Configurations of Linear Subspaces and Rational Invariants

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1. Introduction

Let $\mathrm{Gr}_{n,d}(\mathbb{C})$ denote the Grassmannian of all d-dimensional linear subspaces in $\mathbb{C}^n$, and let $\mathrm{GL}_n(\mathbb{C}) \times (\mathrm{Gr}_{n,d}(\mathbb{C}))^s \to (\mathrm{Gr}_{n,d}(\mathbb{C}))^s$ be the canonical diagonal action. Dolgachev [DB] posed the following question:

> *Is the quotient* $\mathrm{Gr}_{n,2}(\mathbb{C})^s/\mathrm{GL}_n(\mathbb{C})$ (*e.g., in the sense of Rosenlicht*) *always rational?*

Recall that a Rosenlicht quotient of an algebraic variety X acted on by an algebraic group G is an algebraic variety V together with a rational map $X \to V$ whose generic fibers coincide with the G-orbits. Such quotients always exist and are unique up to birational isomorphisms [R]. In the sequel all quotients will be assumed of this type. An algebraic variety Q is *rational* if it is birationally equivalent to $\mathbb{P}^m$ with $m = \dim Q$.

We answer the above question in the affirmative by applying the rationality of the quotient $(\mathrm{GL}_2(\mathbb{C}))^2/\mathrm{GL}_2(\mathbb{C})$, where $\mathrm{GL}_2(\mathbb{C})$ acts diagonally by conjugations (see [P1] and surveys [B; D]).

Theorem 1.1. *For all positive integers n and s, the quotient* $(\mathrm{Gr}_{n,2}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ *is rational. Equivalently, the field of rational* $\mathrm{GL}_n(\mathbb{C})$*-invariants on* $(\mathrm{Gr}_{n,2}(\mathbb{C}))^s$ *is pure transcendental.*

The statement of Theorem 1.1 has been recently proved by Megyesi [M] in the case $n = 4$ and by Dolgachev and Boden [DB] in the case of odd n. Their proofs are independent of the present one.

More generally, we show the birational equivalence between $(\mathrm{Gr}_{n,d}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ and certain quotients of matrix spaces. Let $\mathrm{GL}_n(\mathbb{C}) \times (\mathrm{GL}_n(\mathbb{C}))^s \to (\mathrm{GL}_n(\mathbb{C}))^s$ be the action defined by $(g, M_1, \dots, M_s) \mapsto (gM_1g^{-1}, \dots, gM_sg^{-1})$. The first main result of the present paper consists of the following two statements.

Theorem 1.2.

(1) *Let s and d be arbitrary positive integers, and let $n = rd$ for some integer $r > 1$. Then* $(\mathrm{Gr}_{n,d}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ *is birationally equivalent to* $(\mathrm{GL}_d(\mathbb{C}))^k/\mathrm{GL}_d(\mathbb{C})$, *where*

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$$k = \begin{cases} (r-1)(s-r-1) & \textit{if } \ s > r+1, \\ 1 & \textit{else.} \end{cases}$$

(2) *Let s be an arbitrary positive integer. Let $n = (2r+1)e$ and $d = 2e$ for some integers r and e. Then $(\mathrm{Gr}_{n,d}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ is birationally equivalent to $(\mathrm{GL}_e(\mathbb{C}))^k/\mathrm{GL}_e(\mathbb{C})$, where*

$$k = \begin{cases} 2r-4+(4r-2)(s-r-2) & \textit{if } \ r>1, \ s > r+1, \\ 2(s-4) & \textit{if } \ r=1, \ s>4, \\ 1 & \textit{else.} \end{cases}$$

If n is even then Theorem 1.1 follows from the first part of Theorem 1.2 and from the rationality of the quotient of $(\mathrm{GL}_2(\mathbb{C}))^s$ by $\mathrm{GL}_2(\mathbb{C})$ (see Procesi [P1]); if n is odd then it follows from the second part of Theorem 1.2. Formanek [F1; F2] proved the rationality of $(\mathrm{GL}_n(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ for $n = 3, 4$. Using his result together with Theorem 1.2, we obtain the following.

THEOREM 1.3. *For every positive integer s, we have*:

(1) $(\mathrm{Gr}_{n,3}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ *is rational for* $n = 0 \pmod 3$;
(2) $(\mathrm{Gr}_{n,4}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ *is rational for* $n = 0 \pmod 2$;
(3) $(\mathrm{Gr}_{n,5}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ *is rational for* $n = 3 \pmod 6$;
(4) $(\mathrm{Gr}_{n,8}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ *is rational for* $n = 4 \pmod 8$;

We refer the reader to [BS] for similar equivalences of stable rationalities (an algebraic variety V is *stable rational* if $V \times \mathbb{P}^m$ is rational for some m). We also refer to [GP1; GP2] for the classification of quadruples of linear subspaces of arbitrary dimensions and their invariants. In our situation, however, all rational invariants of the quadruples (i.e., the case $s = 4$) are constant unless $r = 2$ in the first part of Theorem 1.2.

Our method is based on constructing certain normal forms for our algebraic group actions. We call an algebraic variety acted upon algebraically by an algebraic group G a G-space. A G-subspace is a G-invariant locally closed algebraic subvariety of X. We use the standard notation $Gs := \{ gs : g \in G \}$ and $GS := \{ gs : g \in G, \ s \in S \}$, where $S \subset X$ is an arbitrary subset.

DEFINITION 1.1. We say that (S, H) is a *normal form* for (X, G) if $H \subset G$ is a subgroup and $S \subset X$ is a H-subspace such that the following hold:

(1) GS is Zariski dense in X;
(2) $Gs \cap S = Hs$ for all $s \in S$.

Clearly these conditions guarantee the birational equivalence of the quotients X/G and S/H. In Sections 3 and 4 we construct certain normal forms that are isomorphic to the spaces of matrices as in Theorem 1.2. Then, in Section 5, we use these normal forms for explicit computations of generators of the fields of rational invariants in each case of Theorem 1.2.

This method also has applications to biholomorphic automorphisms of nonsmooth bounded domains, where the configurations of linear subspaces appear naturally as collections of tangent subspaces to the so-called characteristic webs. We refer the reader to [Z] for further details.

2. Notation

For brevity we write GL_n and $\mathrm{Gr}_{n,d}^s$ for $\mathrm{GL}_n(\mathbb{C})$ and $(\mathrm{Gr}_{n,d}(\mathbb{C}))^s$, respectively. The actions on the products will be assumed diagonal unless otherwise specified. For E and E' vector spaces with $\dim E \leq \dim E'$, denote by $\mathrm{Gr}_d(E)$ the Grassmannian of all d-dimensional subspaces of E, by $\mathrm{GL}(E)$ the group of linear automorphisms of E, and by $\mathrm{GL}(E_1, E_2)$ the space of all linear embeddings of E_1 into E_2. A G-space X is *homogeneous* (resp. *almost homogeneous*) if G acts transitively on X (on a Zariski dense subset of X).

3. The Case *d* Divides *n*

In this section we study the diagonal GL_n-action on $\mathrm{Gr}_{n,d}^s$ with $n = rd$ for some integer r. Clearly, the set of all r-tuples $(V_1, \dots, V_r) \in \mathrm{Gr}_{n,d}^r$ such that $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$ is Zariski open and GL_n-homogeneous. This can be reformulated in terms of normal forms as follows.

LEMMA 3.1. *Suppose that $r \geq 2$, $s \geq r$, and $(E_1, \dots, E_r) \in \mathrm{Gr}_{n,d}^r$ is such that $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_r$. Define*

$$S_1 = S_1(s) := \{ (V_1, \dots, V_s) \in \mathrm{Gr}_{n,d}^s : (V_1, \dots, V_r) = (E_1, \dots, E_r) \}$$

and the group $H_1 := \mathrm{GL}(E_1) \times \cdots \times \mathrm{GL}(E_r) \subset \mathrm{GL}_n$. Then (S_1, H_1) is a normal form for $(\mathrm{Gr}_{n,d}^s, \mathrm{GL}_n)$.

Now fix the splitting $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_r$ as before. Let $V \in \mathrm{Gr}_{n,d}$ be such that its projection on each E_i $(i = 1, \dots, r)$ is bijective. In this case we say that V is in *general position* with respect to $(E_1, \dots, E_r)$. Clearly the subset of all V, which are in general position, is Zariski open in $\mathrm{Gr}_{n,d}$. Thus, for every $i = 2, \dots, r$, the projection V_i of V on $E_1 \times E_i$ is a graph of a linear isomorphism $\varphi_i \colon E_1 \to E_i$. We claim that this correspondence between elements $V \in \mathrm{Gr}_{n,d}$ and $(r-1)$-tuples of linear isomorphisms

$$(\varphi_2, \dots, \varphi_r) \in \prod_{i=2}^{r} \mathrm{GL}(E_1, E_i)$$

is birational.

LEMMA 3.2. *Under the assumptions of Lemma 3.1, define*

$$\Phi \colon \prod_{i=2}^{r} \mathrm{GL}(E_1, E_i) \to \mathrm{Gr}_{n,d},$$
$$\Phi(\varphi_2, \dots, \varphi_r) := \{ (z \oplus \varphi_2(z) \oplus \cdots \oplus \varphi_r(z)) : z \in E_1 \},$$

and the H_1-action on $\prod_{i=2}^{r} \mathrm{GL}(E_1, E_i)$ by

$$((g_1, \ldots, g_r), (\varphi_2, \ldots, \varphi_r)) \mapsto (g_2 \circ \varphi_2 \circ g_1^{-1}, \ldots, g_r \circ \varphi_r \circ g_1^{-1}).$$

Then Φ is birational and H_1-equivariant. The image of Φ consists of all subspaces $V \in \mathrm{Gr}_{n,d}$ that are in general position with respect to $(E_1, \ldots, E_r)$.

Proof. Since Φ defines a system of standard coordinates on $\mathrm{Gr}_{n,d}$, it is birational. The equivariance and the statement on the image are straightforward. □

Since $\prod_{i=2}^r \mathrm{GL}(E_1, E_i)$ is H_1-homogeneous, we have the following.

COROLLARY 3.1. *Under the assumptions of Lemma 3.1, let $S_1'(s) \subset S_1(s)$ be the Zariski open subset, where V_{r+1} is in general position with respect to $(E_1, \ldots, E_r)$. Then H_1 acts transitively on $S_1'(r+1)$.*

This proves statement 1 of Theorem 1.2 for $s \le r+1$. Suppose now that $s > r+1$ and that $E_{r+1} \in \mathrm{Gr}_{n,d}$ is in general position with respect to $(E_1, \ldots, E_r)$.

LEMMA 3.3. *Define $S_2 = S_2(s) := \{ (V_1, \ldots, V_s) \in S_1(s) : V_{r+1} = E_{r+1} \}$ and the group $H_2 := \mathrm{GL}(E_1)$ with the action on S_2 defined by the homomorphism*

$$H_2 \to H_1, \qquad g \mapsto (g, \varphi_2 \circ g \circ \varphi_2^{-1}, \ldots, \varphi_r \circ g \circ \varphi_r^{-1}),$$

where

$$(\varphi_2, \ldots, \varphi_r) := \Phi^{-1}(E_{r+1}).$$

Then (S_2, H_2) is a normal form for $(\mathrm{Gr}_{n,d}^s, \mathrm{GL}_n)$.

Proof. By Corollary 3.1, $H_1 S_2$ is Zariski open in S_1. By straightforward calculations, (S_2, H_2) is a normal form for (S_1, H_1) and hence, by Lemma 3.1, a normal form for $(\mathrm{Gr}_{n,d}^s, \mathrm{GL}_n)$. □

We now let $V \in \mathrm{Gr}_{n,d}$ be one more subspace in general position with respect to $(E_1, \ldots, E_r)$. Define $(\psi_2, \ldots, \psi_r) := \Phi^{-1}(V)$. Clearly the map

$$\begin{gathered} \prod_{i=2}^r \mathrm{GL}(E_1, E_i) \to \mathrm{GL}(E_1)^{r-1}, \\ (\psi_2, \ldots, \psi_r) \mapsto (\varphi_2^{-1} \circ \psi_2, \ldots, \varphi_r^{-1} \circ \psi_r) \end{gathered} \tag{1}$$

is one-to-one, where $(\varphi_2, \ldots, \varphi_r) = \Phi^{-1}(E_{r+1})$ is fixed. Moreover, we obtain the following lemma.

LEMMA 3.4. *Define $K : \mathrm{GL}(E_1)^{r-1} \to \mathrm{Gr}_{n,d}$ by*

$$K(\chi_2, \ldots, \chi_r) := \Phi(\varphi_2 \circ \chi_2, \ldots, \varphi_r \circ \chi_r),$$

and define the H_2-action on X_2 by

$$(g, (\chi_2, \ldots, \chi_r)) \mapsto (g \circ \chi_2 \circ g^{-1}, \ldots, g \circ \chi_r \circ g^{-1}).$$

Then K is birational and H_2-equivariant. The image of K consists of all subspaces $V \in \mathrm{Gr}_{n,d}$ that are in general position with respect to $(E_1, \ldots, E_r)$.

Proof. The map K is birational as the composition of Φ and the map (1). The other statements are straightforward. $\square$

COROLLARY 3.2. *Under the assumptions of Lemma 3.3, define*

$$\Psi\colon S_2(s) \to (\mathrm{GL}(E_1)^{r-1})^{s-r-1},$$
$$\Psi(E_1, \ldots, E_{r+1}, V_{r+2}, \ldots, V_s) := (K^{-1}(V_{r+2}), \ldots, K^{-1}(V_s)).$$

Then Ψ is birational and H_2-equivariant.

Finally, using Lemma 3.3, we obtain the following.

COROLLARY 3.3. *For $n = rd$, the space $\mathrm{Gr}^s_{n,d}$ is almost GL_n-homogeneous if and only if $s \le r+1$. If $s > r+1$, there is a normal form for $(\mathrm{Gr}^s_{n,d}, \mathrm{GL}_n)$ that is isomorphic to $(\mathrm{GL}_d^{(r-1)(s-r-1)}, \mathrm{GL}_d)$.*

This implies the first part of Theorem 1.2.

4. The Case $n = (2r+1)e$ and $d = 2e$

We start with an r-tuple $(V_1, \ldots, V_r) \in \mathrm{Gr}^r_{n,d}$. Clearly the subset of all r-tuples that form a direct sum is both Zariski open and GL_n-homogeneous. Fix an r-tuple $(E_1, \ldots, E_r)$ in this subset and suppose that $s > r$. A straightforward calculation yields the following.

LEMMA 4.1. *Define*

$$S_1 = S_1(s) := \{\, (V_1, \ldots, V_s) \in \mathrm{Gr}^s_{n,d} : (V_1, \ldots, V_r) = (E_1, \ldots, E_r) \,\}$$

and the group $H_1 := \{\, g \in \mathrm{GL}_n : g(E_i) = E_i$ for all $i = 1, \ldots, r \,\}$ with the diagonal action on S_1. Then (S_1, H_1) is a normal form for $(\mathrm{Gr}^s_{n,d}, \mathrm{GL}_n)$.

Now we wish to parameterize the d-dimensional linear subspaces in $\mathbb{C}^n$ with respect to $(E_1, \ldots, E_r)$. We say that V is in *general position* with respect to $(E_1, \ldots, E_r)$, if $\dim W = e$, where $W := V \cap (E_1 \oplus \cdots \oplus E_r)$ and the projection of W on each E_i $(i = 1, \ldots, r)$ is injective. The subset of all $V \in \mathrm{Gr}_{n,d}$ that are in general position is clearly Zariski open.

We first give another description of the subspaces $W \in \mathrm{Gr}_e(E_1 \oplus \cdots \oplus E_r)$ that are in general position with respect to $(E_1, \ldots, E_r)$. For this, let $A_i = A_i(V) \in \mathrm{Gr}_e(E_i)$ be the projections of W and let $\varphi_i = \varphi_i(V) \in \mathrm{GL}(A_1, E_i)$ be the linear isomorphisms whose graphs are equal to the projections of W on $E_1 \oplus E_i$ $(i = 2, \ldots, r)$. Denote by $X = X(E_1, \ldots, E_r)$ the space of all tuples $(A, \varphi_2, \ldots, \varphi_r)$, where $A \in \mathrm{Gr}_e(E_1)$ and $\varphi_i \in \mathrm{GL}(A, E_i)$, with the standard structure of a quasiprojective variety.

LEMMA 4.2. *Define*

$$\Phi\colon X \to \mathrm{Gr}_e(E_1 \oplus \cdots \oplus E_r),$$
$$\Phi(A, \varphi_2, \ldots, \varphi_r) := \{\, (z \oplus \varphi_2(z) \oplus \cdots \oplus \varphi_r(z)) : z \in A \,\},$$

and the H_1-action on X by

$$(g, (A, \varphi_2, \dots, \varphi_r)) \mapsto (g(A), g \circ \varphi_2 \circ g^{-1}, \dots, g \circ \varphi_r \circ g^{-1}).$$

Then Φ is birational and H_1-equivariant. The image of Φ consists of all subspaces $W \in X$ that are in general position with respect to $(E_1, \dots, E_r)$.

Proof. The proof is straightforward. □

COROLLARY 4.1. *Let $S_1'(s) \subset S_1(s)$ be the Zariski open subset of all s-tuples $\mathcal{V} = (V_1, \dots, V_s)$ such that V_{r+1} is in general position with respect to $(E_1, \dots, E_r)$. Then H_1 acts transitively on $S_1'(r+1)$.*

Proof. It follows from the general position condition that

$$W(\mathcal{V}) := V_{r+1} \cap (E_1 \oplus \cdots \oplus E_r) \in \mathrm{Gr}_e(E_1 \oplus \cdots \oplus E_r),$$

in the notation of Lemma 4.2. Let $\mathcal{V}, \mathcal{V}' \in S_1'(r+1)$ be arbitrary elements. Since X is H_1-homogeneous, there exists $g_1 \in H_1$ such that $g_1(W(\mathcal{V})) = W(\mathcal{V}')$. Then there exists $g_2 \in \mathrm{GL}_n$ with $g_2(V_{r+1}) = V_{r+1}'$ and $g_2|(E_1 \oplus \cdots \oplus E_r) = \mathrm{id}$. By construction, $g_2 \in H_1$ and the proof is finished. □

COROLLARY 4.2. *For $s \le r+1$, $\mathrm{Gr}_{n,d}^s$ is almost GL_n-homogeneous.*

Let $s > r+1$ and $E_{r+1} \in \mathrm{Gr}_{n,d}$ be in general position with respect to $(E_1, \dots, E_r)$. As a direct consequence of Corollary 4.1, we have the following lemma.

LEMMA 4.3. *Define $S_2 := \{ (V_1, \dots, V_s) \in S_1 : V_{r+1} = E_{r+1} \}$ and $H_2 := \{ g \in \mathrm{GL}_n : g(E_i) = E_i$ for all $i = 1, \dots, r+1 \}$, with the diagonal action on S_2. Then (S_2, H_2) is a normal form for $(\mathrm{Gr}_{n,d}^s, \mathrm{GL}_n)$.*

For the sequel we suppose that the subspaces $E_1, \dots, E_{r+1}$ are fixed. Define

$$(A, \varphi_2, \dots, \varphi_r) := \Phi^{-1}(E_{r+1} \cap (E_1 \oplus \cdots \oplus E_r)).$$

Let $V \in \mathrm{Gr}_{n,d}$ be one more subspace that is in general position with respect to $(E_1, \dots, E_r)$. Clearly, V is not uniquely determined by its e-dimensional intersection $Z_1(V) := V \cap (E_1 \oplus \cdots \oplus E_r)$. However, using E_{r+1}, we can consider another intersection $Z_2(V) := V \cap (E_1 \oplus \cdots \oplus E_{r-1} \oplus E_{r+1})$. Then the map

$$Z := (Z_1, Z_2) \colon \mathrm{Gr}_{n,d} \to \mathrm{Gr}_e(E_1 \oplus \cdots \oplus E_r) \times \mathrm{Gr}_e(E_1 \oplus \cdots \oplus E_{r-1} \oplus E_{r+1})$$

is birational with the inverse $Z^{-1} \colon (W, W') \mapsto W + W'$.

Using the construction of X and Φ for the tuple $(E_1, \dots, E_{r-1}, E_{r+1})$ instead of $(E_1, \dots, E_r)$, we obtain $Y := X(E_1, \dots, E_{r-1}, E_{r+1})$ and the H_2-equivariant birational map

$$\Psi \colon Y \to \mathrm{Gr}_e(E_1 \oplus \cdots \oplus E_{r-1} \oplus E_{r+1}).$$

Combining Φ, Ψ, and Z, we obtain the following.

COROLLARY 4.3. *The composition* $\Omega := Z^{-1} \circ (\Phi, \Psi) \colon X \times Y \to \mathrm{Gr}_{n,d}$ *is birational and* H_2*-invariant.*

In the following we treat the cases $r > 1$ and $r = 1$ separately.

4.1. *The case* $r > 1$.

For $V \in \mathrm{Gr}_{n,d}$, set

$$(B, \psi_2, \ldots, \psi_r) := \Phi^{-1}(V \cap (E_1 \oplus \cdots \oplus E_r)) \in X, \tag{2}$$

$$(C, \chi_2, \ldots, \chi_{r-1}, \chi_{r+1}) := \Psi^{-1}(V \cap (E_1 \oplus \cdots \oplus E_{r-1} \oplus E_{r+1})) \in Y. \tag{3}$$

In order to construct a smaller normal form, fix $(A', \varphi_2', \ldots, \varphi_r') \in X$, $A'' \in \mathrm{Gr}_e(E_1)$, and $\chi_{r+1}' \in \mathrm{GL}(A'', E_{r+1})$ such that $E_1 = A \oplus A'$, $A'' \cap A = A'' \cap A' = \{0\}$, $E_i = \varphi_i(A) \oplus \varphi_i'(A')$ for all $i = 2, \ldots, r$, and $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_r \oplus \chi_{r+1}'(A'')$.

LEMMA 4.4. *In the notation just described, define*

$$S := \{\, V \in \mathrm{Gr}_{n,d} : B = A',\ C = A'',\ \psi_2 = \varphi_2', \ldots, \psi_r = \varphi_r',\ \chi_{r+1} = \chi_{r+1}' \,\}.$$

Then $H_2 S$ *is Zariski open in* $\mathrm{Gr}_{n,d}$.

Proof. For $V \in \mathrm{Gr}_{n,d}$ generic, $E_1 = A \oplus B$. Hence there exists a $g \in \mathrm{GL}(E_1)$ with $g|A = \mathrm{id}$ and $g(B) = A'$. Clearly g extends to an isomorphism from the action by H_2. Without loss of generality, $B = A'$. Similarly, there exists an isomorphism $g \in H_2$ such that $g|\varphi_i(A) = \mathrm{id}$ and $g \circ \psi_i = \varphi_i'$ on A' for all $i = 2, \ldots, s$. Thus we may assume that $\psi_i = \varphi_i'$. By Lemma 3.1, we may also assume that $C = A''$. Again, for V generic, $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_r \oplus \chi_{r+1}(A'')$, where $\chi_{r+1}(A'') \subset E_{r+1}$. Hence there exists an isomorphism $g \in H_2$ such that $g|(E_1 \oplus \cdots \oplus E_r) = \mathrm{id}$ and $g \circ \chi_{r+1} = \chi_{r+1}'$. This proves the lemma. $\square$

Let $\alpha \in \mathrm{GL}(A, A')$ be the isomorphism whose graph is A'' with respect to the splitting $E_1 = A \oplus A'$. Define $\beta \colon A \to A''$ by $\beta(z) := z \oplus \alpha(z)$. For every $i = 2, \ldots, r$, consider

$$\xi_i := (\varphi_i \times \varphi_i')^{-1} \circ \chi_i \circ \beta \in \mathrm{GL}(A, A \oplus A')$$

and its components $[\xi_i]_1 \in \mathrm{GL}(A)$ and $[\xi_i]_2 \in \mathrm{GL}(A, A')$. Define the rational map

$$\Xi \colon S \to \mathrm{GL}(A)^{2(r-2)},$$

$$\Xi \colon V \mapsto ([\xi_2]_1, \alpha^{-1} \circ [\xi_2]_2, \ldots, [\xi_{r-1}]_1, \alpha^{-1} \circ [\xi_{r-1}]_2),$$

where S is defined in Lemma 4.4.

Let $g \in \mathrm{GL}(A)$ be arbitrary. Using the isomorphisms $\alpha \colon A \to A'$, $\varphi_i \colon A \to \varphi_i(A)$ and $\varphi_i' \colon A' \to \varphi_i'(A')$ for $i = 2, \ldots, r$, and $\chi_{r+1}' \circ \beta \colon A \to \chi_{r+1}'(A'')$, we can extend g canonically to an isomorphism from the action by H_2. This extension defines a canonical $\mathrm{GL}(A)$-action on $\mathbb{C}^n$ and therefore on S (as defined in Lemma 4.4). We compare this action with the diagonal $\mathrm{GL}(A)$-action on $\mathrm{GL}(A)^{2(r-2)}$ by conjugations as follows.

LEMMA 4.5. *The map* $\Xi\colon \mathrm{Gr}_{n,d} \to \mathrm{GL}(A)^{2(r-2)}$ *is birational and* $\mathrm{GL}(A)$*-equivariant.*

Proof. The inverse of Ξ is given by

$$\Xi^{-1}(\lambda_2, \dots, \lambda_{r-1}, \delta_2, \dots, \delta_{r-1}) = \Omega(x, y),$$

where $\Omega\colon X \times Y \to \mathrm{Gr}_{n,d}$ is birational (by Corollary 4.3) and

$$x := (A', \varphi_2', \dots, \varphi_r'),$$

$$y := (A'', (\varphi_2 \times \varphi_2') \circ (\lambda_2, \alpha \circ \delta_2) \circ \beta^{-1}, \dots, \\ (\varphi_{r-1} \times \varphi_{r-1}') \circ (\lambda_{r-1}, \alpha \circ \delta_{r-1}) \circ \beta^{-1}, \chi_{r+1}').$$

The equivariance is straightforward. □

In the following corollary we use the space S as defined in Lemma 4.4.

COROLLARY 4.4. *Define* $S_3(s) := \{ (V_1, \dots, V_s) \in \mathrm{Gr}_{n,d}^s : V_{r+2} \in S \}$ *and consider the diagonal* $\mathrm{GL}(A)$*-action on* S_3*. Then* $(S_3, \mathrm{GL}(A))$ *is a normal form for* $(\mathrm{Gr}_{n,d}^s, \mathrm{GL}_n)$.

COROLLARY 4.5. *If* $d = 2e$ *and* $n = (2r+1)e$*, the space* $\mathrm{Gr}_{n,d}^{r+2}$ *is almost* GL_n*-homogeneous if and only if* $r = 2$.

Now let $V \in \mathrm{Gr}_{n,d}$ be one more subspace. Using the previously fixed data, we associate with V a tuple of $4r-2$ linear automorphisms of A as follows. In the notation following (2) and (3), let $\tau_1 \in \mathrm{GL}(A, A')$ and $\tau_2 \in \mathrm{GL}(A, A')$ be the linear isomorphisms with the graphs B and C, respectively. As above, let $\alpha \in \mathrm{GL}(A, A')$ be the isomorphism whose graph is A''. For $j = 1, 2$, set $\sigma_j := \alpha^{-1} \circ \tau_j \in \mathrm{GL}(A)$ and $\tau_j'(z) := z + \tau_j(z)$ ($\tau_1'\colon A \to B$, $\tau_2'\colon A \to C$). For $i = 2, \dots, r$,

$$\zeta_i := (\varphi_i \times \varphi_i')^{-1} \circ \psi_i \circ \tau_1' \in \mathrm{GL}(A, A \oplus A');$$

for $i = 2, \dots, r-1$,

$$\theta_i := (\varphi_i \times \varphi_i')^{-1} \circ \chi_i \circ \tau_2' \in \mathrm{GL}(A, A \oplus A').$$

We write $[\zeta_i]_1, [\theta_i]_1 \in \mathrm{GL}(A)$ and $[\zeta_i]_2, [\theta_i]_2 \in \mathrm{GL}(A, A')$ for the corresponding components with respect to the splitting $E_1 = A \oplus A'$. Then we obtain the linear automorphisms of A as $x_i := [\zeta_i]_1$ and $y_i := \alpha^{-1} \circ [\zeta_i]_2$ for $i = 2, \dots, r$; $z_i := [\theta_i]_1$, $t_i := \alpha^{-1} \circ [\theta_i]_2$ for $i = 2, \dots, r-1$.

In order to deal with the remainder term χ_{r+1}, we define $\varphi_{r+1} \in \mathrm{GL}(A, \mathbb{C}^n)$ by

$$\varphi_{r+1}(z) := z \oplus \varphi_2(z) \oplus \cdots \oplus \varphi_r(z).$$

It follows from the construction that $\varphi_{r+1}(A) \in \mathrm{Gr}_e(E_{r+1})$ and $E_{r+1} = \varphi_{r+1}(A) \oplus \chi_{r+1}'(A'')$. Using this splitting we define the remainder isomorphisms

$$\theta_r := (\varphi_{r+1} \times \chi_{r+1}')^{-1} \circ \chi_{r+1} \circ \tau_2' \in \mathrm{GL}(A, A \oplus A'),$$

$z_r := [\theta_r]_1$, and $t_r := \alpha^{-1} \circ [\theta_r]_2 \in \mathrm{GL}(A)$.

LEMMA 4.6. *Let* $\Theta\colon \mathrm{Gr}_{n,d} \to \mathrm{GL}(A)^{4r-2}$ *be given by*

$$\Theta\colon V \mapsto (\sigma_1, \sigma_2, x_2, y_2, z_2, t_2, \ldots, x_r, y_r, z_r, t_r).$$

Then Θ *is birational and* $\mathrm{GL}(A)$*-equivariant*.

Proof. In the foregoing notation, the inverse Θ^{-1} can be calculated as follows:

$$\tau_j' := \alpha \circ \sigma_j + \mathrm{id}, \quad B := \tau_1'(A), \quad C := \tau_2'(A),$$

$$\psi_i = (\varphi_i \times \varphi_i') \circ (x_i, \alpha \circ y_i) \circ (\tau_1')^{-1} \quad \text{for } i = 2, \ldots, r,$$

$$\chi_i = (\varphi_i \times \varphi_i') \circ (z_i, \alpha \circ t_i) \circ (\tau_2')^{-1} \quad \text{for } i = 2, \ldots, r-2,$$

$$\chi_{r+1} = (\varphi_{r+1} \times \chi_{r+1}') \circ (z_r, \alpha \circ t_r) \circ (\tau_2')^{-1},$$

$$V = \Omega((B, \psi_2, \ldots, \psi_r), (C, \chi_2, \ldots, \chi_{r-1}, \chi_{r+1})).$$

The equivariance is straightforward. □

COROLLARY 4.6. *If* $s \geq r+2$ *then the space* $(S_3, \mathrm{GL}(A))$ *is isomorphic to*

$$(\mathrm{GL}_e^{2r-4+(s-r-2)(4r-2)}, \mathrm{GL}_e).$$

Proof. The required birational isomorphism is given by

$$(V_1, \ldots, V_s) \mapsto (\Xi(V_{r+2}), \Theta(V_{r+3}), \ldots, \Theta(V_s)), \tag{4}$$

where Ξ is birational by Lemma 4.5 and Θ is birational by Lemma 4.6. □

This implies the second part of Theorem 1.2 in the case $r > 1$.

4.2. *The Case* $r = 1$

In this case we have $n = 3e$. Recall that we fixed $E_1, E_2 \in \mathrm{Gr}_{n,d}$ in general position such that $\dim A = e$, where $A := E_1 \cap E_2$. Choose another subspace $E_3 \in \mathrm{Gr}_{n,d}$ such that $\dim A_1 = \dim A_2 = e$, where $A_j := E_j \cap E_3$ for $j = 1, 2$. Recall that we defined $H_2 \subset \mathrm{GL}_n$ to be the stabilizer of both E_1 and E_2. Denote by $H_3 \subset H_2$ the stabilizer of E_3. Then $H_3 = \mathrm{GL}(A) \times \mathrm{GL}(A_1) \times \mathrm{GL}(A_2)$ with respect to the splitting $\mathbb{C}^n = A \oplus A_1 \oplus A_2$. The following is straightforward.

LEMMA 4.7. *Suppose that* $s \geq 3$ *and define* $S_3 = S_3(s) := \{ (V_1, \ldots, V_s) \in S_2 : V_3 = E_3 \}$. *Then* (S_3, H_3) *is a normal form for* $(\mathrm{Gr}_{n,d}^s, \mathrm{GL}_n)$.

Now we choose $B_j \in \mathrm{Gr}_e(E_j)$ such that $B_j \cap A = B_j \cap A_j = \{0\}$ for $j = 1, 2$. Then each B_j can be seen as the graph of an isomorphism $\varphi_j \in \mathrm{GL}(A, A_j)$. On the other hand, for $V \in \mathrm{Gr}_{n,d}$ generic, the subspaces $C_j := V \cap E_j$ are graphs of isomorphisms $\psi_j \in \mathrm{GL}(A, A_j)$. Define

$$g = (\mathrm{id}, \psi_1 \circ \varphi_1^{-1}, \psi_2 \circ \varphi_2^{-1}) \in \mathrm{GL}(A \oplus A_1 \oplus A_2).$$

Then $g \in H_3$ and $g(B_j) = C_j$ for $i = 1, 2$; in particular, $V = B_1 + B_2$. Together with Lemma 3.3 this proves the following.

LEMMA 4.8. *Suppose that $s \geq 4$. Define*

$$S_4 = S_4(s) := \{ (V_1, \ldots, V_s) \in S_3 : V_4 \cap E_1 = B_1, V_4 \cap E_2 = B_2 \}$$

and $H_4 := \mathrm{GL}(A)$. Define the H_4-action on S_4 via the homomorphism

$$H_4 \to H_3, \qquad g \mapsto (g, \varphi_1 \circ g \circ \varphi_1^{-1}, \varphi_2 \circ g \circ \varphi_2^{-1}).$$

Then (S_4, H_4) is a normal form for $(\mathrm{Gr}^s_{n,d}, \mathrm{GL}_n)$.

As a special case of Corollary 3.3, we obtain the following.

LEMMA 4.9. *The space $\mathrm{Gr}^s_{3e,2e}$ is almost GL_{3e}-homogeneous if and only if $s \leq 4$. If $s \geq 5$ then there exists a normal form that is isomorphic to $(\mathrm{GL}_e^{2(s-4)}, \mathrm{GL}_e)$, where GL_e acts diagonally by conjugations.*

This implies the second part of Theorem 1.2 in the case $r = 1$.

5. Computation of Rational Invariants

5.1. The Case $n = rd$

In order to compute the rational GL_{rd}-invariants of $\mathrm{Gr}^s_{rd,d}$ we represent the elements of $\mathrm{Gr}^s_{rd,d}$ by the equivalence classes of $rd \times ds$ matrices $M \in \mathbb{C}^{rd\times sd}$, where the equivalence is taken under the right multiplication by GL^s_d. Then the diagonal GL_{rd}-action on $\mathrm{Gr}^s_{rd,d}$ corresponds to the left multiplication on $\mathbb{C}^{rd\times sd}$. We start with $2d \times 2d$ matrices. Define

$$D\colon \mathrm{GL}_{2d} \to \mathrm{GL}_d, \qquad D\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} := A_{11}A_{21}^{-1}A_{22}A_{12}^{-1}. \tag{5}$$

Then D is a rational map that is invariant under the right multiplication by $\mathrm{GL}_d \times \mathrm{GL}_d$ and the left multiplication by $\{e\} \times \mathrm{GL}_d$, where $e \in \mathrm{GL}_d$ is the unit. Moreover, D is equivariant with respect to the left multiplication by $\mathrm{GL}_d \times \{e\}$.

More generally, let $M \in \mathbb{C}^{rd\times sd}$ be a rectangular matrix, where r and s are arbitrary positive integers. We split M into $d \times d$ blocks A_{ij} ($i = 1, \ldots, r$, $j = 1, \ldots, s$). Then, for every $i = 2, \ldots, r$ and $j = 2, \ldots, s$, define

$$D_{ij}\colon \mathbb{C}^{rd\times sd} \to \mathrm{GL}_d, \qquad D_{ij}(M) := D\begin{pmatrix} A_{11} & A_{1j} \\ A_{i1} & A_{ij} \end{pmatrix}.$$

Similarly to D, each D_{ij} is a rational map that is invariant under the right multiplication by GL^s_d and under the left multiplication by $\{e\} \times \mathrm{GL}^{r-1}_d$.

Finally, we construct rational maps that are invariant under the left multiplication by the larger group GL_{rd}. For this, we assume $s > r + 1$; otherwise, the left GL_{rd}-multiplication is almost homogeneous (see Corollary 3.3) and hence all rational invariants are constant. Then every matrix $M \in \mathbb{C}^{rd\times sd}$ from a Zariski open subset can be split into two blocks:

$$M = (AB), \quad A \in \mathrm{GL}_{rd}, \ B \in \mathbb{C}^{rd\times(s-r)d}.$$

Define

$$\varphi(M) := A^{-1}B \in \mathbb{C}^{rd\times(s-r)d}$$

and

$$G_{ij}(M) := D_{ij}(\varphi(M)) \quad \text{for } i = 2, \ldots, r \text{ and } j = 2, \ldots, s-r.$$

In particular, for $d = 1$, $r = 2$ and $s = 4$, we obtain the classical double ratio:

$$\begin{aligned} G_{22}\begin{pmatrix} 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} &= D\begin{pmatrix} z_2 - z_3 & z_2 - z_4 \\ z_1 - z_3 & z_1 - z_4 \end{pmatrix} \\ &= \frac{z_2 - z_3}{z_1 - z_3} : \frac{z_2 - z_4}{z_1 - z_4}. \end{aligned}$$

By the invariance of D_{ij}, every G_{ij} is invariant under the left multiplication by GL_{rd} and under the right multiplication by $\{e\} \times \mathrm{GL}_d^{s-1}$. Moreover, by construction, G_{ij} is equivariant with respect to the subgroup $\mathrm{GL}_d \times \{e\} \subset \mathrm{GL}_d \times \mathrm{GL}_d^{s-1}$, where we take the right action on $\mathbb{C}^{rd\times sd}$ and the conjugation on the image space GL_d. We therefore obtain the following rational $(\mathrm{GL}_{rd} \times \mathrm{GL}_d^s)$-invariants:

$$I_{\alpha\beta} := \mathrm{Tr}(G_{\alpha_1\beta_1} \cdots G_{\alpha_k\beta_k}) = \mathrm{Tr}((D_{\alpha_1\beta_1} \circ \varphi) \cdots (D_{\alpha_k\beta_k} \circ \varphi)), \tag{6}$$

where $\alpha \in \{2, \ldots, r\}^k$ and $\beta \in \{2, \ldots, s-r\}^k$ are arbitrary multi-indices.

Using a result of Procesi [P2], we show that the invariant field is actually generated by these traces of monomials.

THEOREM 5.1. *Let d, r, s be arbitrary positive integers such that $s > r+1$. Then the field of rational $(\mathrm{GL}_{rd} \times \mathrm{GL}_d^s)$-invariants of $rd \times sd$ matrices is generated by the functions $I_{\alpha\beta}$, where $\alpha \in \{2, \ldots, r\}^k$ and $\beta \in \{2, \ldots, s-r\}^k$ and where $k < 2^d$.*

Proof. Set $n := rd$ as before. We write the spaces $E_1, \ldots, E_{r+1}, V_{r+2}, \ldots, V_s \in \mathrm{Gr}_{n,d}$ as in Lemma 3.3 in the form of an $rd \times sd$ matrix with $d \times d$ blocks as

$$M := \begin{pmatrix} E & 0 & \cdots & 0 & E & E & \cdots & E \\ 0 & E & \cdots & 0 & E & V_{2,r+2} & \cdots & V_{2,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E & E & V_{r,r+2} & \cdots & V_{r,s} \end{pmatrix}, \tag{7}$$

where E denotes the identity matrix of the size $d \times d$.

Let $\tilde{S}_2 \subset \mathbb{C}^{rd\times sd}$ be subspace of all matrices of the form (7). Clearly $E_1, \ldots, E_{r+1}$ fulfill the assumptions of Lemma 3.3. It then follows from the lemma that $(\tilde{S}_2, \mathrm{GL}_d)$ is a normal form for $(\mathbb{C}^{rd\times sd}, \mathrm{GL}_{rd} \times \mathrm{GL}_d^s)$, where GL_d acts on $\tilde{S}_2$ as the diagonal subgroup of $\mathrm{GL}_{rd} \times \mathrm{GL}_d^s$ (i.e., each $d \times d$ block is conjugated by the same matrix).

By the definition of a normal form, it is sufficient to prove that the field of invariants of $(\tilde{S}_2, \mathrm{GL}_d)$ is generated by restrictions of the $I_{\alpha\beta}$. Let M be given by (7). By the obvious calculations,

$$\varphi(M) = \begin{pmatrix} E & E & \cdots & E \\ E & V_{2,r+2} & \cdots & V_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ E & V_{r,r+2} & \cdots & V_{r,s} \end{pmatrix} \tag{8}$$

and hence

$$I_{\alpha\beta}(M) = \operatorname{Tr}(V_{\alpha_1,r+\beta_1} \cdots V_{\alpha_k,r+\beta_k}). \tag{9}$$

By a theorem of Procesi [P2], the polynomial invariants of $(r-1)(s-r-1)$-tuples (V_{ij}) with respect to the diagonal GL_d-conjugations are generated by the monomials of the form (9) with $k < 2^d$. Since all points of $(\tilde{S}_2, \mathrm{GL}_d)$ are semistable (see [MFK]), the categorical quotient $\tilde{S}_2//\mathrm{GL}_d$ exists and is given by these monomials. Then the rational invariants on $(\tilde{S}_2, \mathrm{GL}_d)$ are pullbacks of rational functions on $\tilde{S}_2//\mathrm{GL}_d$, and the proof is finished. □

5.2. *The Case $n = 3e$ and $d = 2e$*

In this case the elements of $\mathrm{Gr}^s_{n,d}$ are represented by the equivalence classes of matrices $M \in \mathbb{C}^{3e\times 2se}$ with $3e \times 2e$ blocks $M_1, \ldots, M_s$. As before, we are looking for rational invariants with respect to left multiplications by GL_{3e} and right multiplications by GL^s_d. We start with the case of 3-blocks. Define the rational map

$$\varphi\colon \mathbb{C}^{3e\times 6e} \to \mathbb{C}^{3e\times 6e},$$

$$\varphi\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} := \begin{pmatrix} E' & E' & E' \\ B_1A_1^{-1} & B_2A_2^{-1} & B_3A_3^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} E & 0 & E & 0 & E & 0 \\ 0 & E & 0 & E & 0 & E \\ c_1 & d_1 & c_2 & d_2 & c_3 & d_3 \end{pmatrix}, \tag{10}$$

where $A_1, A_2, A_3 \in \mathrm{GL}_d$ and $B_1, B_2, B_3 \in \mathbb{C}^{e\times 2e}$ and where $E' \in \mathrm{GL}_{2e}$ and $E \in \mathrm{GL}_e$ denote the identity matrices. We see the corresponding subspaces $E_1, E_2, E_3 \in \mathrm{Gr}_{n,d}$ as graphs of linear maps given by the matrices $(c_1\, d_1)$, $(c_2\, d_2)$, and $(c_3\, d_3)$, respectively. Our first goal will be to compute the intersections $A := E_1 \cap E_2$ and $A_j := E_j \cap E_3$ $(j = 1, 2)$. For this, we set

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} := \left(\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}^{-1}, \begin{pmatrix} c_3 & d_3 \\ c_1 & d_1 \end{pmatrix}^{-1}, \begin{pmatrix} c_2 & d_2 \\ c_3 & d_3 \end{pmatrix}^{-1}\right)\begin{pmatrix} E \\ E \end{pmatrix} \in \mathbb{C}^{e\times 3e}.$$

Then $x_1c_1 + y_1d_1 = x_1c_2 + y_1d_2 = E$ and hence the $3e \times e$ matrix

$$\begin{pmatrix} x_1 \\ y_1 \\ E \end{pmatrix}$$

represents the intersection $E_1 \cap E_2$. Similarly the $3e \times e$ matrices

$$\begin{pmatrix} x_2 \\ y_2 \\ E \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_3 \\ y_3 \\ E \end{pmatrix}$$

represent the intersections $E_3 \cap E_1$ and $E_2 \cap E_3$, respectively. Equivalently, the 3-tuple (E_1, E_2, E_3) can be represented by the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 & y_1 & y_2 & y_3 \\ E & E & E & E & E & E \end{pmatrix}. \tag{11}$$

Now take the general matrix $M = (M_1, \ldots, M_s) \in \mathbb{C}^{3e \times 2es}$. Following the construction of Section 4.2, we bring A, A_1, and A_2 (i.e., the matrix (11)) to a normal form. For this, consider the square matrix

$$H(M) := \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ E & E & E \end{pmatrix}$$

and multiply M by $H(M)^{-1}$:

$$H(M)^{-1}M$$
$$:= \begin{pmatrix} E & 0 & 0 & E & 0 & 0 & C_{14} & D_{14} & \cdots & C_{1s} & D_{1s} \\ 0 & E & 0 & 0 & E & 0 & C_{24} & D_{24} & \cdots & C_{2s} & D_{2s} \\ 0 & 0 & E & 0 & 0 & E & C_{34} & D_{34} & \cdots & C_{3s} & D_{3s} \end{pmatrix}. \tag{12}$$

Next we normalize the $3e \times 2e$ blocks:

$$\begin{pmatrix} C_{1i} & D_{1i} \\ C_{2i} & D_{2i} \\ C_{3i} & D_{3i} \end{pmatrix} \begin{pmatrix} C_{1i} & D_{1i} \\ C_{2i} & D_{2i} \end{pmatrix}^{-1} = \begin{pmatrix} E & 0 \\ 0 & E \\ \alpha_{2i-1} & \alpha_{2i} \end{pmatrix}$$

and define

$$\sigma_{2i-1} := \alpha_{2i-1}\alpha_7^{-1} \quad \text{and} \quad \sigma_{2i} := \alpha_{2i}\alpha_8^{-1} \quad \text{for } i = 5, \ldots, s.$$

Comparing this construction with Section 4.2, we conclude that the matrices σ_j $(j = 9, \ldots, 2s)$ represent exactly the $2(s-4)$ matrices in the isomorphic normal form $(\mathrm{GL}_e^{2(s-4)}, \mathrm{GL}_e)$. Using the theorem of Procesi [P2] and arguments similar to those used in Section 5.1, we obtain the following.

THEOREM 5.2. *Let e and s be arbitrary positive integers such that $s > 4$. Then the field of rational $(\mathrm{GL}_{3e} \times \mathrm{GL}_{2e}^s)$-invariants of $3e \times 2es$ matrices is generated by the functions*

$$J_\alpha := \mathrm{Tr}(\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}),$$

where $\alpha \in \{9, \ldots, 2s\}^k$ and $k < 2^e$.

5.3. *The Case $n = (2r+1)e$ and $d = 2e$*

Here an element $(V_1, \ldots, V_s) \in \mathrm{Gr}_{n,d}^s$ is represented by a matrix $M \in \mathbb{C}^{(2r+1)e \times 2se}$ with s blocks $M_1, \ldots, M_s$ of the size $(2r+1)e \times 2e$. Again, the GL_n-invariants on $\mathrm{Gr}_{n,d}^s$ correspond to $(\mathrm{GL}_n \times \mathrm{GL}_{2e}^s)$-invariants on $\mathbb{C}^{(2r+1)e \times 2se}$. Similarly to the previous paragraph, we start with a special case of a $(2r+1)e \times (2r+2)e$ matrix and compute representatives of the intersections $E_{r+1} \cap (E_1 \oplus \cdots \oplus E_r)$,

$E_{r+2} \cap (E_1 \oplus \cdots \oplus E_r)$, and $E_{r+2} \cap (E_1 \oplus \cdots \oplus E_{r-1} \oplus E_{r+1})$ as in Section 4. We start by normalizing the $n \times 2e$ blocks as in (10):

$$\varphi\begin{pmatrix} A_1 & \cdots & A_s \\ B_1 & \cdots & B_s \end{pmatrix} := \begin{pmatrix} E' & \cdots & E' \\ B_1 A_1^{-1} & \cdots & B_s A_s^{-1} \end{pmatrix} = \begin{pmatrix} E' & \cdots & E' \\ C_1 & \cdots & C_s \end{pmatrix}. \tag{13}$$

In order to compute the first intersection $E_{r+1} \cap (E_1 \oplus \cdots \oplus E_r)$, we consider the corresponding system of linear equations:

$$(E' \ldots E')\begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = E' X_{r+1}, \qquad (C_1 \ldots C_r)\begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = C_{r+1} X_{r+1},$$

where each X_i is a $2e \times e$ block. Solving from the first equation $X_{r+1} = X_1 + \cdots + X_r$ and substituting this into the second, we obtain

$$((C_1 - C_{r+1}) \ldots (C_r - C_{r+1}))\begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = 0.$$

This is a system of $(2r+1)e \times e$ equations with $2re \times e$ variables. Hence, for $C_1, \ldots, C_{r+1}$ in general position, this system has a solution that can be represented by rational $2e \times e$ block functions

$$X_i = X_i(M) \quad (i = 1, \ldots, r).$$

Comparing this with the construction of Section 4, we see that the blocks

$$\begin{pmatrix} E' \\ C_i \end{pmatrix}(X_i) \quad (i = 1, \ldots, r),$$

represent the subspaces $A, \varphi_2(A), \ldots, \varphi_r(A)$. For our normalization (Lemma 4.4) we also need the subspaces $A', \varphi_2'(A'), \ldots, \varphi_r'(A')$ and χ_{r+1}', which come from the other intersections. For them we can also solve the corresponding linear systems and obtain rational $2e \times e$ block functions

$$Y_i(M) \quad (i = 1, \ldots, r), \qquad Z_{r+1}(M)$$

such that the blocks

$$\begin{pmatrix} E' \\ C_i \end{pmatrix}(Y_i(M)) \quad (i = 1, \ldots, r), \qquad \begin{pmatrix} E' \\ C_{r+1} \end{pmatrix}(Z_{r+1}(M))$$

represent $A', \varphi_2'(A'), \ldots, \varphi_r'(A')$ and χ_{r+1}', respectively. As before, we put these blocks together in an $n \times n$ matrix:

$$H(M)$$
$$:= \left(\begin{pmatrix} E' & \cdots & E' \\ C_1 & \cdots & C_r \end{pmatrix}\begin{pmatrix} X_1 & Y_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X_r & Y_r \end{pmatrix}, \begin{pmatrix} E' \\ C_{r+1} \end{pmatrix}(Z_{r+1}(M))\right).$$

The $n \times e$ columns of $H(M)$ are exactly the representatives of

$$A,\ A',\ \varphi(A),\ \varphi'(A'),\ \ldots,\ \varphi_r(A),\ \varphi_r'(A'),\ \chi_{r+1}'(A''),$$

in this order. If M is in general position, then $H(M)$ is invertible. As before, consider the matrix $H(M)^{-1}M$, which equals

$$\begin{pmatrix} E & 0 & \cdots & 0 & 0 & 0 & a_1 & b_{1,r+2} & c_{1,r+2} & \cdots & b_{1,s} & c_{1,s} \\ 0 & E & \cdots & 0 & 0 & 0 & a_2 & b_{2,r+2} & c_{1,r+2} & \cdots & b_{1,s} & c_{1,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & E & 0 & 0 & a_{2r-1} & b_{2r-1,r+2} & c_{2r-1,r+2} & \cdots & b_{2r-1,s} & c_{2r-1,s} \\ 0 & 0 & \cdots & 0 & E & 0 & a_{2r} & b_{2r,r+2} & c_{2r,r+2} & \cdots & b_{2r,s} & c_{2r,s} \\ 0 & 0 & \cdots & 0 & 0 & E & a_{2r+1} & b_{2r+1,r+2} & c_{2r+1,r+2} & \cdots & b_{2r+1,s} & c_{2r+1,s} \end{pmatrix}.$$

Moreover, the property $E_{r+1} = \Phi(A, \varphi_2, \ldots, \varphi_r) + \chi_{r+1}'(A'')$ implies

$$a_2 = a_4 = \cdots = a_{2r} = a_{2r+1} = 0.$$

Using the property $W' := \Phi(A', \varphi_2', \ldots, \varphi_r') \in E_{r+2}$, we may assume that W' is represented by the block

$$\begin{pmatrix} b_{1,r+2} \\ \vdots \\ b_{2r+1,r+2} \end{pmatrix}$$

and hence

$$b_{1,r+2} = b_{3,r+2} = \cdots = b_{2r+1,r+2} = 0.$$

Furthermore, the matrix components of the map (4) can be calculated directly as

$$Z(M) = \left(D\begin{pmatrix} a_1 & c_{1,r+2} \\ a_3 & c_{3,r+2} \end{pmatrix},\ D\begin{pmatrix} b_{2,r+2} & c_{2,r+2} \\ b_{4,r+2} & c_{4,r+2} \end{pmatrix},\ \ldots, \right.$$
$$\left. D\begin{pmatrix} a_1 & c_{1,r+2} \\ a_{2r-3} & c_{2r-3,r+2} \end{pmatrix},\ D\begin{pmatrix} b_{2,r+2} & c_{2,r+2} \\ b_{2r-2,r+2} & c_{2r-2,r+2} \end{pmatrix} \right) \quad (14)$$

and

$$\Theta_i(M) = \left(D\begin{pmatrix} a_1 & b_{1,i} \\ a_3 & c_{3,i} \end{pmatrix},\ D\begin{pmatrix} b_{2,r+2} & c_{2,i} \\ b_{4,r+2} & c_{4,i} \end{pmatrix},\ \ldots,\ D\begin{pmatrix} a_1 & b_{1,i} \\ a_{2r-1} & b_{2r-1,i} \end{pmatrix}, \right.$$
$$\left. D\begin{pmatrix} b_{2,r+2} & c_{2,r+2} \\ b_{2r,r+2} & c_{2r,r+2} \end{pmatrix},\ (c_{2r+1,r+2}^{-1} b_{2r+1,i}),\ (c_{2r+1,r+2}^{-1} c_{2r+1,i}) \right) \quad (15)$$

for $i = r+3, \ldots, s$, where D is the generalized double ratio defined by (5).

Comparing this with the proof of Corollary 4.6 and applying the theorem of Procesi [P2] yields the following.

THEOREM 5.3. *Let e, r, s be arbitrary positive integers such that $s \geq r+2$. Then the field of rational* $(\mathrm{GL}_{(2r+1)e} \times \mathrm{GL}_{2e}^s)$*-invariants of* $(2r+1)e \times 2es$ *matrices is generated by the functions*

$$\mathrm{Tr}(\sigma_1 \cdots \sigma_k),$$

where each σ_l is either a component of the map Z in (14) *or of one of the maps $\Theta_{r+3}, \ldots, \Theta_s$ in* (15) *and $k < 2^e$.*

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