the power series for these Bessel functions, which gives

$$\begin{aligned}T_a(x,y) = &-\frac{1}{8\pi^3} \lim_{0<\varepsilon\to 0} \frac{1}{\xi^2 - i\varepsilon\xi^0} \\ &+ \frac{a}{32\pi^3} \lim_{0<\varepsilon\to 0} (\log(a\xi^2 - i\varepsilon\xi^0) + i\pi + c) \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,(l+1)!} \frac{(a\xi^2)^l}{4^l} \\ &- \frac{a}{32\pi^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,(l+1)!} \frac{(a\xi^2)^l}{4^l} (\Phi(l+1) + \Phi(l))\end{aligned} \tag{3.1}$$

with $\xi = (y-x)$, $c = 2C - \log 2$ with Euler's constant C, and the function

$$\Phi(0) = 0, \qquad \Phi(n) = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n \geq 1.$$

The logarithm is understood in the complex plane, which is cut along the positive real axis (so that $\lim_{0<\varepsilon\to 0} \log(x+i\varepsilon) = \log|x|$ is real for $x > 0$). It can be verified explicitly that T_a is a solution of the Klein–Gordon equation $(-\Box_x - a)T_a(x,y) = 0$. Furthermore, one can calculate the Fourier transform $T_a(p)$ with contour integrals. For $p^0 > 0$, the ξ^0-integral can be closed in the lower complex plane, which gives zero. In this way, one immediately verifies that T_a is formed only of negative-energy states.

This formula for T_a looks quite complicated, and we do not need the details in this section. It suffices to observe that T_a has singularities on the light cone of the form of a pole and a δ-distribution,

$$\lim_{0<\varepsilon\to 0} \frac{1}{\xi^2 - i\varepsilon\xi^0} = \frac{\mathrm{PP}}{\xi^2} + i\pi\delta(\xi^2)\varepsilon(\xi^0),$$

where PP denotes the principal value. Furthermore, there are logarithmic and Θ-like contributions, since

$$\lim_{0<\varepsilon\to 0} \log(a\xi^2 - i\varepsilon\xi^0) + i\pi = \log(|a\xi^2|) + i\pi\Theta(\xi^2)\varepsilon(\xi^0),$$

where ε is the step function $\varepsilon(x) = 1$ for $x \geq 0$ and $\varepsilon(x) = -1$ otherwise. The important point for the following is the qualitative observation that the contributions of higher order in a contain more factors ξ^2 and are thus of higher order on the light cone. This yields the possibility of performing the light-cone expansion by expressing ΔT_a in terms of the a-derivatives of T_a. In the following lemma, we combine this idea with the fact that line integrals over the potential should occur according to (2.8). The lemma gives an explicit solution of the inhomogeneous Klein–Gordon equation (2.6) and is the key for the light-cone expansion. We use the notation

$$T_a^{(n)} = \left(\frac{d}{da}\right)^n T_a.$$

LEMMA 3.1. *The formal series*

$$A(x,y) = -\sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (\alpha - \alpha^2)^n (\Box^n V)|_{\alpha y + (1-\alpha)x}\, d\alpha\, T_a^{(n+1)}(x,y) \tag{3.2}$$

satisfies the equation

$$(-\Box_x - a)\, A(x, y) = -V(x) T_a(x, y). \tag{3.3}$$

Proof. In momentum space, T_a has the form

$$T_a(p) = \delta(p^2 - a)\, \Theta(-p^0).$$

Since $a > 0$, the mass shell does not intersect the hyperplane $p^0 = 0$, and we can thus calculate the distributional derivative to

$$\begin{aligned}
&\frac{\partial}{\partial p^j} (\delta^{(n)}(p^2 - a)\, \Theta(-p^0)) \\
&\qquad = 2 p_j \delta^{(n+1)}(p^2 - a)\, \Theta(-p^0) - \delta^{(n)}(p^2 - a) \delta^0_j \delta(p^0) \\
&\qquad = 2 p_j \delta^{(n+1)}(p^2 - a)\, \Theta(-p^0).
\end{aligned}$$

Hence, for the calculation of derivatives we can view T_a as a function of $(p^2 - a)$; that is,

$$\frac{\partial}{\partial p^j} T_a^{(n)}(p) = -2 p_j T_a^{(n+1)}(p). \tag{3.4}$$

This relation can also be used to calculate the derivatives of $T_a^{(n)}$ in position space,

$$\begin{aligned}
\frac{\partial}{\partial x^j} T_a^{(n)}(x, y) &= \int \frac{d^4 p}{(2\pi)^4} T_a^{(n)}(p) (-i p_j) e^{-ip(x-y)} \\
&= \frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{\partial}{\partial p^j} T_a^{(n-1)}(p) e^{-ip(x-y)} \\
&= -\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} T_a^{(n-1)}(p) \frac{\partial}{\partial p^j} e^{-ip(x-y)} \\
&= \frac{1}{2} (y - x)_j T_a^{(n-1)}(x, y).
\end{aligned} \tag{3.5}$$

Using that T_a is a solution of the Klein–Gordon equation, we also have

$$0 = \left(\frac{d}{da} \right)^n (p^2 - a) T_a(p) = (p^2 - a) T_a^{(n)}(p) - n T_a^{(n-1)}(p)$$

and thus

$$(-\Box_x - a) T_a^{(n)}(x, y) = n T_a^{(n-1)}(x, y). \tag{3.6}$$

With the help of (3.5) and (3.6), we can calculate the derivatives of the individual summands in (3.2) as follows:

$$
\begin{aligned}
(-\Box_x - a) \int_0^1 & (\alpha-\alpha^2)^n (\Box^n V)|_{\alpha y+(1-\alpha)x}\, d\alpha\, T_a^{(n+1)}(x,y) \\
= &- \int_0^1 (1-\alpha)^2 (\alpha-\alpha^2)^n (\Box^{n+1} V)|_{\alpha y+(1-\alpha)x}\, d\alpha\, T_a^{(n+1)}(x,y) \\
&- \int_0^1 (1-\alpha)(\alpha-\alpha^2)^n (\partial_j \Box^n V)|_{\alpha y+(1-\alpha)x}\, d\alpha (y-x)^j T_a^{(n)}(x,y) \\
&+ (n+1) \int_0^1 (\alpha-\alpha^2)^n (\Box^n V)|_{\alpha y+(1-\alpha)x}\, d\alpha\, T_a^{(n)}(x,y).
\end{aligned}
$$

In the second summand, we rewrite the partial derivative as a derivative with respect to α,

$$
(y-x)^j\, \partial_j \Box^n V|_{\alpha y+(1-\alpha)x} = \frac{d}{d\alpha} \Box^n V|_{\alpha y+(1-\alpha)x},
$$

and then integrate by parts. This gives

$$
\begin{aligned}
(-\Box_x - a) \int_0^1 & (\alpha-\alpha^2)^n (\Box^n V)|_{\alpha y+(1-\alpha)x}\, d\alpha\, T_a^{(n+1)}(x,y) \\
= &\, \delta_{n,0} V(x) T_a(x,y) \\
&+ n \int_0^1 (1-\alpha)^2 (\alpha-\alpha^2)^{n-1} (\Box^n V)|_{\alpha y+(1-\alpha)x}\, d\alpha\, T_a^{(n)}(x,y) \\
&- \int_0^1 (1-\alpha)^2 (\alpha-\alpha^2)^n (\Box^{n+1} V)|_{\alpha y+(1-\alpha)x}\, d\alpha\, T_a^{(n+1)}(x,y).
\end{aligned}
$$

After dividing by $n!$ and summing over n, the last two summands are telescopic and vanish. This yields (3.3). □

It would be nice if the solution of the inhomogeneous Klein–Gordon equation constructed in the previous lemma coincided with $\Delta T_a(x,y)$. This is really the case, although it is not obvious. In the remainder of this section, we will prove it. The technique is to expand (2.5) in momentum space and to show that the resulting expression is the Fourier transform of (3.2).

Since (2.5) is linear in V, we can assume that V has the form of a plane wave,

$$
V(x) = e^{-iqx}. \tag{3.7}
$$

Transforming $\Delta T_a(x,y)$ to momentum space gives the formula

$$
\begin{aligned}
\int d^4x \int & d^4y\, \Delta T_a(x,y) e^{ip_2 y - ip_1 x} \\
&= -\left(\frac{\mathrm{PP}}{p_2^2 - a} T_a(p_1) + T_a(p_2) \frac{\mathrm{PP}}{p_1^2 - a} \right) \delta^4(q - p_2 + p_1),
\end{aligned}
\tag{3.8}
$$

where p_1, p_2 are the in- and outgoing momenta and where PP denotes the principal value,

$$\frac{\text{PP}}{p^2 - a} = \frac{1}{2} \lim_{0<\varepsilon\to 0} \sum_{\pm} \frac{1}{p^2 - a \pm i\varepsilon p^0}.$$

The factor $\delta^4(q - p_2 + p_1)$ in (3.8) describes the conservation of energy momentum. In order to simplify the notation, we leave out this δ^4-factor and view ΔT_a as a function of only one free variable $p = (p_1 + p_2)/2$,

$$\Delta T_a\Big(p + \frac{q}{2}, p - \frac{q}{2}\Big) := -\frac{\text{PP}}{(p + \frac{q}{2})^2 - a} T_a\Big(p - \frac{q}{2}\Big) - T_a\Big(p + \frac{q}{2}\Big) \frac{\text{PP}}{(p - \frac{q}{2})^2 - a}. \tag{3.9}$$

The transformation to position space is then given by a single integral,

$$\Delta T_a(x, y) = \int \frac{d^4p}{(2\pi)^4} \Delta T_a\Big(p + \frac{q}{2},\ p - \frac{q}{2}\Big) e^{-ip(x-y)} e^{-i(q/2)(x+y)}; \tag{3.10}$$

the inverse transformation to momentum space takes the form

$$\Delta T_a\Big(p + \frac{q}{2},\ p - \frac{q}{2}\Big) = \int d^4y\ \Delta T_a(x, y) e^{ip(x-y)} e^{i(q/2)(x+y)}. \tag{3.11}$$

The first step for the light-cone expansion in momentum space is to rewrite (3.9) in the form

$$\Delta T_a\Big(p + \frac{q}{2},\ p - \frac{q}{2}\Big) = -\frac{\text{PP}}{[(p - \frac{q}{2})^2 - a] + 2pq} T_a\Big(p - \frac{q}{2}\Big) - T_a\Big(p + \frac{q}{2}\Big) \frac{\text{PP}}{[(p + \frac{q}{2})^2 - a] - 2pq}$$

and to use that the expressions within brackets $[\cdots]$ vanish as the arguments of the δ-distributions in $T_a(p \pm q/2)$,

$$\Delta T_a\Big(p + \frac{q}{2},\ p - \frac{q}{2}\Big) = \frac{\text{PP}}{2pq} \Big(T_a\Big(p + \frac{q}{2}\Big) - T_a\Big(p - \frac{q}{2}\Big)\Big).$$

Now we expand T_a in a Taylor series in q,

$$\Delta T_a\Big(p + \frac{q}{2},\ p - \frac{q}{2}\Big) = \frac{\text{PP}}{2pq} 2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \frac{\partial^{2k+1}}{\partial p^{i_1} \cdots \partial p^{i_{2k+1}}} T_a(p) \frac{q^{i_1}}{2} \cdots \frac{q^{i_{2k+1}}}{2}. \tag{3.12}$$

We want to rewrite the p-derivatives as derivatives with respect to a. This can be done by iteratively carrying out the p-derivatives with the differentiation rule (3.4). One must keep in mind that the p-derivatives act either on T_a or on the factors p_j that were previously generated by (3.4), for example,

$$\frac{\partial^2}{\partial p^j \partial p^k} T_a(p) \frac{q^j}{2} \frac{q^k}{2} = -\frac{\partial}{\partial p^j} ((pq) T_a^{(1)}(p)) \frac{q^j}{2} = (pq)^2 T_a^{(2)}(p) - \frac{q^2}{2} T_a^{(1)}(p).$$

In this way, carrying out the p-derivatives in (3.12) gives a sum of many terms. The combinatorics may be described as follows. Each factor p_j that is generated by (3.4) and differentiated thereafter gives a pairing between two of the derivatives $\partial_{i_1}, \ldots, \partial_{i_{2k+1}}$—namely, between the derivative by which it was created and the derivative by which it was subsequently annihilated. The individual expressions obtained after carrying out all the derivatives correspond to the possible configurations of the pairings among the $\partial_{i_1} \cdots \partial_{i_{2k+1}}$. They depend only on the number of pairs and not on their specific configuration. More precisely, every pair increases the degree of the derivative of $T_a^{(\cdot)}$ by one and gives a factor $q^2/2$, whereas the unpaired derivatives also increase the degree of $T_a^{(\cdot)}$ and give a factor pq.

It remains to count how many configurations of such n pairs exist. We use the notation $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$ for the number of possibilities to choose n pairs from a set of $m \geq 2n$ points. The combinatorics becomes clearer if one first selects $2n$ out of the m points and then counts the number of possible pairings among these $2n$ points to $(2n-1)!!$. This explains the formula

$$\begin{bmatrix} m \\ n \end{bmatrix} = \binom{m}{2n}(2n-1)!! = \frac{m!}{(m-2n)!}\,\frac{1}{2^n n!}. \tag{3.13}$$

We conclude that

$$\frac{\partial^{2k+1}}{\partial p^{i_1} \cdots \partial p^{i_{2k+1}}} T_a(p)\,\frac{q^{i_1}}{2} \cdots \frac{q^{i_{2k+1}}}{2} = \sum_{n=0}^{k} (-1)^{1+n}\,\frac{(2k+1)!}{(2k+1-2n)!}\,\frac{1}{2^n n!}\,T_a^{(2k+1-n)}(p)\left(\frac{q^2}{2}\right)^n (pq)^{2k+1-2n}.$$

After substituting into (3.12) and reordering the sums, we obtain

$$\Delta T_a\left(p+\frac{q}{2},\ p-\frac{q}{2}\right) = -\frac{\mathrm{PP}}{pq} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{q^2}{4}\right)^n \sum_{k=0}^{\infty} \frac{(pq)^{2k+1}}{(2k+1)!}\,T_a^{(2k+1+n)}(p). \tag{3.14}$$

Finally, we pull one factor pq out of the sum, which cancels the principal value,

$$\Delta T_a\left(p+\frac{q}{2},\ p-\frac{q}{2}\right) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{q^2}{4}\right)^n \sum_{k=0}^{\infty} \frac{(pq)^{2k}}{(2k+1)!}\,T_a^{(2k+1+n)}(p). \tag{3.15}$$

This is the formula for the light-cone expansion in momentum space.

It remains to show that the Fourier transform (3.10) of (3.15) coincides with (3.2). In order to see a first similarity between these formulas, we substitute the plane-wave ansatz (3.7) into (3.2):

$$A(x,y) = -\sum_{n=0}^{\infty} \frac{(-q^2)^n}{n!} \int_{-1/2}^{1/2} \left(\frac{1}{4} - \tau^2\right)^n e^{-i\tau q(y-x)}\, d\tau\ T_a^{(n+1)}(x,y)\, e^{-i(q/2)(x+y)}. \tag{3.16}$$

The last exponential factor also occurs in (3.10). Furthermore, it is encouraging that both (3.15) and (3.16) contain a power series in q^2. The main difference between the formulas is related to the factor $\exp(-i\tau q(y-x))$ in (3.16): expanding this exponential yields a power series in $q(y-x)$. In momentum space, this corresponds to a power series in $q^j \partial_{p^j}$ (because differentiating the exponential in (3.10) with respect to p gives a factor $i(y-x)$). The expansion (3.15), however, contains a power series in pq, not in $q^j \partial_{p^j}$. The following lemma allows us to transform these expansions into each other.

LEMMA 3.2. *For all* $r \geq 0$,

$$\left(\frac{q^j}{2}\frac{\partial}{\partial p^j}\right)^{2k} T_a^{(r)}(p) = \sum_{l=0}^{k} (-1)^l \begin{bmatrix} 2k \\ l \end{bmatrix} \left(\frac{q^2}{2}\right)^l (pq)^{2k-2l} T_a^{(r+2k-l)}(p), \tag{3.17}$$

$$(pq)^{2k} T_a^{(r+2k)}(p) = \sum_{l=0}^{k} \begin{bmatrix} 2k \\ l \end{bmatrix} \left(\frac{q^2}{2}\right)^l \left(\frac{q^j}{2}\frac{\partial}{\partial p^j}\right)^{2k-2l} T_a^{(r+l)}(p). \tag{3.18}$$

Proof. On the left side of (3.17), we calculate the derivatives inductively using the differentiation rule (3.4). The derivatives either act on $T_a^{(\cdot)}$, which increases the order of the derivative of $T_a^{(\cdot)}$ and generates a factor pq, or they act on previously generated factors pq, which reduces the number of factors pq by one and produces a factor $q^2/2$. This can also be written in the inductive form

$$\begin{aligned} &\left(\frac{q^j}{2}\frac{\partial}{\partial p^j}\right)^l T_a^{(s)}(p) \\ &\qquad = -pq\left(\frac{q^j}{2}\frac{\partial}{\partial p^j}\right)^{l-1} T_a^{(s+1)}(p) - (l-1)\frac{q^2}{2}\left(\frac{q^j}{2}\frac{\partial}{\partial p^j}\right)^{l-2} T_a^{(s+1)}(p). \end{aligned}$$

The combinatorics is described by counting the number of possibilities in forming l pairs among the $2k$ derivatives.

Equation (3.18) follows in the same way from the relation

$$\begin{aligned} &(pq)^l T_a^{(s+l)}(p) \\ &\qquad = -\left(\frac{q^j}{2}\frac{\partial}{\partial p^j}\right)(pq)^{l-1} T_a^{(s+l-1)}(p) - (l-1)\frac{q^2}{2}(pq)^{l-2} T^{(s+l-1)}(p). \qquad \square \end{aligned}$$

After these preparations, we can prove the main result of this section.

THEOREM 3.3 (formal light-cone expansion of ΔT_{m^2}). *For* $m \neq 0$, *the distribution* $\Delta T_{m^2}(x, y)$ *of* (2.5) *has a representation as the formal series*

$$\Delta T_{m^2}(x, y) = -\sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (\alpha - \alpha^2)^n (\Box^n V)|_{\alpha y + (1-\alpha)x}\, d\alpha\, T_{m^2}^{(n+1)}(x, y). \tag{3.19}$$

Proof. We expand the factor $(\frac{1}{4} - \tau^2)^n$ and the exponential $\exp(-i\tau q(y-x))$ in (3.16) and so have

$$A(x,y) = -\sum_{n=0}^{\infty} \frac{(-q^2)^n}{n!} \int_{-1/2}^{1/2} \sum_{l=0}^{n} \binom{n}{l} (-\tau^2)^l \left(\frac{1}{4}\right)^{n-l}$$
$$\times\, e^{-i\tau q(y-x)}\, d\tau\; T_a^{(n+1)}(x,y)\, e^{-i(q/2)(x+y)}$$
$$= -\sum_{n=0}^{\infty} \frac{(-q^2)^n}{n!} \sum_{k=0}^{\infty} \frac{(-iq(y-x))^k}{k!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \left(\frac{1}{4}\right)^{n-l}$$
$$\times \int_{-1/2}^{1/2} \tau^{k+2l}\, d\tau\; T_a^{(n+1)}(x,y)\, e^{-i(q/2(x+y)}. \tag{3.20}$$

Next we carry out the τ-integration. This gives a contribution only for k even,

$$= -\sum_{n=0}^{\infty} \frac{(-q^2)^n}{n!} \sum_{k=0}^{\infty} \frac{(-iq(y-x))^{2k}}{(2k)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \left(\frac{1}{4}\right)^{n-l}$$
$$\times \frac{T_a^{(n+1)}(x,y)}{(2k+2l+1)4^{k+l}}\, e^{-i(q/2)(x+y)}$$
$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{q^2}{4}\right)^n \sum_{k=0}^{\infty} \frac{(-iq(y-x))^{2k}}{4^k(2k)!} \sum_{l=0}^{n} \frac{(-1)^l}{2k+2l+1}$$
$$\times \binom{n}{l} T_a^{(n+1)}(x,y)\, e^{-i(q/2)(x+y)}.$$

Now we transform to momentum space by substituting into (3.11). The factors $-iq(y-x)/2$ can be rewritten as derivatives $(q^j/2)\partial_{p^j}$ acting on the plane wave $e^{ip(x-y)}$. Integrating these p-derivatives by parts gives

$$A\left(p+\frac{q}{2},\ p-\frac{q}{2}\right) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{q^2}{4}\right)^n \sum_{k=0}^{\infty} \frac{1}{(2k)!} \sum_{l=0}^{n} \frac{(-1)^l}{2k+2l+1}$$
$$\times \binom{n}{l} \left(\frac{q^j}{2} \frac{\partial}{\partial p^j}\right)^{2k} T_a^{(n+1)}(p).$$

We shift the indices n and k according to $n-l \to n$ and $k+l \to k$. This changes the range of the l-summation to $l = 0, \ldots, k$. We thus obtain

$$\Delta T_a\left(p+\frac{q}{2},\ p-\frac{q}{2}\right)$$
$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{q^2}{4}\right)^n \sum_{k=0}^{\infty} \frac{1}{2k+1} \sum_{l=0}^{k} \frac{1}{(2k-2l)!\, l!}$$
$$\times \left(\frac{q^2}{4}\right)^l \left(\frac{q^j}{2} \frac{\partial}{\partial p^j}\right)^{2k-2l} T_a^{(n+1+l)}(p)$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{q^2}{4}\right)^n \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \sum_{l=0}^{k} \begin{bmatrix} 2k \\ l \end{bmatrix}$$

$$\times \left(\frac{q^2}{2}\right)^l \left(\frac{q^j}{2} \frac{\partial}{\partial p^j}\right)^{2k-2l} T_a^{(n+1+l)}(p). \qquad (3.21)$$

In this way, we have transformed the line integrals in (3.16) to momentum space. We remark that we did not use the special form of T_a in the calculation from (3.20) to (3.21); for the computation so far, we could replace T_a by any other function.

In the last step, we apply Lemma 3.2: substituting (3.18) into the light-cone expansion (3.15) for ΔT_a also yields the expression (3.21). Thus A coincides with ΔT_a, which concludes the proof. □

4. Resummation of the Noncausal Contribution

In this section, we will put the previous formal calculations on a rigorous basis. The interesting part is to recover the noncausal structure of $\Delta T_{m^2}(x, y)$ by resumming the formal light-cone expansion. We begin with specifying the conditions on the potential V in (2.5).

LEMMA 4.1. *Let $V \in L^1(\mathbb{R}^4)$ be a potential which decays so fast at infinity that the functions $x^i V(x)$ are also L^1. Then $\Delta T_{m^2}(x, y)$ of* (2.5) *is a well-defined tempered distribution on $\mathbb{R}^4 \times \mathbb{R}^4$.*

Proof. It is easier to proceed in momentum space and to show that

$$\Delta T_{m^2}(p_2, p_1)$$
$$= -S_{m^2}(p_2)\tilde{V}(p_2 - p_1)T_{m^2}(p_1) - T_{m^2}(p_2)\tilde{V}(p_2 - p_1)S_{m^2}(p_1) \qquad (4.1)$$

is a well-defined tempered distribution, where $\tilde{V}$ is the Fourier transform of V. The assumption then follows by Fourier transformation.

In momentum space, the conditions on the potential give $\tilde{V} \in C^1(\mathbb{R}^4) \cap L^\infty(\mathbb{R}^4)$. We choose two test functions $f, g \in S(\mathbb{R}^4)$. Then the function $\tilde{V}(p_2 - \cdot) f(\cdot)$ is C^1 and has rapid decay at infinity. Thus the integral over the lower mass shell

$$I(p_2) := \int \frac{d^4 p_1}{(2\pi)^4} \tilde{V}(p_2 - p_1) T_{m^2}(p_1) f(p_1)$$

is finite and depends differentiably on p_2. Consequently, the product gI is in C^1 and has rapid decay, and we can calculate the principal value by

$$\frac{1}{2} \lim_{0<\varepsilon\to 0} \sum_{\pm} \int \frac{d^4 p_2}{(2\pi)^4} \frac{1}{p_2^2 - m^2 \pm i\varepsilon p_2^0} g(p_2) I(p_2),$$

which gives a finite number. This shows that the first summand

$$S_{m^2}(p_2)\tilde{V}(p_2 - p_1)T_{m^2}(p_1)$$

in (4.1) is a well-defined linear functional on $S(\mathbb{R}^4) \times S(\mathbb{R}^4)$. This functional is bounded in the Schwartz norms $\|\cdot\|_{0,0}$, $\|\cdot\|_{4,0}$, $\|\cdot\|_{0,1}$, and $\|\cdot\|_{4,1}$ of f, g, which gives continuity.

For the second summand in (4.1), one can argue in the same way after exchanging p_1 and p_2. □

For the light-cone expansion, we clearly need a smooth potential. Therefore, we will assume in the following that $V \in C^\infty \cap L^1$ and $x^i V(x) \in L^1$.

We come to the mathematical analysis of the light-cone expansion. We again assume that $m \neq 0$; the case $m = 0$ will be obtained at the end of this section in the limit $m \to 0$. In the first step, we disregard the convergence of the infinite sums and check that all the performed operations make sense and that all expressions are well-defined. We start with the end formula (3.19) of the light-cone expansion. The line integrals over the potentials are C^∞-functions in x, y. The factors $T^{(n)}_{m^2}(x, y)$ are tempered distributions, as one sees after differentiating the explicit formula (3.1) with respect to a. Thus (3.19) makes mathematical sense. The calculations leading to this result are not problematic except for the handling of the principal value following (3.9). The easiest method for studying this more rigorously is to regularize the principal value in (3.9) with the replacement

$$\frac{\mathrm{PP}}{x} \longrightarrow \frac{1}{2}\sum_{\pm} \frac{1}{x \pm i\varepsilon} = \frac{x}{x^2+\varepsilon^2}.$$

Then all the subsequent transformations are well-defined, and the critical operation is the cancellation of the principal value against one factor pq before (3.15). In order to justify this operation, we use in (3.14) the exact formula

$$\begin{aligned}
&\frac{pq}{(pq)^2+\varepsilon^2}(pq)^{2k+1} \\
&\quad = \frac{((pq)^2+\varepsilon^2)-\varepsilon^2}{(pq)^2+\varepsilon^2}(pq)^{2k} = \left(1 - \frac{\varepsilon^2}{(pq)^2+\varepsilon^2}\right)(pq)^{2k} \\
&\quad = \cdots = (pq)^{2k} - \varepsilon^2 (pq)^{2k-2} + \cdots + (-1)^k \varepsilon^{2k} + (-1)^{k+1}\frac{\varepsilon^{2k+2}}{(pq)^2+\varepsilon^2}.
\end{aligned}$$

The first summand gives (3.15); the following summands $(-1)^l \varepsilon^{2l} (pq)^{2k-2l}$ contain no principal value and vanish in the limit $\varepsilon \to 0$. Thus it remains to consider the last summand for $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{2k+2}}{(pq)^2+\varepsilon^2}\left(\frac{q^2}{4}\right)^n T^{(2k+1+n)}_{m^2}(p). \tag{4.2}$$

We use that the support of $T_{m^2}(p)$ is on the mass shell $p^2 = m^2$ and apply the relation $\lim_{\varepsilon\to 0} \varepsilon/(x^2+\varepsilon^2) = \pi\delta(x)$,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\varepsilon}{(pq)^2+\varepsilon^2} T^{(2k+1+n)}_a(p) &= \left(\frac{d}{da}\right)^{2k+1+n} \lim_{\varepsilon\to 0} \frac{\varepsilon}{(pq)^2+\varepsilon^2} T_a(p) \\
&= \left(\frac{d}{da}\right)^{2k+1+n} \left(\frac{\pi}{\sqrt{a}}\,\delta\!\left(q\frac{p}{|p|}\right) T_a(p)\right).
\end{aligned}$$

This expression is well-defined for $m \neq 0$. The limit (4.2) contains an additional factor $\varepsilon^{2k+1}(q^2/4)^n$ and thus vanishes.

We conclude that the light-cone expansion is mathematically rigorous except for the formal character of the infinite sums. In the remainder of this section, we will carefully analyze the infinite sum in (3.19). More precisely, we will do the following. As explained in the introduction, the infinite sum in (1.8) is only a notation for the approximation (1.9) of the partial sums. Following this definition, we need only show that the light-cone expansion is well-defined to any order on the light cone; we need not study the convergence of the sum over the order on the light cone. According to the explicit formula (3.1), each factor $T^{(n+1)}_{m^2}$ in (3.19) consists of an infinite number of terms of different order on the light cone (we will see this in more detail in a moment). In order to bring (3.19) into the required form (1.8), we must collect all contributions to a given order on the light cone and form their sum. This procedure is called *resummation* of the light-cone expansion. If the sum over all contributions to a given order on the light cone were finite, this resummation would be trivial; it would just correspond to a rearrangement of the summands. It will turn out, however, that these sums are infinite, and we must find a way to carry them out.

In order to see the basic problem in more detail, we consider the explicit formula (3.1) for T_a. We start with the last summand

$$T_a(x,y) \asymp -\frac{a}{32\pi^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,(l+1)!} \frac{(a\xi^2)^l}{4^l} (\Phi(l+1) + \Phi(l)) \tag{4.3}$$

(the notation "$\asymp$" means that we consider only a certain contribution to T_a). This is a power series in a, and we can calculate its derivatives to

$$\begin{aligned} T^{(n)}_a(x,y) \asymp \frac{1}{16\pi^3} \sum_{l=n-1}^{\infty} \frac{(-1)^l}{l!\,(l+1-n)!} \frac{a^{l+1-n}\xi^{2(l+n-1)}}{4^l} \\ \times (\Phi(l+1) + \Phi(l)), \quad n \geq 1. \end{aligned} \tag{4.4}$$

For increasing n, the derivatives are of higher order on the light cone; more precisely, the contribution (4.4) is of the order $\mathcal{O}((y-x)^{2(n-1)})$. Thus, the contribution of (4.3) to the formal light-cone expansion (3.19) consists, to any order on the light cone, of only a finite number of terms. Thus the resummation is trivial. Of course, we could rearrange the sum by collecting all the summands in (4.3) and (4.4) of a given degree in ξ^2 and writing them together, but this is only a matter of taste and is not really needed.

For the second summand in (3.1),

$$T_a(x,y) \asymp \frac{a}{16\pi^3} (\log(\xi^2 - i0\xi^0) + i\pi + c) \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,(l+1)!} \frac{(a\xi^2)^l}{4^l} \tag{4.5}$$

$$+ \frac{a}{16\pi^3} \log a \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,(l+1)!} \frac{(a\xi^2)^l}{4^l}, \tag{4.6}$$

the situation is more complicated: the contribution (4.5) is a power series in a and can be discussed exactly as (4.3). The only difference (apart from the missing factors Φ) is the prefactor $(\log(\xi^2 - i0\xi^0) + i\pi + c)$, which has a logarithmic pole on the light cone. The contribution (4.6), however, contains a factor $\log a$ and is *not* a power series in a. As a consequence, the higher a-derivatives of (4.6) are not of higher order on the light cone. For example, the contribution to the order $\mathcal{O}((y-x)^2)$ has the form

$$\begin{aligned}
T_a(x,y) &\asymp \frac{1}{16\pi^3} a \log a + \mathcal{O}((y-x)^2) \\
T_a^{(1)}(x,y) &\asymp \frac{1}{16\pi^3}(1+\log a) + \mathcal{O}((y-x)^2) \\
T_a^{(n)}(x,y) &\asymp \frac{(-1)^n}{16\pi^3}(n-2)!\,\frac{1}{a^{n-1}} + \mathcal{O}((y-x)^2), \quad n \geq 2.
\end{aligned}$$

This means that we must resum an infinite number of terms; more precisely,

$$\begin{aligned}
\Delta T_a^{[0]}(x,y) \asymp &-\frac{1}{16\pi^3}\int_0^1 V|_{\alpha y+(1-\alpha)x}\, d\alpha (1+\log a) \\
&+ \frac{1}{16\pi^3}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 (\alpha-\alpha^2)^n(\Box^n V)|_{\alpha y+(1-\alpha)x}\, d\alpha\, \frac{(-1)^n}{a^n} \\
&+ \cdots + \mathcal{O}((y-x)^2).
\end{aligned} \tag{4.7}$$

This is a serious problem. Namely, we can expect the series in (4.7) to converge only if the derivatives $\Box^n V$ do not grow too fast in the order $2n$ of the derivative. It turns out that analyticity of V is necessary for convergence, which is too restrictive.

On a technical level, this convergence problem of the contributions to ΔT_{m^2} to a given order on the light cone is a consequence of the factor $\log a$ in (4.6); we call it the *logarithmic mass problem*. Because $\Delta T_{m^2}(x,y)$ is well-defined by (2.4), it is not a problem of the perturbation expansion but rather shows that the light-cone expansion was not performed properly. The deeper reason for the convergence problem is that we expressed $\Delta T_{m^2}(x,y)$ only in terms of the potential and its derivatives along the line segment $\overline{xy}$. However, the perturbation $\Delta T_{m^2}(x,y)$ is not causal in this sense; it depends on V in the whole Minkowski space (this becomes clear in (2.4) from the fact that the support of $T_{m^2}(x,\cdot)$ is $\mathbb{R}^4$). In a formal expansion, we can express $\Delta T_{m^2}(x,y)$ in terms of $\Box^n V|_{\lambda y+(1-\lambda)x}$, $0 \leq \lambda \leq 1$, but we cannot expect this expansion to converge. The simplest 1-dimensional analog of this situation is the formal Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

of a smooth function. The right side cannot in general converge, because it is not possible to express $f(x)$, $x \neq 0$, in terms of $f^{(n)}(0)$.

The solution to the logarithmic mass problem is to reformulate the problematic contribution of (4.6) to the light-cone expansion (3.19) as a noncausal term that is obviously finite. In some sense, we will simply reverse our former construction of the light-cone expansion. Yet this is not trivial, because the differentiation rule (3.4), which was crucial for rewriting the Taylor expansion (3.12) as an expansion in the mass parameter a, is not valid for (4.6). In the end, we want to write the light-cone expansion in a way which shows that part of the behavior of $\Delta T_{m^2}(x, y)$ can be described with line integrals of the form (3.19) whereas other contributions are noncausal in a specific way.

We work in momentum space. The Fourier transform of the problematic series (4.6) is

$$\begin{aligned} J_a(p) &= \int d^4x\, e^{ipx}\, \frac{1}{16\pi^3} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,(l+1)!} \frac{a\xi^2}{4^l} \\ &= \pi \sum_{l=0}^{\infty} \frac{a^l}{4^l l!\,(l+1)!} \Box^l \delta^4(p). \end{aligned} \tag{4.8}$$

Notice that this expression is highly singular at $p = 0$; especially, it is not a distribution. However, it is well-defined as a distribution on analytic functions in p. This comprises all functions with compact support in position space, which is a sufficiently large function space for the following. Furthermore, we introduce the series

$$L_a(p) = \pi \sum_{l=0}^{\infty} \frac{a^l}{4^l (l!)^2} \Box^l \delta^4(p) \tag{4.9}$$

and set

$$J_a^{(n)} = \left(\frac{d}{da}\right)^n J_a, \qquad L_a^{(n)} = \left(\frac{d}{da}\right)^n L_a.$$

LEMMA 4.2. *The series* (4.8) *and* (4.9) *satisfy the relations*

$$J_a(p) = \int_0^1 L_{\tau a}(p)\, d\tau, \tag{4.10}$$

$$\frac{\partial}{\partial p^j} L_a^{(n)}(p) = -2p_j L_a^{(n+1)}(p). \tag{4.11}$$

Proof. Equation (4.10) is verified by integrating the power series (4.9) and comparing with (4.8). The distribution $p_j \delta^4(p)$ vanishes identically. Since the derivatives of distributions are defined in the weak sense, it follows that

$$0 = \Box^{n+1}(p_j \delta^4(p)) = p_j \Box^{n+1} \delta^4(p) + 2(n+1) \frac{\partial}{\partial p^j} \Box^n \delta^4(p)$$

and thus

$$\frac{\partial}{\partial p^j} \Box^n \delta^4(p) = -\frac{1}{4(n+1)} (\Box^{n+1} \delta^4(p)) 2p_j.$$

Applying this relation to every term of the series (4.9) yields that

$$\partial_{p^j} L_a = -2p_j L_a^{(1)},$$

and (4.11) follows by differentiating with respect to a. □

The function L_a is useful because (4.11) coincides with the differentiation rule (3.4) for T_a. This implies that all the formulas for T_a, especially the manipulations of Lemma 3.2, are also valid for L_a.

The following technical lemma is the key for handling the logarithmic mass problem.

LEMMA 4.3 (resummation of the noncausal contribution). *If V is the plane wave* (3.7), *then*

$$-\sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (\alpha - \alpha^2)^n (\Box^n V)|_{\alpha y + (1-\alpha)x} \, d\alpha \left(\frac{d}{da} \right)^{n+1} (a \log(a) J_a(p)) \tag{4.12}$$

$$= -\frac{1}{2} \frac{d}{da} \int_{-1}^{1} d\mu \Big((a+b) \log(a+b) \times \int_0^1 L_{\tau a + (\tau-1)b - \mu pq}(p) \, d\tau \Big) \Big|_{a=m^2-q^2/4,\, b=\mu^2 q^2/4} . \tag{4.13}$$

Proof. The series (4.12) is obtained from the formula (3.2) for $A(x, y)$ by the replacement $T_a \to a \log(a) J_a$. As remarked in the proof of Theorem 3.3, all the transformations from (3.20) to (3.21) are also valid if we replace T_a by any other function. Therefore,

$$(4.12) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{q^2}{4} \right)^n \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \sum_{l=0}^{k} \begin{bmatrix} 2k \\ l \end{bmatrix} \left(\frac{q^2}{2} \right)^l \times \left(\frac{q^j}{2} \frac{\partial}{\partial p^j} \right)^{2k-2l} \left(\frac{d}{da} \right)^{n+1+l} (a \log(a) J_a(p)).$$

We carry out the sum over n by redefining a as $a = m^2 - q^2/4$ and substitute (4.10) as follows:

$$(4.12) = -\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \sum_{l=0}^{k} \begin{bmatrix} 2k \\ l \end{bmatrix} \left(\frac{q^2}{2} \right)^l \times \left(\frac{q^j}{2} \frac{\partial}{\partial p^j} \right)^{2k-2l} \left(\frac{d}{da} \right)^{1+l} \left(a \log(a) \int_0^1 L_{\tau a}(p) \, d\tau \right).$$

Using that $L_a(p)$ and $T_a(p)$ obey the same differentiation rules (4.11) and (3.4), respectively, we can apply relation (3.17) with $T_a^{(r)}$ replaced by $L_a^{(r)}$ to obtain

$$(4.12) = -\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \sum_{l=0}^{k} \begin{bmatrix} 2k \\ l \end{bmatrix} \sum_{s=0}^{k-l} (-1)^s \begin{bmatrix} 2k-2l \\ s \end{bmatrix} \left(\frac{q^2}{2} \right)^{l+s} \times (pq)^{2k-2l-2s} \left(\frac{d}{da} \right)^{1+l} \left(a \log(a) \int_0^1 L_{\tau a}^{(2k-2l-s)}(p) \, d\tau \right).$$

We introduce the index $r = l + s$, replace s by $r - l$, and substitute the combinatorial formula (3.13):

$$(4.12) = -\sum_{k=0}^{\infty} \frac{(-1)^r}{2k+1} \sum_{r=0}^{k} \frac{1}{r!\,(2k-2r)!} \left(\frac{q^2}{4}\right)^r (pq)^{2k-2r}$$
$$\times \sum_{l=0}^{r} (-1)^l \binom{r}{l} \left(\frac{d}{da}\right)^{1+l} \left(a \log(a) \int_0^1 L_{\tau a}^{(2k-2r+(r-l))}(p)\, d\tau \right).$$

The last sum can be eliminated using the combinatorics of the product rule,

$$\sum_{l=0}^{r} (-1)^l \binom{r}{l} \left(\frac{d}{da}\right)^l \left(a \log(a) L_{\tau a}^{(2k-2r+(r-l))}(p) \right)$$
$$= (-1)^r \left(\frac{d}{db}\right)^r \left((a+b) \log(a+b) L_{\tau(a+b)-b}^{(2k-2r)}(p) \right)\Big|_{b=0}.$$

Furthermore, we shift the index k according to $k - r \to k$, yielding

$$(4.12) = -\sum_{k=0}^{\infty} \frac{1}{2k+2r+1} \frac{(pq)^{2k}}{(2k)!} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{q^2}{4}\right)^r$$
$$\times \left(\frac{d}{da}\right) \left(\frac{d}{db}\right)^r \left((a+b) \log(a+b) \int_0^1 L_{\tau a+(\tau-1)b}^{(2k)}(p)\, d\tau \right)\Bigg|_{b=0}.$$

Without the factor $(2k+2r+1)^{-1}$, we had two separate Taylor series which could easily be carried out explicitly. The coupling of the two series by this factor can be described with an additional line integral,

$$(4.12) = -\frac{1}{2} \frac{d}{da} \int_{-1}^{1} d\mu \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{q^2}{4}\right)^r \mu^{2r}$$
$$\times \left(\frac{d}{db}\right)^r \left((a+b) \log(a+b) \int_0^1 L_{\tau a+(\tau-1)b+\mu pq}(p)\, d\tau \right)\Bigg|_{b=0}.$$

We finally carry out the remaining Taylor sum. □

The result of this lemma is quite complicated. The important point is that the convergence problems of the infinite series (4.12) have disappeared in (4.13), which is obviously finite. Namely, the a-derivative of the integrand in (4.13) has at most logarithmic singularities. These singularities are integrable and disappear when the μ-integration is carried out.

After these preparations, we can state the main theorem. Since the resulting expansion is regular in the limit $m \to 0$, it is also valid for $m = 0$.

THEOREM 4.4 (light-cone expansion of ΔT_{m^2}). *The distribution* ΔT_{m^2} *of* (2.5) *has the representation*

$$\Delta T_{m^2}(x,y) = -\sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (\alpha - \alpha^2)^n (\Box^n V)|_{\alpha y+(1-\alpha)x}\, d\alpha\, T_{m^2}^{\mathrm{reg}(n+1)}(x,y) \quad (4.14)$$
$$+ N_{m^2}(x,y), \quad (4.15)$$

with

$$T_a^{\mathrm{reg}} = T_a - a\log(a)J_a, \qquad T_a^{\mathrm{reg}(n)} = \left(\frac{d}{da}\right)T_a^{\mathrm{reg}},$$

and the Bessel function J_a *of* (4.8). *The series* (4.14) *is well-defined in the sense of Definition 1.1. The contribution* $N_{m^2}(x, y)$ *is a smooth function in* x, y *and has a representation as the Fourier integral*

$$N_{m^2}(x, y) = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \tilde{V}(q)\, N_{m^2}(p, q)\, e^{-ip(x-y)} e^{-i(q/2)(x+y)} \tag{4.16}$$

with

$$N_a(p, q) = -\frac{1}{2}\frac{d}{da}\int_{-1}^{1} d\mu \log\left(a - (1-\mu^2)\frac{q^2}{4}\right)\beta J_\beta(p)\Big|_{\beta=-\mu^2q^2/4+\mu pq}^{\beta=a-q^2/4+\mu pq}. \tag{4.17}$$

Proof. By definition, T_a^{reg} differs from T_a by the contribution (4.6). Thus, an explicit formula for T_a^{reg} is obtained from (3.1) if we replace the factor $\log(a\xi^2 - i\varepsilon\xi^0)$ in the second line by $\log(\xi^2 - i\varepsilon\xi^0)$. As a consequence, T_a^{reg} is a power series in a, and the higher-order contributions in a are of higher order on the light cone. This justifies the infinite sum in (4.14) in the sense of Definition 1.1. Furthermore, the difference between the formal light-cone expansions (3.19) and (4.14) coincides with the contribution (4.12), which was resummed in Lemma 4.3. We carry out the τ-integration in (4.13) using the series expansions (4.8) and (4.8), which gives (4.17).

The q-integral in (4.16) is well-defined since $\tilde{V}(q)$ is C^1 and decays sufficiently fast at infinity. Finally, the p-integration can be carried out with the δ^4-distributions in (4.8), which gives a smooth function $N_{m^2}(x, y)$. □

We call (4.14) and (4.15) the *causal* and *noncausal contributions*, respectively.

We could proceed by studying the noncausal contribution more explicitly in position space. For the purpose of this paper, however, it is sufficient to notice that $N(x, y)$ is smooth on the light cone.

5. The Light-Cone Expansion of the Dirac Sea

Having performed the light-cone expansion for ΔT_{m^2}, we now return to the study of the Dirac sea (1.7). From the theoretical point of view, the light-cone expansion for ΔP_{m^2} is an immediate consequence of Theorem 4.4 and formula (2.4): we substitute the light-cone expansion (4.14)–(4.15) into (2.4). Calculating the partial derivatives ∂_x and ∂_y of the causal contribution (4.14) gives expressions of the form

$$\Delta P_{m^2}(x, y) \asymp \int_0^1 \mathcal{P}(\alpha)\, D^a \Box^b V|_{\lambda y+(1-\lambda)x}\, d\alpha\, D^c T_{m^2}^{\mathrm{reg}(n+1)}(x, y), \tag{5.1}$$

which are again causal in the sense that they depend on the potential and its partial derivatives only along the line segment $\overline{xy}$ (here $\mathcal{P}(\alpha)$ denotes a polynomial in α; D^a stands for any partial derivatives of the order a). Since (5.1) contains distributional derivatives of $T_{m^2}^{\mathrm{reg}(n+1)}$, it is in general more singular on the light cone than the corresponding contribution to ΔT_{m^2}. On the other hand, the partial derivatives of the noncausal contribution $N_{m^2}(x, y)$ can be calculated with (4.17) and yield smooth functions in x, y. We conclude that the qualitative picture of Theorem 4.4, especially the splitting into a causal and a noncausal contributions, is also valid for the Dirac sea.

The situation becomes more complicated if one wants to go beyond this qualitative picture and is interested in explicit formulas for the Dirac sea. The problem is to find an effective and reliable method for calculating the partial derivatives and for handling the combinatorics of the Dirac matrices. Before entering these computational details, we explain how the qualitative picture of the light-cone expansion can be understood directly from the integral formula (1.7). The tempered distributions $s(x, y)$ and $P(x, y)$ are regular functions for $(y-x)^2 \neq 0$ and are singular on the light cone (this can be seen explicitly from e.g. (2.1) and (3.1)). Integrals of the form

$$\int P(x, z) f(z)\, d^4z \quad \text{or} \quad \int s(x, z) f(z)\, d^4z$$

with a smooth function f (which decays sufficiently fast at infinity) give smooth functions in x. The integral in (1.7) is more complicated because it contains two distributional factors, s and P. This causes complications only if the singularities of s and P meet—that is, if z lies on the intersection $L_x \cap L_y$ of the light cones around x, y, where

$$L_x = \{y \in \mathbb{R}^4,\ (y-x)^2 = 0\}.$$

If $y-x$ is timelike or spacelike then $L_x \cap L_y$ is a 2-sphere or a hyperboloid (respectively), either of which depends smoothly on x, y. As a consequence, the integral over these singularities can be carried out in (1.7) and gives a smooth function. On the light cone $(y-x)^2 = 0$, however, $L_x \cap L_y$ does not depend smoothly on x, y. More precisely, in the limit $0 < (y-x)^2 \to 0$, the 2-sphere $L_x \cap L_y$ degenerates to the line segment $\{\lambda y + (1 - \lambda x),\ 0 \leq \lambda \leq 1\}$. The limit $0 > (y-x)^2 \to 0$, on the other hand, gives the degenerated hyperboloid $\{\lambda y + (1 - \lambda x),\ \lambda \leq 0 \text{ or } \lambda \geq 1\}$. This simple consideration explains why the singularities of $\Delta P(x, y)$ occur on the light cone and makes it plausible that the behavior of the singularities is characterized by the potential and its derivatives along the line $xy = \{\lambda y + (1-\lambda)x,\ \lambda \in \mathbb{R}\}$. Clearly, $V(z)$ also enters into $\Delta P_{m^2}(x, y)$ for $z \notin xy$, but this noncausal contribution is not related to the discontinuity of $L_x \cap L_y$ on the light cone and is therefore smooth. The special form of the singularities,

$$\Delta P(x, y) \sim D^a \log((y-x)^2 - i0(y-x)^0)(y-x)^{2n}, \tag{5.2}$$

is less obvious. That the potential enters only along the line segment $\overline{xy}$ can be understood only from the special form of (1.7); it is a consequence of the causality principle for the Dirac sea that was introduced in [1]. In fact, it gives an easy way to understand the meaning of "causality" of the perturbation expansion for the Dirac sea.

We finally describe our method for explicitly calculating $\Delta P(x, y)$. As in Theorem 4.4, we will not study the noncausal contribution; we are content with the fact that it is bounded and smooth. In other words, we consider only the singular contribution (5.1) to the Dirac sea. Since the difference between $T^{(n)}_{m^2}$ and $T^{\mathrm{reg}(n)}_{m^2}$ is smooth, we can just as well consider the formal light-cone expansion (3.19) and calculate modulo smooth terms on the light cone. This has the advantage that we can work with the useful differentiation rule (3.5). The calculation can be split into several steps, which may be listed as follows.

(1) *Calculation of the partial derivatives* with the product rule and the differentiation formulas

$$\frac{\partial}{\partial x^j} T^{(n)}_{m^2}(x,y) = -\frac{\partial}{\partial y^j} T^{(n)}_{m^2}(x,y) \overset{(3.5)}{=} \frac{1}{2}(y-x)_j\, T^{(n-1)}_{m^2}(x,y), \tag{5.3}$$

$$\frac{\partial}{\partial y^j} \int_0^1 \mathcal{P}(\alpha)\, D^a \Box^b V|_{\alpha y+(1-\alpha)x} = \int_0^1 \alpha \mathcal{P}(\alpha)\, \partial_j\, D^a \Box^b V|_{\alpha y+(1-\alpha)x}, \tag{5.4}$$

$$\begin{aligned} &\frac{\partial}{\partial x^j} \int_0^1 \mathcal{P}(\alpha)\, D^a \Box^b V|_{\alpha y+(1-\alpha)x} \\ &\quad = \int_0^1 (1-\alpha)\mathcal{P}(\alpha)\, \partial_j\, D^a \Box^b V|_{\alpha y+(1-\alpha)x}, \end{aligned} \tag{5.5}$$

$$\frac{\partial}{\partial x^j}(y-x)_k = -\frac{\partial}{\partial y^j}(y-x)_k = -g_{jk}. \tag{5.6}$$

(2) *Simplification of the Dirac matrices* with the anti-commutation relations $\{\gamma^j, \gamma^k\} = 2g^{jk}$. This leads to a contraction of tensor indices. The generated factors $(y-x)^2$ and $(y-x)^j \partial_j V$ are simplified in the calculation steps (3) and (4).

(3) *Absorption of the factors* $(y-x)^2$. We calculate the Laplacian by iterating (5.3),

$$\Box_x T^{(n+2)}_{m^2}(x,y) = -2T^{(n+1)}_{m^2}(x,y) + \tfrac{1}{4}(y-x)^2 T^{(n)}_{m^2}(x,y),$$

and then combine it with (3.6), which gives the rule

$$(y-x)^2 T^{(n)}_{m^2}(x,y) = -4n T^{(n+1)}_{m^2}(x,y) - 4m^2 T^{(n+2)}_{m^2}(x,y). \tag{5.7}$$

(4) *Partial integration of the tangential derivatives,*

$$\begin{aligned} &\int_0^1 \mathcal{P}(\alpha)(y-x)^j\, \partial_j\, D^a \Box^b V|_{\alpha y+(1-\alpha)x} \\ &\quad = \int_0^1 \mathcal{P}(\alpha) \frac{d}{d\alpha} D^a \Box^b V|_{\alpha y+(1-\alpha)x} \\ &\quad = \mathcal{P}(\alpha)\, D^a \Box^b V|_{\alpha y+(1-\alpha)x} \Big|_{\alpha=0}^{\alpha=1} - \int_0^1 \mathcal{P}'(\alpha)\, D^a \Box^b V|_{\alpha y+(1-\alpha)x}. \end{aligned} \tag{5.8}$$

After these steps, $\Delta P_{m^2}(x,y)$ consists of many terms of the form

$$\Delta P_{m^2}(x,y) \asymp (\text{causal expression in } D^a \Box^b V) \times T^{(n)}_{m^2}(x,y), \quad n \geq -1.$$

It remains to insert the series representations for $T^{(n)}_{m^2}(x,y)$. It is useful first to introduce the short notation $z^n = \xi^{2n}$ $(n \geq 0)$ and

$$z^{-2} := \frac{1}{2} \lim_{0<\varepsilon\to 0} \sum_{\pm} \frac{1}{(\xi^2 - i\varepsilon\xi^0)^2},$$

$$z^{-1} := \frac{1}{2} \lim_{0<\varepsilon\to 0} \sum_{\pm} \frac{1}{\xi^2 - i\varepsilon\xi^0},$$

$$\log z := \frac{1}{2} \lim_{0<\varepsilon\to 0} \sum_{\pm} \log(\xi^2 - i\varepsilon\xi^0).$$

(5) *Substitution of the explicit formulas*

$$T^{(-1)}_{m^2}(x,y) \asymp -\frac{1}{2\pi^3} z^{-2} - \frac{m^2}{8\pi^3} z^{-1} - \frac{1}{8\pi^3} \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{4^{l+1}} \frac{m^{2l+4}}{l!\,(l+2)!} z^l \log z, \tag{5.9}$$

$$T_{m^2}(x,y) \asymp -\frac{1}{8\pi^3} z^{-1} - \frac{1}{8\pi^3} \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{4^{l+1}} \frac{m^{2l+2}}{l!\,(l+1)!} z^l \log z, \tag{5.10}$$

$$T^{(n)}_{m^2}(x,y) \asymp -\frac{1}{8\pi^3} \sum_{l=n-1}^{\infty} \frac{(-1)^{l+1}}{4^{l+1}} \frac{m^{2l+2-2n}}{l!\,(l+1-n)!} z^l \log z \quad (n \geq 1), \tag{5.11}$$

where again we have used the notation of (3.1) (we take only the singular contribution on the light cone; $T^{(-1)}_{m^2}$ is defined via (5.3)).

In this way, the calculation of the causal contribution is reduced to a small number of symbolic computation rules (5.3)–(5.11), which can be applied mechanically. This makes it possible to use a computer program for the calculation. The C++ program "class_commute" was designed for this task (commented source code available from the author on request). It computes the causal contribution for a general perturbation (1.3) to any order on the light cone. The formulas to the order $\mathcal{O}((y-x)^0)$ modulo the noncausal contribution are listed in the appendix.

6. Outlook

In this paper, the light-cone expansion was performed for the Dirac sea to first order in the external potential. The presented method can be generalized in several directions and applied to related problems, which we now briefly outline.

First of all, the method is not restricted to the Dirac and Klein–Gordon equations; it can also be used for the analysis of other scalar and matrix hyperbolic equations (in any space–time dimension). The consideration (2.8), which gives the basic explanation for the line integrals in the light-cone expansion, can be applied to any hyperbolic equation (in curved space–time, the line integrals must be replaced by integrals along null geodesics; see e.g. [5]). Thus, the behavior of the solution near the light cone is again described by an infinite series of line integrals. The line integrals might be unbounded, however, which leads to additional convergence problems (e.g., one can replace the integrals in (3.2) by $\frac{1}{2}\int_{-\infty}^{\infty} \varepsilon(\alpha)\, d\alpha \cdots$, which gives a different formal solution of (3.3)).

Furthermore, the light-cone expansion can be generalized to symmetric eigensolutions and the fundamental solutions. The formal light-cone expansion of Section 3 applies in the same way to any Lorentzian invariant family T_{m^2} of solutions of the Klein–Gordon equation, that is, to a linear combination of

$$T_{m^2}(p) = \delta(p^2 - m^2) \qquad \text{and} \tag{6.12}$$

$$T_{m^2}(p) = \delta(p^2 - m^2)\varepsilon(p^0). \tag{6.13}$$

The second case (6.13) allows us to generalize the light-cone expansion to the Green's function. The advanced Green's function $S^\vee_{m^2}$ of the Klein–Gordon operator, for example, can be derived from T_{m^2} of (6.13) by

$$S^\vee_{m^2}(x, y) = 2\pi i T_{m^2}(x, y)\Theta(y^0 - x^0).$$

This relation even remains valid in the perturbation expansion—for example, to first order,

$$\Delta S^\vee_{m^2}(x, y) = 2\pi i \Delta T_{m^2}(x, y)\Theta(y^0 - x^0) \tag{6.14}$$

(for a derivation of this formula in the context of the Dirac equation, see [1]). Thus the light-cone expansion for ΔT_{m^2} immediately yields corresponding formulas for the Green's function.

In contrast to the formal light-cone expansion of Section 2, the resummation of the noncausal contribution depends much on the particular problem. An analysis in position space according to [4] might be helpful for the understanding of the noncausality. For T_{m^2} according to (6.13), for example, there is no noncausal contribution at all, which also simplifies the analysis of the Green's functions.

By iteration, the method can also be applied to higher-order Feynman diagrams and even makes it possible to sum up certain classes of Feynman diagrams explicitly. For the Dirac Green's function and the Dirac sea, this is explained in detail in [3].

A. Some Formulas of the Light-Cone Expansion

The following formulas give $\Delta P(x, y)$ to first order in the external potential (1.3) up to contributions of the order $\mathcal{O}((y-x)^0)$ on the light cone. For the causal line integrals, we use the short notation

$$\int_x^y f \cdots := \int_0^1 f|_{\alpha y + (1-\alpha)x} \cdots d\alpha.$$

A.1. Electromagnetic Potential

$$\begin{aligned} \Delta P(x, y) = &-\frac{e}{4\pi^3}\int_x^y A_j \xi^j \not{\xi} z^{-2} \\ &- \frac{e}{16\pi^3}\int_x^y (\alpha^2 - \alpha)\not{\xi}\xi^k j_k z^{-1} \\ &+ \frac{e}{16\pi^3}\int_x^y (2\alpha - 1)\xi^j \gamma^k F_{kj} z^{-1} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{ie}{32\pi^3} \int_x^y \varepsilon^{ijkl} F_{ij} \xi_k \gamma^5 \gamma_l z^{-1} \\
&- \frac{e}{128\pi^3} \int_x^y (\alpha^4 - 2\alpha^3 + \alpha^2) \not{\xi} \xi_k \Box j^k \log z \\
&+ \frac{e}{128\pi^3} \int_x^y (4\alpha^3 - 6\alpha^2 + 2\alpha) \xi^j \gamma^k (\Box F_{kj}) \log z \\
&+ \frac{ie}{128\pi^3} \int_x^y (\alpha^2 - \alpha) \varepsilon^{ijkl} (\Box F_{ij}) \xi_k \gamma^5 \gamma_l \log z \\
&+ \frac{e}{16\pi^3} \int_x^y (\alpha^2 - \alpha) \gamma^k j_k \log z \\
&+ \frac{ie}{8\pi^3} m \int_x^y A_j \xi^j z^{-1} \\
&- \frac{e}{64\pi^3} m \int_x^y F_{ij} \sigma^{ij} \log z \\
&- \frac{ie}{32\pi^3} m \int_x^y (\alpha^2 - \alpha) j_k \xi^k \log z \\
&- \frac{e}{16\pi^3} m^2 \int_x^y A_j \xi^j \not{\xi} z^{-1} \\
&- \frac{e}{64\pi^3} m^2 \int_x^y (2\alpha - 1) \gamma^i F_{ij} \xi^j \log z \\
&- \frac{ie}{128\pi^3} m^2 \int_x^y \varepsilon^{ijkl} F_{ij} \xi_k \gamma^5 \gamma_l \log z \\
&+ \frac{e}{64\pi^3} m^2 \int_x^y (\alpha^2 - \alpha) j_k \xi^k \not{\xi} \log z \\
&- \frac{ie}{32\pi^3} m^3 \int_x^y A_j \xi^j \log z \\
&+ \frac{e}{128\pi^3} m^4 \int_x^y A_j \xi^j \not{\xi} \log z \\
&+ (\text{noncausal contributions}) + \mathcal{O}(\xi^2),
\end{aligned}
$$

with the electromagnetic field tensor $F_{jk} = \partial_j A_k - \partial_k A_j$ and the electromagnetic current $j^k = \partial_l F^{kl}$.

A.2. Axial Potential

$$
\begin{aligned}
\Delta P(x, y) = &\frac{e}{4\pi^3} \int_x^y B_j \xi^j \gamma^5 \not{\xi} z^{-2} \\
&+ \frac{e}{16\pi^3} \int_x^y (\alpha^2 - \alpha) \gamma^5 \not{\xi} \xi^k j_k z^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{e}{16\pi^3} \int_x^y (2\alpha-1)\xi^j \gamma^5 \gamma^k F_{kj} z^{-1} \\
&- \frac{ie}{32\pi^3} \int_x^y \varepsilon^{ijkl} F_{ij} \xi_k \gamma_l z^{-1} \\
&+ \frac{e}{128\pi^3} \int_x^y (\alpha^4 - 2\alpha^3 + \alpha^2) \gamma^5 \not{\xi} \xi_k \Box j^k \log z \\
&- \frac{e}{128\pi^3} \int_x^y (4\alpha^3 - 6\alpha^2 + 2\alpha) \xi^j \gamma^5 \gamma^k (\Box F_{kj}) \log z \\
&- \frac{ie}{128\pi^3} \int_x^y (\alpha^2 - \alpha) \varepsilon^{ijkl} (\Box F_{ij}) \xi_k \gamma_l \log z \\
&- \frac{e}{16\pi^3} \int_x^y (\alpha^2 - \alpha) \gamma^5 \gamma^k j_k \log z \\
&- \frac{ie}{8\pi^3} m \int_x^y \gamma^5 \frac{1}{2} [\not{\xi}, \not{B}] z^{-1} \\
&- \frac{e}{64\pi^3} m \int_x^y (2\alpha - 1) F_{jk} \gamma^5 \sigma^{jk} \log z \\
&+ \frac{ie}{32\pi^3} m \int_x^y \partial_j B^j \gamma^5 \log z \\
&+ \frac{e}{32\pi^3} m \int_x^y (\alpha^2 - \alpha) \Box B_j \xi_k \gamma^5 \sigma^{jk} \log z \\
&+ \frac{e}{16\pi^3} m^2 \int_x^y B_j \xi^j \gamma^5 \not{\xi} z^{-1} \\
&- \frac{e}{16\pi^3} m^2 \int_x^y \gamma^5 \not{B} \log z \\
&+ \frac{e}{64\pi^3} m^2 \int_x^y (2\alpha - 1) F_{jk} \xi^k \gamma^5 \gamma^j \log z \\
&+ \frac{ie}{128\pi^3} m^2 \int_x^y \varepsilon^{ijkl} F_{ij} \xi_k \gamma_l \log z \\
&- \frac{e}{64\pi^3} m^2 \int_x^y (\alpha^2 - \alpha) j_k \xi^k \gamma^5 \not{\xi} \log z \\
&+ \frac{ie}{32\pi^3} m^3 \int_x^y \gamma^5 \frac{1}{2} [\not{\xi}, \not{B}] \log z \\
&- \frac{e}{128\pi^3} m^4 \int_x^y A_j \xi^j \gamma^5 \not{\xi} \log z \\
&+ (\text{noncausal contributions}) + \mathcal{O}(\xi^2),
\end{aligned}
$$

with the axial field tensor $F_{jk} = \partial_j B_k - \partial_k B_j$ and the axial current $j^k = \partial_l F^{kl}$.

A.3. Scalar Potential

$$
\begin{aligned}
\Delta P(x,y) = {} & \frac{1}{16\pi^3}(\Phi(y)+\Phi(x))z^{-1} \\
& + \frac{i}{16\pi^3}\int_x^y (\partial_j \Phi)\xi_k \sigma^{jk} z^{-1} \\
& + \frac{i}{64\pi^3}\int_x^y (\alpha^2-\alpha)(\partial_j \Box\Phi)\xi_k \sigma^{jk}\log z \\
& - \frac{1}{64\pi^3}\int_x^y \Box\Phi \log z \\
& + \frac{i}{8\pi^3} m \int_x^y \Phi \not{\xi} z^{-1} \\
& + \frac{i}{32\pi^3} m \int_x^y (2\alpha-1)(\not{\partial}\Phi)\log z \\
& + \frac{i}{32\pi^3} m \int_x^y (\alpha^2-\alpha)(\Box\Phi)\not{\xi}\log z \\
& - \frac{1}{64\pi^3} m^2 (\Phi(y)+\Phi(x))\log z \\
& - \frac{1}{16\pi^3} m^2 \int_x^y \Phi \log z \\
& - \frac{i}{64\pi^3} m^2 \int_x^y (\partial_j \Phi)\xi_k \sigma^{jk}\log z \\
& - \frac{i}{32\pi^3} m^3 \int_x^y \Phi \not{\xi}\log z \\
& + (\text{noncausal contributions}) + \mathcal{O}(\xi^2).
\end{aligned}
$$

A.4. Pseudoscalar Potential

$$
\begin{aligned}
\Delta P(x,y) = {} & -\frac{i}{16\pi^3}(\Xi(y)+\Xi(x))\gamma^5 z^{-1} \\
& + \frac{1}{16\pi^3}\int_x^y (\partial_j \Xi)\xi_k \gamma^5 \sigma^{jk} z^{-1} \\
& + \frac{1}{64\pi^3}\int_x^y (\alpha^2-\alpha)(\partial_j \Box\Xi)\xi_k \gamma^5 \sigma^{jk}\log z \\
& + \frac{i}{64\pi^3}\int_x^y \Box\Xi\gamma^5 \log z \\
& - \frac{1}{32\pi^3} m\gamma^5 \int_x^y (\not{\partial}\Xi)\log z
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i}{64\pi^3} m^2 (\Xi(y)+\Xi(x))\gamma^5 \log z \\
&-\frac{1}{64\pi^3} m^2 \int_x^y (\partial_j \Xi)\xi_k \gamma^5 \sigma^{jk} \log z \\
&+ (\text{noncausal contributions}) + \mathcal{O}(\xi^2).
\end{aligned}
$$

A.5. Bilinear Potential

$$
\begin{aligned}
\Delta P(x,y) = &-\frac{1}{2\pi^3}\int_x^y H_{ij}\xi^i\xi_k\sigma^{jk} z^{-2} \\
&+\frac{1}{16\pi^3}(H_{jk}(y)+H_{jk}(x))\sigma^{jk} z^{-1} \\
&-\frac{1}{4\pi^3}\int_x^y H_{jk}\sigma^{jk} z^{-1} \\
&+\frac{i}{8\pi^3}\int_x^y \xi_j H^{jk}_{,k} z^{-1} \\
&+\frac{1}{8\pi^3}\int_x^y (2\alpha-1)(\xi^k H_{jk,i}+\xi_i H^k_{jk,})\sigma^{ij} z^{-1} \\
&+\frac{1}{8\pi^3}\int_x^y (\alpha^2-\alpha)(\Box H_{ij})\xi^i\xi_k\sigma^{jk} z^{-1} \\
&-\frac{1}{16\pi^3}\int_x^y \varepsilon^{ijkl} H_{ij,k}\xi_l \gamma^5 z^{-1} \\
&+\frac{1}{8\pi^3}\int_x^y (\alpha^2-\alpha)\partial_j H^l_{kl,}\sigma^{jk} \log z \\
&-\frac{1}{16\pi^3}\int_x^y \left(\alpha^2-\alpha+\tfrac{1}{4}\right)(\Box H_{jk})\sigma^{jk} \log z \\
&-\frac{1}{64\pi^3}\int_x^y (\alpha^2-\alpha)\varepsilon^{ijkl}(\Box H_{ij,k})\xi_l\gamma^5 \log z \\
&+\frac{i}{32\pi^3}\int_x^y (\alpha^2-\alpha)\xi^j(\Box H^k_{jk,}) \log z \\
&+\frac{1}{64\pi^3}\int_x^y (\alpha^2-\alpha)^2(\Box^2 H_{jk})\xi^j\xi_l\sigma^{kl} \log z \\
&+\frac{1}{32\pi^3}\int_x^y (2\alpha^3-3\alpha^2+\alpha)(\xi^k\Box H_{jk,i}+\xi_i\Box H^k_{jk,})\sigma^{ij} \log z \\
&+\frac{i}{8\pi^3} m \int_x^y \varepsilon^{ijkl} H_{ij}\xi_k\gamma^5\gamma_l z^{-1} \\
&-\frac{1}{16\pi^3} m \int_x^y H^k_{jk,}\gamma^j \log z
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{i}{32\pi^3} m \int_x^y (2\alpha - 1)\, \varepsilon^{ijkl} H_{ij,k} \gamma^5 \gamma_l \log z \\
&+ \frac{i}{32\pi^3} m \int_x^y (\alpha^2 - \alpha)\, \varepsilon^{ijkl} (\Box H_{ij}) \xi_k \gamma^5 \gamma_l \log z \\
&- \frac{1}{8\pi^3} m^2 \int_x^y H_{ij} \xi^i \xi_k \sigma^{jk} z^{-1} \\
&- \frac{1}{64\pi^3} m^2 (H_{jk}(y) + H_{jk}(x)) \sigma^{jk} \log z \\
&- \frac{i}{32\pi^3} m^2 \int_x^y \xi_j H^{jk}_{,k} \log z \\
&- \frac{1}{32\pi^3} m^2 \int_x^y (2\alpha - 1)(\xi^k H_{jk,i} + \xi_i H^k_{jk,}) \sigma^{ij} \log z \\
&- \frac{1}{32\pi^3} m^2 \int_x^y (\alpha^2 - \alpha)(\Box H_{ij}) \xi^i \xi_k \sigma^{jk} \log z \\
&+ \frac{1}{64\pi^3} m^2 \int_x^y \varepsilon^{ijkl} H_{ij,k} \xi_l \gamma^5 \log z \\
&- \frac{i}{32\pi^3} m^3 \int_x^y \varepsilon^{ijkl} H_{ij} \xi_k \gamma^5 \gamma_l \log z \\
&+ \frac{1}{64\pi^3} m^4 \int_x^y \varepsilon^{ijkl} H_{ij} \xi^i \xi_k \sigma^{jk} \log z \\
&+ \text{(noncausal contributions)} + \mathcal{O}(\xi^2).
\end{aligned}
$$

B. Perturbation by a Gravitational Field

In this appendix, we outline how the light-cone expansion can be extended to a perturbation by a gravitational field. For the metric, we consider a perturbation h_{jk} of the Minkowski metric $\eta_{jk} = \text{diag}(1, -1, -1, -1)$,

$$
g_{jk}(x) = \eta_{jk} + h_{jk}(x).
$$

We describe gravitation with the linearized Einstein equations (see e.g. [6]). According to the usual formalism, we raise and lower tensor indices with respect to the Minkowski metric. Using the transformation of h_{jk} under infinitesimal coordinate transformations, we can assume [6, Par. 105] that

$$
\partial^k h_{jk} = \frac{1}{2} \partial_j h \quad \text{with} \;\; h := h^k_k.
$$

In the so-called symmetric gauge, the Dirac operator takes the form

$$
i\partial\!\!\!/_x - \frac{i}{2} \gamma^j h_{jk} \eta^{kl} \frac{\partial}{\partial x^l} + \frac{i}{8} (\partial\!\!\!/ h)
$$

(see [2]). In contrast to (1.2), the perturbation is now itself a differential operator.

One complication arises from the fact that the integration measure in curved space is $\sqrt{|g|}\, d^4x = (1+h/2)\, d^4x$, whereas the formula (1.7) for the perturbation of the Dirac sea is valid only if one has the integration measure d^4x of Minkowski space. Therefore we first transform the system such that the integration measure becomes d^4x, then apply (1.7), and finally transform back to the original integration measure $\sqrt{|g|}\, d^4x$. Since the scalar product

$$\int \bar{\Psi}\Phi\sqrt{|g|}\, d^4x = \int \overline{(|g|^{1/4}\Psi)}(|g|^{1/4}\Phi)\, d^4x$$

is coordinate-invariant, the transformation to the measure d^4x is accomplished by

$$\begin{aligned}\Psi(x) \to \hat{\Psi}(x) &= |g|^{1/4}(x)\Psi(x);\\ i\partial\!\!\!/_x - \frac{i}{2}\gamma^j h_j^k \partial_k + \frac{i}{8}(\partial\!\!\!/ h) \to &|g|^{1/4}\left(i\partial\!\!\!/_x - \frac{i}{2}\gamma^j h_j^k \partial_k + \frac{i}{8}(\partial\!\!\!/ h)\right)|g|^{-1/4}\\ &= i\partial\!\!\!/_x - \frac{i}{2}\gamma^j h_j^k \partial_k - \frac{i}{8}(\partial\!\!\!/ h).\end{aligned}$$

The perturbation $\Delta P^{(d^4x)}$ of the transformed system is given by (1.7),

$$\begin{aligned}\Delta P^{(d^4x)}(x,y) = -\int d^4z\Bigg(&s(x,z)\left(\frac{i}{2}\gamma^j h_j^k \frac{\partial}{\partial z^k} - \frac{i}{8}(\partial\!\!\!/ h)(z)\right)P(z,y)\\ &+ P(x,z)\left(\frac{i}{2}\gamma^j h_j^k \frac{\partial}{\partial z^k} - \frac{i}{8}(\partial\!\!\!/ h)(z)\right)s(z,y)\Bigg). \qquad \text{(B.1)}\end{aligned}$$

The formula for the transformation of the Dirac sea to the original integration measure $\sqrt{|g|}d^4x$ is

$$P(x,y) + \Delta P(x,y) = |g|^{-1/4}(x)|g|^{-1/4}(y)(P(x,y) + \Delta P^{(d^4x)}(x,y)).$$

Thus

$$\Delta P(x,y) = \Delta P^{(d^4x)}(x,y) - \tfrac{1}{4}(h(x)+h(y))P(x,y).$$

The factors $P(z,y)$ and $s(z,y)$ in (B.1) depend only on $(z-y)$, that is,

$$\frac{\partial}{\partial z^k}P(z,y) = -\frac{\partial}{\partial y^k}P(z,y), \qquad \frac{\partial}{\partial z^k}s(z,y) = -\frac{\partial}{\partial y^k}s(z,y),$$

so we may rewrite the z-derivatives as y-derivatives, which can be pulled out of the integral. Furthermore, the relations

$$\begin{aligned}\int d^4z\, P(x,z)(i\partial\!\!\!/_z h(z))s(z,y) &= \int d^4z\, P(x,z)[(i\partial\!\!\!/_z - m), h(z)]s(z,y)\\ &= -P(x,y)h(y)\end{aligned}$$

and

$$\int d^4z\, s(x,z)(i\partial\!\!\!/_z h(z))P(z,y) = h(x)P(x,y)$$

allow us to simplify the factors $(\partial\!\!\!/ h)$ in the integral. In the resulting formula for $\Delta P(x,y)$, one recovers the perturbation by an electromagnetic potential. More precisely,

$$\Delta P(x,y) = \left(-\frac{1}{8}h(x) - \frac{3}{8}h(y)\right) P(x,y) - i\,\frac{\partial}{\partial y^k}\Delta P[\gamma^j h_j^k](x,y), \qquad \text{(B.2)}$$

where $\Delta P[\gamma^j h_j^k](x,y)$ is the perturbation (1.7) of the Dirac sea corresponding to the electromagnetic potential $\mathcal{B} = \gamma^j h_j^k$. The light-cone expansion of $\Delta P(x,y)$ is obtained by substituting the light-cone expansion of $\Delta P[\gamma^j h_j^k](x,y)$ into (B.2) and calculating the y-derivatives. To the order $\mathcal{O}((y-x)^0)$ on the light cone, this gives the following formula for the light-cone expansion of the Dirac sea in the gravitational field:

$$\begin{aligned}
\Delta P(x,y) = &-\frac{i}{8\pi^3}\left(\int_x^y h_j^k\right)\xi^j\,\frac{\partial}{\partial y^k}\,\not{\xi}\,z^{-2} \\
&- \frac{i}{16\pi^3}\left(\int_x^y (2\alpha-1)\gamma^i\xi^j\xi^k(h_{jk,i}-h_{ik,j})\right)z^{-2} \\
&- \frac{1}{32\pi^3}\left(\int_x^y \varepsilon^{ijlm}(h_{jk,i}-h_{ik,j})\xi^k\xi_l\rho\gamma_m\right)z^{-2} \\
&+ \frac{i}{16\pi^3}\left(\int_x^y (\alpha^2-\alpha)\xi^j\xi^k R_{jk}\right)\not{\xi}\,z^{-2} \\
&- \frac{i}{128\pi^3}\left(\int_x^y (\alpha^4-2\alpha^3+\alpha^2)\not{\xi}\,\xi^j\xi^k\Box R_{jk}\right)z^{-1} \\
&+ \frac{i}{128\pi^3}\left(\int_x^y (6\alpha^2-6\alpha+1)\not{\xi}\,R\right)z^{-1} \\
&- \frac{i}{128\pi^3}\left(\int_x^y (4\alpha^3-6\alpha^2+2\alpha)\xi^j\xi^k\gamma^l R_{j[k,l]}\right)z^{-1} \\
&- \frac{1}{64\pi^3}\left(\int_x^y (\alpha^2-\alpha)\varepsilon^{ijlm}R_{ki,j}\xi^k\xi_l\rho\gamma_m\right)z^{-1} \\
&+ \frac{i}{32\pi^3}\left(\int_x^y (\alpha^2-\alpha)\xi^j\gamma^k G_{jk}\right)z^{-1} \\
&+ \frac{i}{32\pi^3}m\left(\int_x^y h_{ki,j}\right)\xi^k\sigma^{ij}z^{-1} \\
&+ \frac{1}{32\pi^3}m\int_x^y (\alpha^2-\alpha)R_{jk}\xi^j\xi^k z^{-1} \\
&- \frac{i}{64\pi^3}m^2\int_x^y (2\alpha-1)(h_{jk,i}-h_{ik,j})\gamma^i\xi^j\xi^k z^{-1} \\
&+ \frac{1}{64\pi^3}m^2\int_x^y \varepsilon^{ijlm}h_{jk,i}\xi^k\xi_l\rho\gamma_m z^{-1} \\
&+ \frac{i}{64\pi^3}m^2\int_x^y (\alpha^2-\alpha)R_{jk}\xi^j\xi^k\not{\xi}\,z^{-1} + \mathcal{O}(\xi^0),
\end{aligned}$$

where R_{jk} and R are the (linearized) Ricci tensor and scalar curvature, respectively. A general difference to the formulas of Appendix A is that $\Delta P(x, y)$ now has a stronger singularity on the light cone. This is a consequence of the y-derivative in (B.2). The leading singularity of $\Delta P(x, y)$ can be understood as describing the "deformation" of the light cone by the gravitational field in linear approximation.

We finally remark that this method works also for the higher-order perturbation theory as developed in [3]. It can likewise be used to perform the light-cone expansion of higher-order Feynman diagrams in the presence of a gravitational field.

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Max Planck Institute for
Mathematics in the Sciences
Inselstr. 22-26
04103 Leipzig
Germany

Felix.Finster@mis.mpg.de