Birational Maps, Positive Currents, and Dynamics

JEFFREY DILLER

1. Introduction

There has been a great deal of recent research in multivariable complex dynamics, most of it devoted to either polynomial diffeomorphisms of $\mathbf{C}^2$ or holomorphic maps of $\mathbf{P}^n$. Pluripotential theory plays a prominent supporting role in nearly all this work. Our concern in this paper and its predecessor [Dil] is to extend the application of pluripotential theory to study dynamics of birational maps of $\mathbf{P}^2$.

Anyone who seeks to understand the dynamics of a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ faces an immediate problem: birational maps are not generally maps. That is, except when f_+ has degree $d = 1$, there exists a finite non-empty set I^+ of points where f_+ cannot be defined continuously. In a precise sense, f_+ "blows up" each of these points of indeterminacy to an entire algebraic curve. Nevertheless, we believe that it is worthwhile to pretend as far as possible that birational maps really are diffeomorphisms.

Maintaining this pretense means (among other things) that we must generalize operations like pushforward and pullback that are natural for diffeomorphisms. Since we intend to use pluripotential theory, it is particularly important to make sense of these operations as they apply to positive currents. Already in [Dil] we observed that there are at least two reasonable ways for a birational map to act on a positive closed $(1,1)$ current T. In order to distinguish between these actions, we refer to them as pushforward $f_{+*}T$ and pullback f_+^*T, respectively. Intuitively speaking, the first action discounts any contribution from the indeterminacy set whereas the second (defined by pulling back a potential function) takes the fullest possible account of such contributions. Theorem 2.3 gives a precise condition for agreement between pushforward by a birational map and pullback by its inverse. Namely, one has agreement if and only if the so-called Lelong numbers of T vanish at each point in I^+.

Our first application of Theorem 2.3 is to a natural current associated with iterates of a birational map. By pulling back and rescaling the Fubini–Study Kähler form Θ, one obtains a positive closed $(1,1)$ current

$$\mu^+ = \lim_{n\to\infty} \frac{1}{d^n} f_+^{n*}\Theta.$$

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That this limit exists was proven in [Dil]. It is immediate, moreover, that $f_+^*\mu^+ = d \cdot \mu^+$. Here we show (Theorem 2.9) that μ^+ is *extremal*—that is, the only positive closed (1, 1) currents dominated by μ^+ are multiples of μ^+. Aside from its dynamical significance as a sort of ergodicity property of μ^+, Theorem 2.9 provides a new source of naturally arising, nonalgebraic extremal currents.

Bedford and Smillie [BS1; BS2] considered the action of polynomial automorphisms on certain *nonclosed* (1, 1) currents and thereby obtained many useful dynamical results. Following their lead, in Section 3 we consider the action of a birational map on a current of the form ψT, where $\psi : \mathbf{P}^2 \to \mathbf{C}$ is a cutoff function and T is a positive closed (1, 1) current. We show under fairly general conditions that the sequence $(1/d^n) f^n_{+*}(\psi T)$ converges to a multiple of μ^-, that is, the invariant current associated with the inverse of f_+. The main novelties in this section are the definition and means we provide for making sense of $f^n_{+*}(\psi T)$. Once these are in place, Theorem 2.9 and the methods of Bedford and Smillie combine to give the desired results.

In the final section of the paper, we give several applications of the results from Section 3. We show that $\operatorname{supp}\mu^+$ is nowhere dense unless it contains $\operatorname{supp}\mu^-$. With stronger hypotheses on f_+, we are able to generalize some results of Bedford and Smillie for polynomial diffeomorphisms. Namely, $\operatorname{supp}\mu^+$ coincides with the boundary of the basin of any attracting periodic point and with the closure of the stable manifold of any saddle periodic point. In a somewhat different vein, we show that $\operatorname{supp}\mu^+ \cap \operatorname{supp}\mu^-$ consists only of nonwandering points and exhibit another ergodic-type property of μ^+ that fails even in fairly simple examples outside the birational setting.

2. Birational Maps Acting on Positive Closed (1, 1) Currents

Let $\pi : \mathbf{C}^3 \setminus \{\mathbf{0}\} \to \mathbf{P}^2$ be the canonical projection giving homogeneous coordinates on $\mathbf{P}^2$. Any rational map $f : \mathbf{P}^2 \to \mathbf{P}^2$ can be regarded as the natural relation induced by a homogeneous polynomial map $\tilde{f} : \mathbf{C}^3 \to \mathbf{C}^3$. Clearly, f does not change if we multiply each of the coordinates of $\tilde{f}$ by the same homogeneous polynomial. Therefore, we will assume that $\tilde{f}$ is a *minimal* representative for f in the sense that the coordinate functions of f have lowest possible degree. Under this assumption, we define the (algebraic) degree of f to be the degree of $\tilde{f}$.

The critical set $\mathcal{C}$ of f is an algebraic curve equal to the image under π of the critical set of $\tilde{f}$. It can happen that $\tilde{f}^{-1}(\mathbf{0})$ is nontrivial even when $\tilde{f}$ is minimal. In this case $f(\pi(\tilde{p}))$ is ill-defined whenever $\tilde{f}(\tilde{p}) = \mathbf{0}$. The set $I = \pi(\tilde{f}^{-1}(\mathbf{0})) \subset \mathbf{P}^2$ of all such *points of indeterminacy* is always finite, and we will persist in writing $f : \mathbf{P}^2 \to \mathbf{P}^2$ as if f were well-defined everywhere.

A rational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is *birational* if there exists another rational map $f_- : \mathbf{P}^2 \to \mathbf{P}^2$ and an algebraic curve V such that $f_+ \circ f_- = f_- \circ f_+ = \mathrm{id}$ on $\mathbf{P}^2 \setminus V$. The use of $+/-$ superscripts to distinguish a birational map from its rational inverse emphasizes the fact that f_+ and f_- are not, strictly speaking, set theoretic inverses. We will use $+/-$ subscripts and superscripts in all of what follows to distinguish objects corresponding to f_+ from objects corresponding to f_-.

For instance, I^- denotes the indeterminacy set for f_-. The following proposition (see [Dil] for a proof) describes the relationship between indeterminacy and critical sets for a birational map.

Proposition 2.1. *The following statements are true for any birational map* $f_+ \colon \mathbf{P}^2 \to \mathbf{P}^2$.

(1) $I^+ \subset \mathcal{C}^+$, *and each irreducible component of* $\mathcal{C}^+$ *contains a point of* I^+.
(2) *Given any irreducible curve* $V \subset \mathcal{C}^+$, $f_+(V)$ *is a single point in* I^-; *likewise, given any* $p^- \in I^-$, $f_+^{-1}(p^-)$ *is a component of* $\mathcal{C}^+$.
(3) $f_+ \colon \mathbf{P}^2 \setminus \mathcal{C}^+ \to \mathbf{P}^2 \setminus \mathcal{C}^-$ *is a biholomorphism.*

We make an important technical distinction between the image of a closed set K under f_+ and its preimage under f_-. We declare that $f_+(K) = \overline{f_+(K \setminus I^+)}$ and $f_-^{-1}(K) = \overline{\{ p \in \mathbf{P}^2 \setminus I^- : f_-(p) \in K \}}$. In general, $f_+(K) \subset f_-^{-1}(K)$, but the inclusion can be strict if $K \cap I^+ \neq \emptyset$.

Degree-1 birational maps of $\mathbf{P}^2$ are dynamically rather simple, so we assume in what follows that all birational maps under consideration have degree greater than 1. Such maps will necessarily have non-empty critical sets and thus (by Proposition 2.1) non-empty indeterminacy sets as well. Therefore, one must be rather careful when using a birational map to transform an analytic object such as a form or a current.

We want specifically to consider actions of birational maps on positive closed (1, 1) currents. Before doing so, however, we fix some notation and recall a couple of facts about positive currents on $\mathbf{P}^2$. For more thorough background on positive currents, we refer the reader to the book by Klimek [Kli] and survey articles by Demailly [Dem] and Skoda [Sko].

The *mass* of a positive current T on a set $K \subset \mathbf{P}^2$ is

$$\mathrm{M}_K[T] = \sup\{ T(\varphi) : |\varphi| \le 1,\ \operatorname{supp}\varphi \subset K \}.$$

Of course, this definition implies the choice of an Hermitian metric on a neighborhood of $\bar{K}$, but for any two such choices the resulting mass norms are comparable. Where we do not indicate otherwise, we imply the use of the Fubini–study metric on $\mathbf{P}^2$, letting Θ denote the associated Kähler form. It turns out that

$$\|T\| \stackrel{\text{def}}{=} \mathrm{M}_{\mathbf{P}^2}[T] = \int_{\mathbf{P}^2} T \wedge \Theta.$$

A positive closed (1, 1) current can be expressed locally as $dd^c u$ for some plurisubharmonic u. Fornæss and Sibony [FS2] have in fact observed that there is a correspondence between positive closed (1, 1)-currents T on $\mathbf{P}^2$ and *homogeneous potentials*—that is, plurisubharmonic functions $\tilde{u} \colon \mathbf{C}^3 \to \mathbf{R} \cup \{-\infty\}$ satisfying $\tilde{u}(\lambda\tilde{p}) = \tilde{u}(\tilde{p}) + c\log|\lambda|$ for every $\lambda \in \mathbf{C}$, $\tilde{p} \in \mathbf{C}^3$ and some $c \ge 0$. They show that, given T, there exists a $\tilde{u}$ such that $\pi^*T = dd^c\tilde{u}$. It follows that $c = \|T\|$ and $\tilde{u}$ is unique up to addition of a constant. Likewise, any homogeneous potential $\tilde{u}$ induces a positive closed (1, 1) current T on $\mathbf{P}^2$. If $U \subset \mathbf{P}^2$ and $\sigma : U \to \mathbf{C}^3$ is a holomorphic section, then $T|_U$ is given by $dd^c(\tilde{u} \circ \sigma)$. Homogeneity guarantees that this definition does not depend on the choice of section.

In [Dil] we discussed two actions of a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ on a positive closed $(1,1)$ current T with homogeneous potential $\tilde{u}$. First of all, we defined the "pullback" by $\pi^* f_+^* T = dd^c(\tilde{u} \circ \tilde{f})$. Besides its consistency with notation used in related papers (e.g. [HuPa; FS2]), this definition of $f_+^* T$ has the advantage that mass transforms predictably according to the formula $\| f_+^* T \| = (\deg f_+) \| T \|$. As we hope will emerge in what follows, $f_+^* T$ is in some sense the largest reasonable notion of the preimage of T, generalizing the notion of the total transform of an algebraic curve by a rational map.

We also defined the pushforward of T by f_+. Taking advantage of the fact that $f_+ : \mathbf{P}^2 \setminus \mathcal{C}^+ \to \mathbf{P}^2 \setminus \mathcal{C}^-$ is a biholomorphism, we first push the restriction $T|_{\mathbf{P}^2 \setminus \mathcal{C}^+}$ forward to a positive closed $(1,1)$ current on $\mathbf{P}^2 \setminus \mathcal{C}^-$. We then extend T by zero across $\mathcal{C}^-$. Thanks to an extension theorem of Harvey and Polking [HaPo], the result is a well-defined positive closed $(1,1)$ current on $\mathbf{P}^2$. We denote this current by $f_{+*} T$. It should be clear that $f_{+*} T$ is the smallest reasonable notion of the image of T, analogous to the proper transform of a curve by a rational map. Before stating the next proposition, we recall that T is *extremal* among positive closed $(1,1)$ currents if every decomposition $T = T_1 + T_2$ into a sum of positive closed currents is trivial, that is, if $T_j = c_j T$.

PROPOSITION 2.2. *If T is extremal, then so is $f_{+*}T$. If $T|_{\mathcal{C}^+} = 0$ and $f_{+*}T$ is extremal, then so is T.*

Proof. First assume that T is extremal. Let $f_{+*}T = S_1 + S_2$ be a decomposition in which S_1 dominates no positive multiple of $f_{+*}T$. It is clear that pushforward acts linearly and preserves positivity, so item (5) of Proposition 4.7 in [Dil] gives that $T \geq f_{-*} f_{+*} T \geq f_{-*} S_1$. That is, $T = f_{-*} S_1 + (T - f_{-*} S_1)$, so $f_{-*} S_1 = cT$. Using the same fact from [Dil], we then conclude that $S_1 \geq c f_{+*} T$ and therefore $c = 0$. The only nontrivial elements in the kernel of f_{-*} are supported on $\mathcal{C}^-$, and by definition $f_{+*}T$ has no support on this set. Hence $S_1 = 0$.

Now assume that $f_{+*}T$ is extremal and that T has no mass concentrated on $\mathcal{C}^+$. Let $T = T_1 + T_2$ be a decomposition, and note that $f_{+*}T = f_{+*}T_1 + f_{+*}T_2$. It follows that $f_{+*}T_1 = c f_{+*} T$. From the aforementioned fact in [Dil], we have $T_1 = cT$. □

As with images and preimages of closed sets, it is not always the case that $f_{+*}T = f_-^* T$. The main result of this section is a necessary and sufficient condition for equality. To state it, we recall that the *Lelong number* of a positive closed current T at $p \in \mathbf{P}^2$ is given in local coordinates z centered at p by

$$\nu(T, p) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{\|z\| < r} T \wedge \theta,$$

where $\theta = dd^c \|z\|^2$. If $T = dd^c u$ near p, then the Lelong number can be computed from u by

$$\nu(T, p) = \sup\{ \gamma \geq 0 : u(q') \leq \gamma \log \operatorname{dist}(p, q) + O(1) \} \qquad (1)$$

(see [Dem, eq. (5.5e)]).

THEOREM 2.3. *Suppose that T is a positive closed $(1,1)$ current on $\mathbf{P}^2$ and that $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is birational. Then $f_-^* T - f_{+*}T$ is a nonnegative linear combination of currents of integration over components of $\mathcal{C}^-$. Furthermore, $f_{+*}T = f_-^* T$ if and only if $\nu(T, p) = 0$ for every $p \in I^+$.*

This theorem is a consequence of the following result about Lelong numbers.

THEOREM 2.4. *Suppose that T is a positive closed $(1,1)$ current on $\mathbf{P}^2$. Then $\nu(f_-^* T, p) \neq 0$ if and only if either $p \in I^-$ or $\nu(T, f_-(p)) \neq 0$.*

An earlier version of this paper contained a proof of this result. However, recent papers of Favre [Fa2] and Kiselman [Kis] generalize the result to arbitrary rational maps of $\mathbf{P}^n$, so for brevity's sake we refer the reader to those papers for the proof.

Proof of Theorem 2.3. Since $f_+ : \mathbf{P}^2 \setminus \mathcal{C}^+ \to \mathbf{P}^2 \setminus \mathcal{C}^-$ is a biholomorphism, $f_{+*}T$ and $f_-^* T$ coincide with the usual notions of pushforward and pullback on $\mathbf{P}^2 \setminus \mathcal{C}^-$. In particular, they coincide with each other on this set. Hence $f_-^* T - f_{+*}T$ is supported on $\mathcal{C}^-$. The restriction of $f_{+*}T$ to $\mathcal{C}^-$ is trivial by definition, so $f_-^* T - f_{+*}T$ is positive. A well-known theorem [Siu] implies that $f_-^* T - f_{+*}T = \sum_{V \subset \mathcal{C}^-} c_V [V]$, where $V \subset \mathcal{C}^-$ is an irreducible component and $c_V \geq 0$. If $c_V > 0$ then $\nu(T, p) > 0$ for every $p \in V$. Therefore, we can apply Theorem 2.4 to any $p \in V \setminus I^-$ and conclude that $\nu(T, f_-(p)) > 0$. Since $f_-(p) \in I^+$, the "only if" portion of the theorem holds.

If, on the other hand, $f_-^* T = f_{+*}T$, then it follows that the restriction of $f_-^* T$ to $\mathcal{C}^-$ is trivial. Thus, by Siu's results again, $\nu(f_-^* T, p) = 0$ for every $p \in \mathcal{C}^-$ outside a countable subset. Each $p \in I^+$ is the f_--image of some nontrivial algebraic curve in $\mathcal{C}^-$, by Proposition 2.1—in particular, $p = f_-(q)$ for some q such that $\nu(f_-^* T, q) = 0$. Therefore, Theorem 2.4 implies that $\nu(T, p) = 0$ as well. $\square$

2.1. Application: Invariant Currents are Extremal

We now present a dynamical application of Theorem 2.3. For this, it is necessary to recall some results from [Dil].

PROPOSITION 2.5. *The following statements are equivalent for a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ with degree $d \geq 2$ and inverse f_-:*

(1) $\deg(f_+^n) = d^n$ *for all* n;
(2) $I^+ \cap f_+^n(I^-) = \emptyset$ *for all* n;
(3) $f_-^n(I^+) \cap f_+^m(I^-) = \emptyset$ *for all* $n, m \geq 0$.

We will call a birational map *algebraically stable* if it satisfies any of the equivalent conditions in the conclusion of this proposition. This accords with a recent survey article [Sib] wherein the term is applied to any rational map whose iterates have maximal degree growth. It turns out that an algebraically stable map admits a dynamically invariant *Green's function*.

THEOREM 2.6. *If f_+ is algebraically stable, then the sequence*

$$\lim_{n\to\infty} \frac{1}{d^n} \log\|\tilde{f}_+^n\|$$

converges pointwise and in L^1_{loc} to a plurisubharmonic function $\tilde{G}^+$ satisfying

(1) $\tilde{G}^+ \circ \tilde{f}_+(\tilde{p}) = d \cdot \tilde{G}^+(\tilde{p})$ *and*
(2) $\tilde{G}^+(\lambda\tilde{p}) = \tilde{G}^+(\tilde{p}) + \log|\lambda|$

for all $\tilde{p} \in \mathbf{C}^3$ and all $\lambda \in \mathbf{C}$.

Green's functions for holomorphic maps of $\mathbf{P}^n$ were first introduced by Hubbard and Papadopol [HuPa] and further studied in the more general setting of rational maps by Fornæss and Sibony [FS2]. In [Dil] we proved the theorem just stated, and then Favre [Fa1] gave a quite different proof. Sibony [Sib] has recently given a very elegant proof for existence of a Green's function that applies to *any* algebraically stable rational map of $\mathbf{P}^n$.

We note that f_+ determines $\tilde{G}^+$ only up to an additive constant. By replacing $\tilde{f}_+$ with a small multiple of $\tilde{f}_+$, one can arrange that the sequence defining $\tilde{G}^+$ is actually decreasing. We refer to the unique induced current $\pi^*\mu^+ = dd^c\tilde{G}^+$ as the *escape current* for f_+. We showed in our previous paper that μ^+ transforms well under f_+ and that μ^+ attracts a large set of currents under pullback.

THEOREM 2.7. *The current μ^+ for an algebraically stable birational map has the following properties*:

(1) μ^+ *has no support concentrated on any algebraic curve* (*see* [FS2]);
(2) $\mu^+ = f_+^*\mu^+/d = f_{-*}\mu^+/d = d \cdot f_{+*}\mu^+$.

Suppose that $W \subset \mathbf{P}^2$ is a (*possibly empty*) *open set containing all superattracting periodic points of an algebraically stable birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$. Suppose that $\{T_n\}$ is a sequence of positive closed* $(1,1)$ *currents such that* $\operatorname{supp} T_n \cap W = \emptyset$ *and that $\|T_n\| = c$ is constant with respect to n. Then*

$$\lim_{n\to\infty} \frac{1}{d^n} f_+^{n*} T_n = c\mu^+.$$

For any $1 \le n \le \infty$, set $I_n^+ = \bigcup_{j=0}^{n-1} f_-^n(I^+)$. An immediate consequence of Theorem 2.3 and Theorem 2.7(2) is the following.

COROLLARY 2.8. *If $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is algebraically stable, then $\nu(\mu^+, p) = 0$ for each $p \in I_\infty^-$.*

Another consequence of Theorems 2.3 and 2.7—and the main result of this section—is an "ergodic" property for μ^+.

COROLLARY 2.9. *If $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ is algebraically stable, then μ^+ is extremal in the cone of positive closed* $(1,1)$ *currents.*

This corollary is proven for Hénon maps in [FS1, Sec. VII.3]. Our proof is a generalization of the one given there.

Proof. Suppose that μ^+ dominates a positive closed current T. (We will show $T = c\mu^+$.) It follows from Theorems 2.7 and 2.3 that both $f^n_{+*}T$ and f^{n*}_+T are dominated by multiples of μ^+ for all $n \geq 0$. Consequently, Corollary 2.8 implies that $\nu(f^n_{+*}T, p) = 0$ at every point in I^-_∞.

We showed in [Dil] that $T - f^n_{-*}f^n_{+*}T$ is positive and concentrated on an algebraic curve. Since μ^+ concentrates no support on any algebraic curve, we must actually have $T = f^n_{-*}f^n_{+*}T$ for all n. Theorem 2.3 implies further that $T = f^{n*}_+f^n_{+*}T$. In particular,

$$\|T\| = \|f^{n*}_+f^n_{+*}T\| = d^n \cdot \|f^n_{+*}T\|.$$

Hubbard and Papadopol [HuPa] showed that, if iterates of f form a normal family on an open set W, then $\operatorname{supp}\mu^+ \cap W = \emptyset$. Therefore, there is a neighborhood W of any superattracting cycle such that

$$(\operatorname{supp} f^n_{+*}T) \cap W \subset (\operatorname{supp}\mu^+) \cap W = \emptyset$$

for all n. We can now apply the last part of Theorem 2.7 to the sequence $T_n = d^n f^n_{+*}T$ to conclude that

$$\|T\|\mu^+ = \lim_{n\to\infty} \frac{1}{d^n} f^{n*}_+(d^n f^n_{+*}T) = T. \qquad \square$$

3. Pushforwards of Nonclosed Positive Currents

For many purposes, the last part of Theorem 2.7 is not strong enough. In this section we will extend that statement to include closed currents that have been “truncated” by contraction with cutoff functions. That is, for birational maps we will prove the analog of Theorem 1.6 in [BS2]. Actually, we will prove two such analogs: one imposes a weak hypothesis on the map but a somewhat restrictive hypothesis on the current; the other places less restriction on the current but only in exchange for a stronger hypothesis concerning the map. Substantial technical details aside, the proof that we give—especially Lemmas 3.2 and 3.3—largely follows the one given in [BS2]. However, at the conclusion of the proof, our approach diverges from [BS2] and instead follows [FS1, Sec. VII.3] more closely.

Throughout this section, let $U \subset \mathbf{P}^2$ be a given open set, T a positive closed $(1,1)$ current on U, $\psi\colon U \to \mathbf{C}$ a smooth function with compact support, and $f_+\colon \mathbf{P}^2 \to \mathbf{P}^2$ a birational map. First, we borrow an idea from [RS] to provide a workable definition of $f_{+*}(\psi T)$. Let $\Gamma \subset \mathbf{P}^2 \times \mathbf{P}^2$ be the irreducible analytic subvariety obtained as the closure of the graph of $f_+|_{\mathbf{P}^2\setminus I^+}$. Let $\alpha, \beta\colon \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$ be projection onto the first and second coordinates. Since Γ might be singular, we consider a desingularization $\tilde\Gamma \to \Gamma$ of Γ. Abusing notation slightly, we continue to use α and β to denote the pullback to $\tilde\Gamma$ of the projection functions. It is evident that the exceptional set of $\alpha\colon \tilde\Gamma \to \mathbf{P}^2$ is the 1-dimensional “vertical” curve

$\alpha^{-1}(I^+)$. It is also clear that $\alpha\colon \tilde{\Gamma} \setminus \alpha^{-1}(I^+) \to \mathbf{P}^2 \setminus I^+$ is a biholomorphism. Therefore, we can lift T to a positive closed (1, 1) current $\alpha^* T$ on $\alpha^{-1}(U) \subset \tilde{\Gamma}$ by pushing forward with α^{-1} on $U \setminus I^+$ and then extending trivially across $\alpha^{-1}(I^+)$. The extension theorem of [HaPo] guarantees that $\alpha^* T$ is positive and closed on $\alpha^{-1}(U)$. We define $f_{+*}(\psi T)$ by its action on test forms:

$$\langle f_{+*}(\psi T), \varphi \rangle = \langle \alpha^* T, (\psi \circ \alpha)\beta^* \varphi \rangle.$$

In what follows we may assume, with no loss of generality, that ψ is real and nonnegative. Clearly, this assumption implies that both ψT and $f_{+*}(\psi T)$ are positive currents.

If $U = \mathbf{P}^2$ and $\psi \equiv 1$, then the definition of pushforward we have just given coincides with the one given in Section 2. In fact, more is true.

PROPOSITION 3.1. *Suppose that $\chi_j\colon \mathbf{P}^2 \to [0, 1]$ are smooth functions such that χ_j vanishes on a neighborhood of $\mathcal{C}^-$ and* supp $(1 - \chi_j)$ *decreases to $\mathcal{C}^-$ as $j \to \infty$. Then, for any test form φ, we have*

$$\langle f_{+*}(\psi T), \varphi \rangle = \lim_{j\to\infty} \langle f_{+*} T, \chi_j (\psi \circ f_-)\varphi \rangle = \lim_{j\to\infty} \langle \psi T, f_+^*(\chi_j \varphi) \rangle.$$

The pushforward in the middle expression and the pullback in the right-hand expression can be understood to take place with respect to a biholomorphic map.

Proof. What is needed is to show that $f_{+*}(\psi T)$ concentrates no mass on $\mathcal{C}^-$. Note that $\beta^{-1}(\mathcal{C}^-) = \alpha^{-1}(\mathcal{C}^+)$ can be divided into two components: $\overline{\beta^{-1}(\mathcal{C}^- \setminus I^-)} = \alpha^{-1}(I^+)$ and $\beta^{-1}(I^-) = \overline{\alpha^{-1}(\mathcal{C}^+ \setminus I^+)}$. We have that $\alpha^* T$ concentrates no support on the first component by definition, and $\beta^* \varphi$ is identically zero on the second component. Therefore, the restriction of $\alpha^*(\psi T)$ to $\beta^{-1}(\mathcal{C}^-)$ contributes nothing to the pairing $\langle \alpha^*(\psi T), \beta^* \varphi \rangle$. □

In order to state and prove the following lemma, we recall that one can sometimes use an "integration by parts" construction to define wedge products of positive closed currents (see [BT; FS3] for details). Namely, if $T = dd^c u$ and $S = dd^c v$ are positive closed (1, 1) currents on an open subset $V \subset \mathbf{C}^2$ and if u is continuous, then one declares

$$\int_U \varphi T \wedge S \stackrel{\text{def}}{=} \langle S, u\, dd^c \varphi \rangle$$

for any $\varphi \in C_0^\infty(U)$. This continuously extends the usual notion of wedge product of smooth currents in the sense that, if u_j and v_j are smooth plurisubharmonic functions decreasing (resp.) to u and v, then $dd^c u_j \wedge dd^c v_j \to S \wedge T$ weakly. In particular, $T \wedge S$ is a positive Borel measure. For purposes of this paper, we will say that a wedge product $T \wedge S$ of positive closed currents is *admissible* if, near each point, at least one of the currents has a continuous local potential. In particular, a necessary condition for admissibility of $T \wedge \mu^+$ is that T have continuous potentials in a neighborhood of each point in the *extended indeterminacy set* $\mathcal{I}^+ = \overline{I_\infty^+}$.

LEMMA 3.2. *Suppose that f_+ is algebraically stable of degree d and that T admits a wedge product with μ^+. Then*

$$\lim_{n\to\infty} \frac{1}{d^n} \int_{\mathbf{P}^2} f^n_{+*}(\psi T) \wedge \Theta = \int_{\mathbf{P}^2} \psi T \wedge \mu^+.$$

In particular, there is a constant C such that $\mathrm{M}_{\mathbf{P}^2}[f^n_{+*}(\psi T)] \le Cd^n$ for all n.

Proof. Let $\mathcal{C}^+_n = \bigcup_0^{n-1} f_+^{-1}(\mathcal{C}^+)$ and $\mathcal{C}^-_n$ denote the critical sets of f^n_+ and f^n_-, respectively. Let $\chi_j : \mathbf{P}^2 \to [0,1]$ be a sequence of smooth functions such that $\chi_j \equiv 0$ in a neighborhood of $\mathcal{C}^-_n$ and $\operatorname{supp}(1-\chi_j)$ decreases to $\mathcal{C}^-_n$. Then, by Proposition 3.1, we have

$$\frac{1}{d^n}\langle f^n_{+*}(\psi T), \Theta\rangle = \lim_{j\to\infty} \frac{1}{d^n}\langle \psi T, (\chi_j \circ f^n_+) f^{n*}_+ \Theta\rangle.$$

Local potentials for $f^{n*}_+\Theta$ are unbounded only at points in I^+_n, so T admits a wedge product with $f^{n*}_+\Theta$ (viewed as a positive closed (1, 1) current). Recall further that local potentials for $(1/d^n) f^{n*}_+\Theta$ may be taken to decrease to local potentials for μ^+. Therefore, we continue to compute

$$\begin{aligned}\lim_{j\to\infty} \frac{1}{d^n}\langle \psi T, (\chi_j \circ f^n_+) f^{n*}_+ \Theta\rangle &= \lim_{j\to\infty} \frac{1}{d^n}\int_{\mathbf{P}^2} (\chi_j \circ f^n_+)\psi T \wedge f^{n*}_+\Theta \\ &\le \frac{1}{d^n}\int_{\mathbf{P}^2} \psi T \wedge f^{n*}_+\Theta \to \int_{\mathbf{P}^2} \psi T \wedge \mu^+.\end{aligned}$$

It remains to show that the last inequality is actually an equality—in other words, that the measure $T \wedge f^{n*}_+\Theta$ concentrates no mass on $\mathcal{C}^+_n = (f^n_+)^{-1}(\mathcal{C}^-_n)$. Note first that $f^{n*}_+\Theta$ is smooth everywhere except at I^+_n. Therefore, it follows directly from the integration-by-parts definition of wedge product that $T \wedge f^{n*}_+\Theta$ will not concentrate mass on $\mathcal{C}^+_n \setminus I^+_n$ unless T does. But if T concentrates mass on $\mathcal{C}^+_n$, then the theorem of Siu (mentioned in the proof of Theorem 2.3) implies that T dominates a multiple of the current of integration over some component of $\mathcal{C}^+_n$. This would be inconsistent with the assumption that local potentials for T are continuous near points in I^+_n. Therefore, we need worry only about mass focused at points in I^+_n. However, continuous local potentials for T near I^+_n rule out point masses, which can be established by essentially the same argument used to prove Corollary 2.5 in [BT]. □

LEMMA 3.3. *Given the hypotheses of Lemma 3.2, the sequences $(1/d^n)\partial f^n_{+*}(\psi T)$ and $(1/d^n) dd^c f^n_{+*}(\psi T)$ tend to zero in the mass norm as $n \to \infty$.*

Proof. Let λ be a test 1-form on $\mathbf{P}^2$ such that $\|\lambda\|_\infty \le 1$. Let $\tilde{\Gamma}$ be the desingularization of the graph of f^n_+, with coordinate projections α and β. Choose a compactly supported smooth function $\psi_1 : U \to [0,1]$ such that $\psi_1 \equiv 1$ on a neighborhood of $\operatorname{supp} \psi$. Then

$$
\begin{aligned}
|\langle f^n_{+*}\psi T, d\lambda\rangle| &= |\langle \alpha^* T, d(\psi\circ\alpha)\wedge\beta^*\lambda\rangle| \\
&\le \langle \alpha^* T, \alpha^*(d\psi)\wedge\alpha^*(d^c\psi)\rangle^{1/2}\langle \alpha^* T, -i(\psi_1\circ\alpha)\beta^*\lambda\wedge\beta^*\bar\lambda\rangle^{1/2} \\
&= \langle T, d\psi\wedge d^c\psi\rangle^{1/2}\langle f^n_{+*}(\psi_1 T), -i\lambda\wedge\bar\lambda\rangle^{1/2} \\
&\le C d^{n/2}
\end{aligned}
$$

The first inequality is essentially Schwarz's inequality. The second equality follows from the facts that T concentrates no mass on I^+, that by definition $\alpha^* T$ concentrates no mass on $\alpha^{-1}(I^+)$, and that $\alpha\colon \tilde\Gamma\setminus\alpha^{-1}(I^+)\to \mathbf{P}^2\setminus I^+$ is a biholomorphism. Thus $\langle\alpha^* T, \alpha^*(d\psi\wedge d^c\psi)\rangle = \langle T, d\psi\wedge d^c\psi\rangle$, as asserted. The last inequality follows from the previous lemma. Since d^n tends to ∞ with n, we see that $(1/d^n)\partial f^n_{+*}(\psi T)$ tends to zero at a rate independent of λ—that is, in the mass norm.

If $\rho\colon \mathbf{P}^2\to\mathbf{C}$ is a test function with $\|\rho\|_\infty\le 1$, then we have

$$
\begin{aligned}
|\langle f^n_{+*}(\psi T), dd^c\rho\rangle| &= \langle \alpha^* T, (\rho\circ\beta)dd^c(\psi\circ\alpha)\rangle \\
&= \langle (\alpha^* T)\wedge\alpha^*(dd^c\psi), \rho\circ\beta\rangle \\
&\le \mathrm{M}[\alpha^* T\wedge\alpha^* dd^c\psi] \\
&= \mathrm{M}[T\wedge dd^c\psi].
\end{aligned}
$$

The last equality holds because $\alpha^* T\wedge\alpha^* dd^c\psi$ puts no mass on $\alpha^{-1}(I^+)$. Dividing through by d^n finishes the proof. □

We continue to assume that the hypotheses of Lemma 3.2 hold. Let $\mathcal{S}$ denote the set of all limit points of the sequence of currents $\{(1/d^n)f^n_{+*}(\psi T)\}$. Lemma 3.3 implies that all elements of $\mathcal{S}$ are closed. Lemma 3.2 implies that $\mathcal{S}$ is non-empty and that any $S\in\mathcal{S}$ satisfies

$$
\|S\| = \int_{\mathbf{P}^2}\psi T\wedge\mu^+.
$$

The first of our two convergence theorems addresses the case where T is defined on all of $\mathbf{P}^2$.

THEOREM 3.4. *Suppose that f_+ is algebraically stable and T extends to a positive closed current on $\mathbf{P}^2$. If T admits a wedge product with μ^+ (at least near* supp ψ*) then, in the weak topology on currents, we have*

$$
\lim_{n\to\infty}\frac{1}{d^n}f^n_{+*}(\psi T) = \left(\int_{\mathbf{P}^2}\psi T\wedge\mu^+\right)\cdot\mu^-.
$$

Proof. Clearly ψT is dominated by $\|\psi\|_\infty T$. Remark 4.16 from [Dil] (a variant on the last part of Theorem 2.7) implies that

$$
\lim_{n\to\infty}\frac{1}{d^n}f^n_{+*}T = \lim_{n\to\infty}\frac{1}{d^n}f^{n*}_- T = \|T\|\cdot\mu^-.
$$

Therefore, any element of $\mathcal{S}$ is dominated by $\|T\| \cdot \|\psi\|_\infty \cdot \mu^-$. Corollary 2.9 then implies that any element of $\mathcal{S}$ is a multiple of μ^-; that is, $S = \|S\| \cdot \mu^-$. The remarks preceding the statement of this theorem determine $\|S\|$ uniquely. $\square$

The second of our convergence theorems addresses the case where T is not globally defined on $\mathbf{P}^2$, but our proof requires that f_+ be *completely separating*: that the iterates of f_- form a normal family on a neighborhood of $\mathcal{I}^+$. We recall from [Dil] that this condition automatically implies that f_+ is algebraically stable, but it allows for stronger conclusions about μ^+. In particular, iterates of f_+ form a normal family on the complement of $\operatorname{supp} \mu^+$, and any positive closed $(1,1)$ current with support contained in $\operatorname{supp} \mu^+$ is actually a multiple of μ^+.

THEOREM 3.5. *If f_+ is completely separating, then the conclusion of Theorem 3.4 remains true for any T that admits a wedge product with μ^+.*

Proof. Since iterates of f_+ act normally on $\mathbf{P}^2 \setminus \operatorname{supp} \mu^+$, it is clear that currents in $\mathcal{S}$ all have support contained in $\operatorname{supp} \mu^+$. Hence all currents in $\mathcal{S}$ are multiples of μ^+ with total mass determined by Lemma 3.2. $\square$

4. More Applications: Support of μ^+

Following Proposition 2.1, we described an action of a birational map on the collection of closed subsets of $\mathbf{P}^2$. There is, however, no reasonable sense in which a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ sends open sets to open sets. In fact, if $U \subset \mathbf{P}^2$ is open then it is not hard to show that $f_+(U)$ is open if and only if U either avoids or contains $\mathcal{C}^+$. On the other hand, f_+ does preserve *closures of* open sets. It is always true that $f_+(\bar{U}) = \overline{\operatorname{int} f_+(\bar{U})}$. Moreover, this action is bijective, since $f_+(f_-(\bar{U})) = \bar{U}$. Therefore, it makes sense to talk about open sets whose closures are invariant under a birational map. Theorem 3.4 implies that such sets can intersect the supports of μ^+ and μ^- in only a rather limited number of ways.

THEOREM 4.1. *Suppose that $U \subset \mathbf{P}^2$ is an open set whose closure is invariant under an algebraically stable birational map. If* $\operatorname{supp} \mu^+ \cap U \neq \emptyset$, *then* $\operatorname{supp} \mu^- \subset \bar{U}$.

Proof. Assume there is a point $p \in \operatorname{supp} \mu^+ \cap U$. Pick a smooth function ψ: $\mathbf{P}^2 \to [0,1]$ such that $\operatorname{supp} \psi \subset U$ and $\psi(p) = 1$. Then

$$c \stackrel{\text{def}}{=} \int_{\mathbf{P}^2} \psi \Theta \wedge \mu^+ > 0.$$

Hence, by Theorem 3.4,

$$\frac{1}{d^n} f^n_{+*}(\psi\Theta) \to c\mu^-;$$

since $\operatorname{supp} f^n_{+*}(\psi\Theta) \subset \bar{U}$ for every n, we have $\operatorname{supp} \mu^- \subset \bar{U}$ as well. $\square$

We do not yet know whether sets like $\mathcal{I}^+$, $\operatorname{supp}\mu^+$, and so forth can have interior. The theorem just proved shows that if such sets *do* have interior then the map f_+ must be rather special.

COROLLARY 4.2. *The following are true for an algebraically stable birational map.*

(1) *If* $\operatorname{supp}\mu^+$ *omits one point in* $\operatorname{supp}\mu^-$, *then* $\operatorname{supp}\mu^+$ *is nowhere dense.*
(2) *If* $\mathcal{I}^+$ *omits one point in* $\operatorname{supp}\mu^-$ (*or, more particularly, in* $\mathcal{I}^-$), *then* $\mathcal{I}^+$ *is nowhere dense.*
(3) *If both* $\operatorname{supp}\mu^+$ *and* $\operatorname{supp}\mu^-$ *have non-empty interior, then*

$$\operatorname{supp}\mu^+ = \overline{\operatorname{int}\operatorname{supp}\mu^+} = \overline{\operatorname{int}\operatorname{supp}\mu^-} = \operatorname{supp}\mu^-.$$

(4) *If both* $\mathcal{I}^+$ *and* $\mathcal{I}^-$ *have non-empty interior, then* $\operatorname{supp}\mu^+ = \operatorname{supp}\mu^- = \mathcal{I}^+ = \mathcal{I}^-$.

Proof. (1) The complement of $\operatorname{supp}\mu^+$ is an open set with invariant closure. If $\operatorname{supp}\mu^-$ intersects this set, then $\operatorname{supp}\mu^+$ is contained in its closure. (2) The complement of $\mathcal{I}^+$ is another open set with invariant closure. If $\operatorname{supp}\mu^-$ intersects this set, then $\mathcal{I}^+ \subset \operatorname{supp}\mu^+$ lies in its boundary. Statements (3) and (4) follow immediately from (1) and (2). □

Stronger conclusions are possible if we impose stronger hypotheses concerning separation of the indeterminacy sets of f_+ and f_-. We call a birational map f_+ *separating* if $\mathcal{I}^+ \cap \mathcal{I}^- = \emptyset$. By Proposition 2.5 and the paragraph preceding Theorem 3.5, it is clear that this requirement falls somewhere in between the conditions that f be algebraically stable and that f be completely separating. We remark that examples from [Dil] show that the three categories of birational maps are actually, as well as apparently, different.

COROLLARY 4.3. *If* f_+ *is separating and* p *is an attracting periodic point, then* $\operatorname{supp}\mu^+$ *lies in the boundary of any connected component of the basin of* p. *In particular,* $\operatorname{supp}\mu^+$ *is nowhere dense.*

Proof. Iterates of f_+ form a normal family on the interior of the basin of p, so $\operatorname{supp}\mu^+$ does not intersect the interior of the basin. On the other hand, we showed in [Dil] that attracting periodic points of separating birational maps belong to $\operatorname{supp}\mu^-$. Since the closure of the basin of p is invariant under f_+, Theorem 4.1 implies that $\operatorname{supp}\mu^+$ lies in the closure of the basin. Applying this reasoning to f_+^k (where k is the period of p) shows that $\operatorname{supp}\mu^+$ lies in the boundary of any connected component of the basin of p. □

COROLLARY 4.4. *If* f_+ *is completely separating, then* $\operatorname{supp}\mu^-$ *is nowhere dense. Moreover,* $\operatorname{supp}\mu^+$ *is equal to the boundary of any connected component of any attracting basin*.

Proof. Since iterates of f_- form a normal family in a neighborhood of $\mathcal{I}^+$, we have that $\mathcal{I}^+ \cap \operatorname{supp}\mu^-$ is empty. On the other hand, $\mathcal{I}^+ \subset \operatorname{supp}\mu^+$. Thus, the first statement follows from Corollary 4.2. The second statement follows from Corollary 4.3, the remarks about completely separating maps preceding Theorem 3.5, and the fact that iterates of f_+ do not form a normal family on any open set intersecting the boundary of an attracting basin. □

For completely separating birational maps, we can apply Theorem 3.5 instead of Theorem 3.4. This gives a description of supp μ^+ in terms of stable manifolds.

COROLLARY 4.5. *Suppose that f_+ is completely separating. Then* supp μ^- *is equal to the closure of the unstable manifold of any saddle periodic point.*

Proof. Let p be a saddle periodic point, and let $W^u_{\mathrm{loc}}(p)$ be a local unstable manifold through p. Let $\chi : \mathbf{P}^2 \to [0,1]$ be a smooth function supported in a small neighborhood of p and such that supp $\chi \cap W^u_{\mathrm{loc}}(p)$ is relatively compact in $W^u_{\mathrm{loc}}(p)$.

Since f_+ is completely separating, we know that $p \notin \mathcal{I}^+$ (iterates of f_- cannot form a normal family near p). From [Dil] we know that $\tilde{G}^+$ is continuous over points in $\mathbf{P}^2 \setminus \mathcal{I}^+$ (this is true even for separating maps). Therefore, the current of integration over $W^u_{\mathrm{loc}}(p)$ admits a wedge product with μ^+. On the other hand, we also showed in [Dil] that $\tilde{G}^+$ cannot be pluriharmonic on $\pi^{-1}(W^u_{\mathrm{loc}}(p))$. Therefore,

$$c = \int_{\mathbf{P}^2} \chi[W^u_{\mathrm{loc}}(p)] \wedge \mu^+ > 0.$$

We apply Theorem 3.5 to conclude that

$$\lim_{n\to\infty} \frac{1}{d^{kn}} f^{kn}_{+*}(\chi[W^u_{\mathrm{loc}}(p)]) = c\mu^-,$$

where k is the period of p. Since $f^{kn}_{+*}(\chi[W^u_{\mathrm{loc}}(p)])$ is supported on the global unstable manifold of p for all n, we have that μ^- is supported on the closure of the unstable manifold. The opposite inclusion was proved in [Dil] for separating birational maps. □

REMARK 4.6. The previous theorem is not true for all algebraically stable birational maps. Example 7.9 in [Dil] presents an algebraically stable map with a saddle fixed point whose unstable manifold has closure equal to a line in $\mathbf{P}^2$. Hence, by Theorem 2.7(1), supp μ^+ must contain points not included in the closure of this unstable manifold. We wonder if there are examples similar to this one except that the unstable manifold is not contained in an algebraic curve.

We now return to the theme of recurrence and ergodicity touched on in the latter part of Section 2, recalling the notion of a nonwandering point. Let $f : X \to X$ be a continuous map of a metric space X. We define a pre-order $\prec$ for points in X by saying that $q \prec p$ if, for any neighborhoods $U \ni p$ and $V \ni q$, there exist arbitrarily large n such that $f^n(U)$ intersects V. We call $p \in X$ *nonwandering* if $p \prec p$. We extend these definitions to birational maps $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ by disregarding any points of indeterminacy for f^n_+ in the intersection $f^n_+(U) \cap V$.

COROLLARY 4.7. *Given an algebraically stable birational map* $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$, *we have* $q \prec p$ *for every* $p \in \operatorname{supp} \mu^+$ *and* $q \in \operatorname{supp} \mu^-$. *In particular, every point in* $\operatorname{supp} \mu^+ \cap \operatorname{supp} \mu^-$ *is nonwandering.*

Proof. Let $U \ni p$ and $V \ni q$ be neighborhoods, and let ψ be a positive test function supported on U and nonvanishing at p. Then $\langle \mu^+, \psi\Theta \rangle$ is positive. Hence, by Theorem 3.4,

$$\lim_{n\to\infty} \frac{1}{d^n} f^n_{+*}(\psi\Theta) = c\mu^- \quad \text{for some } c > 0.$$

Therefore, we must have $\emptyset \neq f^n_+(\operatorname{supp}\psi) \cap V \subset f^n_+(U) \cap V$ for n sufficiently large. This proves the first statement in the corollary. The second statement follows by taking $p = q$ in the first. □

We close this paper with another ergodiclike result—similar to and depending on Corollary 2.9—for μ^+ and two related examples. If f_+ is separating then the wedge product $\mu = \mu^+ \wedge \mu^-$ is admissible. This follows from the fact from [Dil] that, for separating maps, $\tilde{G}^+$ is continuous on $\pi^{-1}(\mathbf{P}^2 \setminus \mathcal{I}^+)$. We intend to study the measure μ in its own right in a later paper, but for now we observe the following.

COROLLARY 4.8. *Suppose that* f_- *is separating and that* $\psi : \mathbf{P}^2 \to \mathbf{C}$ *is smooth. Then*

$$\lim_{n\to\infty} \frac{f^n_{+*}(\psi\mu^-)}{\deg f^n_+} = \left(\int_{\mathbf{P}^2} \psi\mu\right) \cdot \mu^-$$

in the weak topology on currents.

Proof. Since f_+ is separating, so is f_-. In particular, local potentials for μ^- are continuous on a neighborhood of every point in $\mathcal{I}^+$. Therefore, all hypotheses of Theorem 3.4 are fulfilled by setting $T = \mu^-$. □

EXAMPLE 4.9. It is important in Corollary 4.8 that we require some regularity from ψ. Suppose that f_+ is a polynomial diffeomorphism of $\mathbf{C}^2$ with at least two attracting periodic points. Then $\operatorname{supp} \mu^-$ intersects the basins of both points. If χ is the characteristic function for one of the basins, we have that $\chi\mu^- = f_{+*}\chi\mu^-/\deg f_+$ is a nontrivial forward invariant current.

EXAMPLE 4.10. It is also important that f_+ be birational. We illustrate this point with an example in which μ^+ plays the role that μ^- played in Corollary 4.8. Consider the holomorphic map $f : \mathbf{P}^2 \to \mathbf{P}^2$ whose restriction to $\mathbf{C}^2$ is $f(x, y) = (x^2, y^2)$. The restriction of the current μ^+ (usually denoted by T in this context) to $\mathbf{C}^2$ is $dd^c \log\max\{1, |x|, |y|\}$. In particular, the restriction of μ^+ to $U = \{|x| < 1\}$ is $dd^c \log^+|y|$, which simply doubles under pullback by f_+. Let ρ: $[0, 1] \to [0, 1]$ be any smooth function such that $\rho(0) = 1$ and $\rho(1) = 0$. If $\psi = \rho(|x|)$ then we have that $\psi \circ f^n$ tends uniformly to 1 on compact subsets of U. Since μ^+ has finite mass, we have $f^{n*}(\psi\mu^+)/2^n \to \mu^+|_U$ as n tends to infinity.

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Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556

Jeffrey.A.Diller.1@nd.edu