Pseudodifferential Operators with Homogeneous Symbols

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1. Introduction

The study of pseudodifferential operators with symbols in the exotic classes $S^m_{1,1}$ has received a lot of attention. These are operators of the form

$$(Tf)(x) = \int_{\mathbf{R^n}} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\, d\xi,$$

where the symbol $a(x,\xi)$ is a $C^\infty(\mathbf{R^n}\times\mathbf{R^n})$ function satisfying

$$|\partial_\xi^\beta \partial_x^\gamma a(x,\xi)| \le C_{\beta,\gamma}(1+|\xi|)^{m-|\beta|+|\gamma|},$$

for all β and γ n-tuples of nonnegative integers. The interest in such operators is due in part to the role they play in the paradifferential calculus of Bony [1]. The fact that not all such operators of order zero are bounded on L^2 complicates their study. Nevertheless, the exotic pseudodifferential operators do preserve spaces of smooth functions. See, for example, Meyer [12], Paivarinta [14], Bourdaud [2], as well as Stein [16] and the references therein.

The continuity results are often obtained by making use of the so-called singular integral realization of the operators. This involves proving estimates on the Schwartz kernels of the pseudodifferential operators similar to those of the kernels of Calderón–Zygmund operators. There is, however, an alternative approach working directly with the symbols of the pseudodifferential operators. This approach has been pursued by Hörmander in [9] and [10] for L^2-based Sobolev spaces. The ideas in those papers combined with wavelets techniques were later extended by Torres [17] to L^p-based Sobolev spaces and other more general spaces of smooth functions.

In this note we consider C^∞ symbols $a(x,\xi)$ in $\mathbf{R^n}\times(\mathbf{R^n}\setminus\{\mathbf{0}\})$ that satisfy the following conditions: For all n-tuples of nonnegative integers β and γ there exist positive constants $C_{\beta,\gamma}$ such that

$$|\partial_\xi^\beta \partial_x^\gamma a(x,\xi)| \le C_{\beta,\gamma}|\xi|^{m-|\beta|+|\gamma|} \tag{1}$$

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for $(x,\xi)\in\mathbf{R^n}\times(\mathbf{R^n}\setminus\{\mathbf{0}\})$. We call such symbols *homogeneous* symbols of type (1, 1) and order m. The class of all such symbols will be denoted by $\dot{S}^m_{1,1}$.

Our purpose is to show boundedness for pseudodifferential operators with symbols in $\dot{S}^m_{1,1}$ on homogeneous function spaces. Our results are motivated by an observation of Grafakos [8], who proved boundedness of pseudodifferential operators with symbols in $\dot{S}^m_{1,1}$ on homogeneous Lipschitz spaces. The proof in [8] follows more or less the singular integral approach of Stein [16]. In this paper we use a wavelet approach, borrowing ideas from [17].

The appropriate setting for our results is in the context of the homogeneous Triebel–Lizorkin $\dot{F}^{\alpha,q}_p(\mathbf{R^n})$ spaces and Besov–Lipschitz spaces $\dot{B}^{\alpha,q}_p(\mathbf{R^n})$ (these spaces will be defined shortly). As with inhomogeneous symbols, it is possible to show that, for $x\neq y$, the Schwartz kernels of the operators of order zero satisfy estimates of the form

$$|\partial_x^\gamma K(x,y)|+|\partial_y^\gamma K(x,y)|\le C_\gamma|x-y|^{-n-|\gamma|} \tag{2}$$

and even better estimates for $|x-y|$ large (see e.g. [13, p. 294]). It is then possible to analyze boundedness properties of the operators using versions of the T1 theorem of David and Journé [4]. Moreover, such types of results are applied to pseudodifferential operators with inhomogeneous symbols in the book by Meyer [13, p. 329] in the context of Besov spaces with smoothness $\alpha>0$ and $p,q>1$ (cf. also [11]). The arguments in [13] may be adapted to pseudodifferential operators with homogeneous symbols and the same range of parameters for the Besov spaces. Our approach, however, will be based on some very simple calculations that notoriously work for the full scale of both $\dot{F}^{\alpha,q}_p(\mathbf{R^n})$ and $\dot{B}^{\alpha,q}_p(\mathbf{R^n})$ spaces.

Let $\mathcal{S}(\mathbf{R^n})$ be the space of Schwartz test functions and denote its dual by $\mathcal{S}'(\mathbf{R^n})$, the space of tempered distributions. In this paper the Fourier transform of a function $f\in\mathcal{S}(\mathbf{R^n})$ is given by $\hat{f}(\xi)=\int f(x)e^{-ix\cdot\xi}\,dx$, and $\mathcal{S}_0(\mathbf{R^n})$ is used to denote the subspace of $\mathcal{S}(\mathbf{R^n})$ consisting of all functions whose Fourier transform vanishes to infinite order at zero. The dual space of $\mathcal{S}_0(\mathbf{R^n})$ with respect to the topology inherited from $\mathcal{S}(\mathbf{R^n})$ is $\mathcal{S}'/\mathcal{P}(\mathbf{R^n})$, the set of tempered distributions modulo polynomials. The Triebel–Lizorkin and Besov–Lipschitz spaces are defined as follows. Let φ be a function in $\mathcal{S}(\mathbf{R^n})$ satisfying $\operatorname{supp}\hat{\varphi}\subset\{\xi: 1/2\le|\xi|\le 2\}$ and $|\hat{\varphi}|\ge C>0$ for $3/5\le|\xi|\le 5/3$. Define $\varphi_j(\xi)=2^{jn}\varphi(2^j\xi)$. For α real, $0<p,q<\infty$, and f in $\mathcal{S}'/\mathcal{P}(\mathbf{R^n})$, define the Triebel–Lizorkin and Besov–Lipschitz norms of f by

$$\|f\|_{\dot{F}^{\alpha,q}_p(\mathbf{R^n})}=\left\|\left(\sum_{j=-\infty}^{\infty}(2^{j\alpha}|\varphi_j*f|)^q\right)^{1/q}\right\|_{L^p},$$

and

$$\|f\|_{\dot{B}^{\alpha,q}_p(\mathbf{R^n})}=\left(\sum_{j=-\infty}^{\infty}(\|2^{j\alpha}(\varphi_j*f)\|_{L^p})^q\right)^{1/q},$$

respectively. Note that these definitions are given modulo all polynomials, so strictly speaking an element of the spaces $\dot{F}^{\alpha,q}_p(\mathbf{R^n})$ and $\dot{B}^{\alpha,q}_p(\mathbf{R^n})$ is an equivalent class of distributions. It can be shown that the definition of these spaces is independent of φ. The space $\mathcal{S}_0(\mathbf{R^n})$ is dense in all of these spaces. For these and further properties of these function spaces, we refer to the books [15] and [19].

We have the following results for operators with symbols in $\dot{S}^m_{1,1}$ acting on the spaces $\dot{F}^{\alpha,q}_p(\mathbf{R^n})$ and $\dot{B}^{\alpha,q}_p(\mathbf{R^n})$.

THEOREM 1.1. *Let $0 < p, q < \infty$. For $\alpha > n(\max(1, p^{-1}, q^{-1}) - 1)$, every pseudodifferential operator*

$$(Tf)(x) = \int_{\mathbf{R^n}} a(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi \quad (f \in \mathcal{S}_0)$$

with symbol $a(x,\xi)$ in $\dot{S}^m_{1,1}$ extends to a bounded operator that maps the space $\dot{F}^{\alpha+m,q}_p(\mathbf{R^n})$ to $\dot{F}^{\alpha,q}_p(\mathbf{R^n})$. For $\alpha \le n(\max(1, p^{-1}, q^{-1}) - 1)$, every T as above that satisfies $T^*x^\gamma = 0$ for all $|\gamma| \le n(\max(1, p^{-1}, q^{-1}) - 1) - \alpha$ extends to a bounded operator from $\dot{F}^{\alpha+m,q}_p(\mathbf{R^n})$ to $\dot{F}^{\alpha,q}_p(\mathbf{R^n})$.

THEOREM 1.2. *Let $0 < p, q < \infty$. For $\alpha > n(\max(1, p^{-1}, q^{-1}) - 1)$, every pseudodifferential operator*

$$(Tf)(x) = \int_{\mathbf{R^n}} a(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi \quad (f \in \mathcal{S}_0) \tag{3}$$

with symbol $a(x,\xi)$ in $\dot{S}^m_{1,1}$ extends to a bounded operator that maps the space $\dot{B}^{\alpha+m,q}_p(\mathbf{R^n})$ to $\dot{B}^{\alpha,q}_p(\mathbf{R^n})$. For $\alpha \le n(\max(1, p^{-1}, q^{-1}) - 1)$, every T as above that satisfies $T^*x^\gamma = 0$ for all $|\gamma| \le n(\max(1, p^{-1}, q^{-1}) - 1) - \alpha$ extends to a bounded operator from $\dot{B}^{\alpha+m,q}_p(\mathbf{R^n})$ to $\dot{B}^{\alpha,q}_p(\mathbf{R^n})$.

We end this section with the following observation. Note that if $m - |\gamma| + |\beta| < 0$ then $\partial^\gamma_\xi \partial^\beta_x a(x,\xi)$ is singular at $\xi = 0$ and, for a general function f in $\mathcal{S}$, the integral in (3) is not absolutely convergent. For this reason it is natural initially to define the operator T on $\mathcal{S}_0$.

2. Proofs of the Theorems

As we have just discussed, it is natural to consider T initially defined on $\mathcal{S}_0$. Moreover, we have the following lemma.

LEMMA 2.1. *Let $a(x,\xi)$ be a symbol in $\dot{S}^m_{1,1}$. Then the pseudodifferential operator with symbol $a(x,\xi)$ maps $\mathcal{S}_0$ to $\mathcal{S}$. In particular, its formal transpose T^* maps $\mathcal{S}'$ to $\mathcal{S}'/\mathcal{P}$.*

Proof. Let f be a function in $\mathcal{S}_0$ and let Δ_ξ be the Laplace operator in the variable ξ. Because

$$(I - \Delta_\xi)^N(e^{ix\cdot\xi}) = (1 + |x|^2)^N e^{ix\cdot\xi}$$

for any positive integer N, an integration by parts gives

$$(Tf)(x) = \int_{\mathbf{R^n}} e^{ix\cdot\xi}\frac{(I - \Delta_\xi)^N}{(1 + |x|^2)^N}(a(x,\xi)\hat{f}(\xi))\,d\xi. \tag{4}$$

Since $\hat{f}$ vanishes to infinity order at the origin, the conditions on the symbol $a(x,\xi)$ and an application of Leibniz's rule give

$$|(Tf)(x)| \le \frac{\mathcal{R}_{m,N}(f)}{(1+|x|^2)^N}, \tag{5}$$

where $\mathcal{R}_{m,N}$ is an appropriate seminorm in $\mathcal{S}$. A similar computation applies to the derivatives $\partial^\gamma(Tf)$, thus proving the lemma. □

For an n-tuple of integers k and an integer j, denote by Q_{jk} the dyadic cube $\{(x_1,\dots,x_n)\in\mathbf{R^n} : k_i \le 2^j x_i < k_i+1\}$, by $x_{Q_{jk}}$ its "lower left corner" $2^{-j}k$, and by $l(Q_{jk})$ its side length 2^{-j}. For Q dyadic let $\varphi_Q(x) = |Q|^{1/2}\varphi_j(x-x_Q)$. A function φ as in the definition of the homogeneous spaces can be chosen so that, for $f\in\mathcal{S}'$,

$$f = \sum_Q \langle f,\varphi_Q\rangle \varphi_Q, \tag{6}$$

where $\langle f,\varphi_Q\rangle$ simply denotes the action of the distribution f on the test function φ_Q. For f in $\dot{F}_p^{\alpha,q}(\mathbf{R^n})$ or $\dot{B}_p^{\alpha,q}(\mathbf{R^n})$, the convergence in (6) is in the (quasi-)norm of the spaces; for f in $\mathcal{S}_0$, the convergence is in the topology of $\mathcal{S}$. See [6] and [7].

It follows from Lemma 2.1 that the action of a pseudodifferential operator on $\mathcal{S}_0$ can be expressed as

$$Tf = \sum_Q \langle f,\varphi_Q\rangle T\varphi_Q. \tag{7}$$

The operator given by (7) is the one that is extended to the whole homogeneous space in our theorems. We now turn to the proofs.

The map

$$S_\varphi(f) = \{\langle f,\varphi_Q\rangle\}_Q \tag{8}$$

is called the φ-transform (or sequence of nonorthogonal wavelet coefficients). It is well known by the work in [5; 6] that the homogeneous φ-transform provides a characterization of the spaces $\dot{F}_p^{\alpha,q}(\mathbf{R^n})$ and $\dot{B}_p^{\alpha,q}(\mathbf{R^n})$ via the equivalence of norms

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R^n})} \sim \left\| \left(\sum_j \left(\sum_{l(Q)=2^{-j}} |Q|^{-\alpha/n-1/2} |\langle f,\varphi_Q\rangle| \chi_Q \right)^q \right)^{1/q} \right\|_{L^p}$$

and

$$\|f\|_{\dot{B}_p^{\alpha,q}(\mathbf{R^n})} \sim \left(\sum_j \left\| \sum_{l(Q)=2^{-j}} |Q|^{-\alpha/n-1/2} |\langle f,\varphi_Q\rangle| \chi_Q \right\|_{L^p}^q \right)^{1/q},$$

where χ_Q is the characteristic function of Q.

For α, p, q as before let $J = n/\min(1,p,q)$ and let $[\alpha]$ be the integer part in α. A smooth molecule for $\dot{F}_p^{\alpha,q}(\mathbf{R^n})$ or $\dot{B}_p^{\alpha,q}(\mathbf{R^n})$ associated with a dyadic cube Q with side length $l(Q)$ is a function m_Q satisfying:

$$\int x^\gamma m_Q(x)\,dx = 0 \quad \text{if } |\gamma| \le [J-n-\alpha], \tag{9}$$

$$|m_Q(x)| \le |Q|^{-1/2}(1+l(Q)^{-1}|x-x_Q|)^{-\max(J+\varepsilon,\,J+\varepsilon-\alpha)}, \tag{10}$$

$$|\partial^\gamma m_Q(x)| \le |Q|^{-1/2-|\gamma|/n}(1+l(Q)^{-1}|x-x_Q|)^{-J-\varepsilon}, \quad |\gamma|\le[\alpha]+1. \tag{11}$$

The importance of these functions is due to the fact that, if

$$f = \sum_Q s_Q m_Q$$

in $\mathcal{S}'$, where $\{m_Q\}$ is a family of smooth molecules for $\dot{F}_p^{\alpha,q}(\mathbf{R^n})$ or $\dot{B}_p^{\alpha,q}(\mathbf{R^n})$, then

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R^n})} \le C \left\| \left(\sum_j \left(\sum_{l(Q)=2^{-j}} |Q|^{-\alpha/n-1/2} |s_Q| \chi_Q \right)^q \right)^{1/q} \right\|_{L^p}$$

or

$$\|f\|_{\dot{B}_p^{\alpha,q}(\mathbf{R^n})} \le C \left(\sum_j \left\| \sum_{l(Q)=2^{-j}} |Q|^{-\alpha/n-1/2} |S_Q| \chi_Q \right\|_{L^p}^q \right)^{1/q}.$$

For these results we refer again to [6] and [7].

Now, let T be a linear continuous operator from $\mathcal{S}_0$ to $\mathcal{S}'$. Assume that

$$T\varphi_Q = C|Q|^{-m/n} m_Q, \tag{12}$$

where $\{m_Q\}_Q$ is a family of smooth molecules for $\dot{F}_p^{\alpha,q}(\mathbf{R^n})$ or $\dot{B}_p^{\alpha,q}(\mathbf{R^n})$. Then, using the φ-transform, it is easy to see that T can be extended as a bounded operator from $\dot{F}_p^{\alpha+m,q}(\mathbf{R^n})$ to $\dot{F}_p^{\alpha,q}(\mathbf{R^n})$ or from $\dot{B}_p^{\alpha+m,q}(\mathbf{R^n})$ to $\dot{B}_p^{\alpha,q}(\mathbf{R^n})$.

Suppose that T is a pseudodifferential operator whose symbol $a(x,\xi)$ is in $\dot{S}^m_{1,1}$. By (12) it will suffice to show that, for a fixed dyadic cube Q, $T\varphi_Q$ is a scaled multiple of a molecule. A simple dilation shows that

$$(T\varphi_Q)(x) = \int e^{ix\cdot\xi} a(x,\xi) \widehat{\varphi_Q}(\xi)\, d\xi = 2^{jn/2} (T_Q\varphi)(2^j x - k) \tag{13}$$

if $Q = Q_{jk}$, where we set

$$(T_Q f)(x) = \int e^{ix\cdot\xi} a(2^{-j}(x+k), 2^j\xi) \hat{f}(\xi)\, d\xi.$$

Let us fix a multi-index γ. We have

$$(\partial^\gamma T_Q\varphi)(x) = \int_{\mathbf{R^n}} e^{ix\cdot\xi} \sum_{\delta\le\gamma} C_\delta (i\xi)^\delta \partial_x^{\gamma-\delta} a(2^{-j}(x+k), 2^j\xi) \hat{\varphi}(\xi)\, d\xi \tag{14}$$

for certain C_δ constants, where $\delta \le \gamma$ simply means that $\delta_j \le \gamma_j$ for all $j = 1,\dots,n$.

Now fix $N > \max(J, J-\alpha)/2$. An integration by parts gives

$$\begin{aligned}
&(\partial^\gamma T_Q\varphi)(x) \\
&\quad = \int_{\mathbf{R^n}} e^{ix\cdot\xi} \frac{(I-\Delta_\xi)^N}{(1+|x|^2)^N} \sum_{\delta\le\gamma} C_\delta (i\xi)^\delta \partial_x^{\gamma-\delta} a(2^{-j}(x+k), 2^j\xi) \hat{\varphi}(\xi)\, d\xi.
\end{aligned} \tag{15}$$

By Leibniz's rule, there exist constants $K_{\alpha,\beta}$ such that

$$\begin{aligned}
&(I-\Delta_\xi)^N \partial_x^{\gamma-\delta} (a(2^{-j}(x+k), 2^j\xi) \hat{\varphi}(\xi)(i\xi)^\delta) \\
&\qquad = \sum_{|\alpha+\beta|=2N} K_{\alpha,\beta} \partial_\xi^\beta \partial_x^{\gamma-\delta} \big(a(2^{-j}(x+k), 2^j\xi) \partial_\xi^\alpha (\hat{\varphi}(\xi)(i\xi)^\delta) \big).
\end{aligned}$$

Using the estimates (1), we conclude that

$$\begin{aligned}|\partial_\xi^\beta \partial_x^{\gamma-\delta} a(2^{-j}(x+k), 2^j\xi)| &\le C_{\beta,\gamma-\delta} 2^{j|\beta|} 2^{-j(|\gamma|-|\delta|)} |2^j\xi|^{m+(|\gamma|-|\delta|)-|\beta|} \\ &\le C 2^{jm} |\xi|^{m+(|\gamma|-|\delta|)-|\beta|}.\end{aligned}$$

Summing over all α, β, δ as before and using that $|\xi| \sim 1$, we conclude that the integrand in (15) is pointwise bounded by $C_\gamma 2^{jm}(1+|x|^2)^{-N}$ and thus

$$|(\partial^\gamma T_Q \varphi)(x)| \le C_\gamma 2^{jm}(1+|x|^2)^{-N}. \tag{16}$$

We now dilate and translate (16) to deduce that

$$\begin{aligned}|(\partial^\gamma T_Q \varphi)(x)| &\le C 2^{jn/2} 2^{j|\gamma|} 2^{jm}(1+|2^j x - k|^2)^{-N} \\ &\le C|Q|^{-m/n-1/2-|\gamma|/n}(1+l(Q)^{-1}|x-x_Q|)^{-\max(J+\varepsilon, J+\varepsilon-\alpha)}.\end{aligned}$$

We must now check the vanishing moment condition for $T\varphi_Q$. If $[J-n-\alpha] < 0$ then this condition is vacuous. If $[J-n-\alpha] \ge 0$ then this is an easy consequence of the hypothesis $T^* x^\gamma = 0$, since

$$\int x^\gamma (T\varphi_Q)(x)\, dx = \langle x^\gamma, T\varphi_Q\rangle = \langle T^*(x^\gamma), \varphi_Q\rangle = 0.$$

Both theorems are now proved. □

We conclude this section with some remarks.

1. For $p > 1$ let p' denote $p/(p-1)$. It can be shown that the spaces $\dot F_p^{\alpha,q}(\mathbf{R^n})$ can actually be considered as spaces of distributions modulo polynomials of degree less than or equal to $[\alpha - n/p]$ (see [6, p. 154]). For $1 < p, q < \infty$, if T maps $\dot F_p^{\alpha+m,q}(\mathbf{R^n})$ to $\dot F_p^{\alpha,q}(\mathbf{R^n})$ then, by duality, T^* maps $\dot F_{p'}^{-\alpha,q'}(\mathbf{R^n})$ to $\dot F_{p'}^{-\alpha-m,q'}(\mathbf{R^n})$. It follows that for T^* to be even well-defined on $\dot F_{p'}^{-\alpha,q'}(\mathbf{R^n})$ it must annihilate polynomials of degree $[-\alpha - n/p']$. The conditions on T^* in the second part of Theorems 1.1 and 1.2 for $\alpha < 0$ then become necessary for $p \to 1^+$.

2. For more general operators T with kernels satisfying estimates (2) and the usual weak boundedness property assumed in the T1 theorem (which is always satisfied by pseudodifferential operators of order zero and their transposes), the results in [13] state that the conditions $T(x^\gamma) = 0$ for all $|\gamma| \le [\alpha]$ imply that T is bounded on $\dot B_p^{\alpha,q}(\mathbf{R^n})$ for $\alpha > 0$ and $p, q \ge 1$. Let now $\alpha < 0$. If T is a pseudodifferential operator in $\dot S^0_{1,1}$, then T^* is not necessarily a pseudodifferential operator. Nevertheless, its kernel $K^*(x,y)$ is $K(y,x)$ and hence still satisfies the estimates (2) by symmetry. The results in [13] state that if T^* annihilates polynomials of degree less than or equal to $[-\alpha]$ then T^* is bounded on $\dot B_p^{-\alpha,q}$, and then by duality $T = T^{**}$ is bounded in $\dot B_p^{\alpha,q}$, which agrees with our results. Such duality arguments are not available for other values of the parameters, but our proof is still valid.

3. For $m \ne 0$ we could have precomposed the operator with an appropriate power of the Laplacian and reduced our proof to the case $m = 0$. There are also versions of the T1 theorem for general operators whose kernels satisfy appropriate m-versions of the estimates (2). In principle, such results could be applied to

operators in $\dot{S}^m_{1,1}$ and some values of the parameters α, p, and q, but they lead to weaker results than the ones we have presented here (cf. [18]).

4. Note that our approach does not require the giving of a precise meaning on the action of an operator T (satisfying (2)) on polynomials, as is usually required in versions of the T1 theorem. We only need to know that T^* acts on polynomials, which is automatic by duality.

5. Operators with homogeneous symbols of degree m in ξ satisfying estimates (1) were studied by Calderón and Zygmund [3]. For this subclass of symbols, a partial calculus holds but does not extend to the whole class $\dot{S}^m_{1,1}$ (see [16, p. 268]).

3. Examples and Applications

Symbols in $\dot{S}^m_{1,1}$ that are independent of x exist in abundance. For instance, it is easy to see that the reciprocal of an elliptic polynomial of n variables that is homogeneous of degree $m > 0$ is in $\dot{S}^{-m}_{1,1}$.

An example of a homogeneous symbol in the class $\dot{S}^0_{1,1}$ is the following:

$$\sum_{k=-\infty}^{+\infty} e^{i2^k x\cdot\xi}\phi(2^{-k}\xi),$$

where ϕ is a smooth bump supported away from the origin. More generally, suppose that the sequence of smooth functions $\{m_k(x)\}_{k\in\mathbf{Z}}$ in $\mathbf{R^n}$ satisfies

$$\|\partial^\beta m_k\|_\infty \le C_\alpha 2^{|\alpha|k} \tag{17}$$

for all α n-tuples of nonnegative integers and k any integer. Then the symbol

$$a(x,\xi) = \sum_{k=-\infty}^{+\infty} m_k(x)\phi(2^{-k}\xi)$$

is in $\dot{S}^0_{1,1}$.

We now give an application. Let Δ_j be the Littlewood–Paley operators defined by $\widehat{\Delta_j g}(\xi) = \hat{g}(\xi)\phi(2^{-j}\xi)$, where ϕ is a smooth bump (supported away from the origin) that satisfies $\sum_{j\in\mathbf{Z}}\phi(2^{-j}\xi) = 1$ for all $\xi \ne 0$.

Suppose now that f is a function on $\mathbf{R^n}$ that satisfies

$$\sum_{j\in\mathbf{Z}} \|\Delta_j f\|_\infty \le C(f) < \infty. \tag{18}$$

Let F be a C^∞ function on $\mathbf{R^n}$ with $F(0) = 0$. Suppose that f is in some of the homogeneous function space discussed in the previous section with index of smoothness $\alpha > 0$. Denote such space by $X^{\alpha,q}_p$. We claim that $F(f)$ lies in the same space $X^{\alpha,q}_p$. For the proof of this we borrow the ideas of Bony as presented in [12].

For an integer k define

$$f_k = \sum_{j=-\infty}^{k} \Delta_j f \tag{19}$$

and write $f = \lim_{k\to\infty} f_k$, with uniform convergence because of (18). The functions f_k have Fourier transforms with compact support and they are smooth (actually, analytic of exponential type). Moreover, by (18), they are uniformly bounded. Using (18) and Bernstein's inequality we obtain the following estimates for their derivatives:

$$\|\partial^\alpha f_k\|_\infty \le C_\alpha 2^{|\alpha|k} \|f_k\|_\infty \le C(f) C_\alpha 2^{|\alpha|k}. \tag{20}$$

Write now

$$F(f) = \lim_{N\to\infty} \sum_{k=-N}^{N} F(f_k) - F(f_{k-1}) = \sum_{k=-\infty}^{+\infty} F(f_k) - F(f_{k-1}), \tag{21}$$

where convergence is justified from the fact that F is continuous, $F(0) = 0$, and that $f_N \to f$ and $f_{-N} \to 0$ uniformly as $N \to +\infty$. Next apply the mean value theorem to write (21) as

$$F(f)(x) = \sum_{k=-\infty}^{+\infty} m_k(x)(\Delta_k f)(x),$$

where

$$m_k = \int_0^1 F'(tf_k + (1-t) f_{k-1})\, dt. \tag{22}$$

Using (20), the smoothness of F, and the chain rule, we see that the functions m_k satisfy (17). We conclude that the symbol

$$a(x,\xi) = \sum_{j=-\infty}^{+\infty} m_j(x)\phi(2^{-j}\xi)$$

is in $\dot{S}^0_{1,1}$. We have that $F(f) = T_a f$. It follows from Theorems 1.1 and 1.2 that the function $F(f)$ is in $X_p^{\alpha,q}$, provided $\alpha > 0$ and $p, q \ge 1$, or if $\alpha > n(\max(1, p^{-1}, q^{-1}) - 1)$. Observe that, owing to the nonlinearity of the problem, in the estimate

$$\|F(f)\|_{X_p^{\alpha,q}} \le C_f \|f\|_{X_p^{\alpha,q}} \tag{23}$$

the constant C_f depends on f. In fact, both the functions m_k and the symbol a depend on f. If we assume that $F(t) = t^D$ then, after carefully examining the arguments given here and the proofs of the theorems (given in Section 2), we see that C_f in (23) is controlled by a suitable (large) power of the bound $C(f)$ in (18).

Finally, observe that the left-hand side of (18) is the $\dot{B}^{0,1}_\infty$ norm of f. By some well-known facts about functions of exponential type (see [19]),

$$\|\Delta_j f\|_\infty \le C 2^{jn/p} \|\Delta_j f\|_p.$$

From this one obtains the inequality (Sobolev–Besov embedding)

$$\sum_{j\in\mathbf{Z}} \|\Delta_j f\|_\infty \le C \sum_{j\in\mathbf{Z}} 2^{jn/p} \|\Delta_j f\|_p.$$

Then, in particular, the application described can be used in $\dot{B}_p^{\alpha,q}(\mathbf{R^n})$ with $\alpha = n/p$ and $q = 1$, yielding (23) with C_f controlled by a power of $\|f\|_{\dot{B}_p^{n/p,1}}$.

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