$L_0^\infty(G)^*$ as the Second Dual of the Group Algebra $L^1(G)$ with a Locally Convex Topology

AJIT IQBAL SINGH*

Isik, Pym and Ulger [8] give a good account of the structure of the second dual $L^1(G)^{**}$ of the group algebra $L^1(G)$ of a compact group G. Lau and Pym [10] investigate the general case of a locally compact group G. They introduce a subalgebra L_G, the norm closure of elements in $L^1(G)^{**}$ with compact carriers, and identify it with $L_0^\infty(G)^*$ via restriction on the subspace $L_0^\infty(G)$ of bounded measurable functions on G that vanish at infinity. For $L_0^\infty(G)^*$, they are able to recover most of the results obtained for $L^1(G)^{**}$ in the compact case. Therefore, they suggest in [10] that the sensible replacement for $L^1(G)^{**}$ should be $L_0^\infty(G)^*$. The purpose of this paper is to give a locally convex topology τ on $L^1(G)$ under which $L_0^\infty(G)$ (with $\|\cdot\|_\infty$) is its strong dual and thus present $L_0^\infty(G)^*$ as the second dual of $(L^1(G), \tau)$. We show that, except for the trivial case of G finite, there are uncountably many such topologies, and we discuss various levels of continuity of multiplication.

As far as possible, we follow [10] in our notation and refer to [5] for basic functional analysis and to [7] for basic harmonic analysis (see also [12]). In particular, λ is the left Haar measure on the locally compact group G for a Borel measurable subset K of G. Moreover, $f \in L^\infty(G)$, $\|f\|_K = \operatorname{ess\,sup}\{\,|f(x)| : x \in K\,\}$, and $L_0^\infty(G) = \{\, f \in L^\infty(G) : \text{for } K \text{ compact, } \|f\|_{G\setminus K} \to 0 \text{ as } K \uparrow G\,\}$. It follows that $(L^1(G), L_0^\infty(G))$ is a dual pair.

Let σ and μ denote (resp.) the weak topology $\sigma(L^1(G), L_0^\infty(G))$ and the Mackey topology $\mu(L^1(G), L_0^\infty(G))$ on $L^1(G)$. Let σ^* denote the weak*-topology $\sigma(L_0^\infty(G), L^1(G))$ on $L_0^\infty(G)$, and let $L_{00}^1(G)$ be the subalgebra of $L^1(G)$ consisting of those f that vanish outside some compact subset of G.

Let $\mathcal{S}$ and $\mathcal{R}$ be (resp.) the sets of increasing sequences (K_n) in $\mathcal{K}$ and (a_n) in $(0, \infty)$ with $a_n \to \infty$. For $((K_n), (a_n)) \in \mathcal{S} \times \mathcal{R}$, let

$$U((K_n), (a_n)) = \{\, \phi \in L^1(G) : \|\phi\chi_{K_n}\|_1 \le a_n,\ n \in \mathbb{N}\,\}.$$

Then $\mathcal{U} = \{\, U((K_n), (a_n)) : ((K_n), (a_n)) \in \mathcal{S} \times \mathcal{R}\,\}$ is a base of neighborhoods of zero for a locally convex topology β^1 on $L^1(G)$. It is similar to the strict topology β defined by Buck [1].

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*Née Ajit Kaur Chilana.

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1. REMARKS. (i) If G is σ-compact then there exists a $(K_n) \in \mathcal{S}$ with $\bigcup\{ K_n : n \in \mathbb{N} \} = G$ satisfying the condition that each K in $\mathcal{K}$ is contained in some K_n. Therefore, a base of neighborhoods for β^1 is also given by

$$\mathcal{U} = \{ U((K_n), (a_n)) : (a_n) \in \mathcal{R} \}.$$

(ii) If G is infinite then there is a $(K_n) \in \mathcal{S}$ with $\lambda(K_n \setminus K_{n-1}) > 0$ for each n, where $K_0 = \phi$. It is easy to see this if G is not compact because, for a K in $\mathcal{K}$, $G \setminus K$ is a non-empty open subset of the locally compact space G and thus contains a compact subset L with non-empty interior. Alternatively, we can use the proof of [7, item (11.43)(e)]. On the other hand, if G is compact then G is not discrete, so by regularity of λ there is a decreasing sequence (U_n) of open neighborhoods of the identity e satisfying $0 < \lambda U_{n+1} < \lambda U_n$ for each n in $\mathbb{N}$. We may take $K_n = G \setminus U_n$ for n in $\mathbb{N}$.

(iii) The construction in [7, item (11.43)] can by modified to give the following stronger form of (ii) to be used later: If G is not compact then there exist (A_n) in $\mathcal{S}$ and sequences (B_n) and (C_n) in $\mathcal{K}$ that satisfy the following conditions.

(a) $A_n B_n^{-1} \subset C_n$.
(b) The B_n are mutually disjoint.
(c) The C_n are mutually disjoint.
(d) $\inf_n \lambda C_n \geq \inf_n \lambda B_n^{-1} > 0$.
(e) If G is unimodular then, for each n,

$$\lambda B_n \leq 1 \quad \text{and} \quad \lambda\big(\textstyle\bigcup_n A_n\big) = \infty.$$

Let U and V be compact symmetric neighborhoods of e in G with $V^2 \subset U$ and $\lambda V \leq 1$. Since G is not compact, for any finite subset F of G there is a z in G with z not in the set $\bigcup\{ x^{-1}UyU : x, y \in F \}$. Hence, taking $x_0 = e$, we can inductively construct a sequence (x_n) in G with

$$x_{2^n} \notin \bigcup\{ x_j^{-1}Ux_kU : 0 \leq j,\ k < 2^n \} \quad \text{for } n \text{ in } \mathbb{N} \cup \{0\},$$

$$x_{2^k+j} = x_j x_{2^k} \quad \text{for } 1 \leq j < 2^k \text{ and } k \text{ in } \mathbb{N}.$$

For $n \in \mathbb{N}$, we put

$$A_n = \bigcup\{ Vx_j : 0 \leq j < 2^n \},$$

$$B_n = V(x_{2^n})^{-1}, \quad \text{and}$$

$$C_n = \bigcup\{ Vx_jV : 2^n \leq j < 2^{n+1} \}.$$

(iv) We can strengthen (ii) in another way by modifiying the construction in [7, item (11.43)(e)]. Suppose G is not compact. Let V be a compact symmetric neighborhood of e and let $(K_n) \in \mathcal{S}$. Then there are sequences (x_n) in G and $(L_n) \in \mathcal{S}$ such that, for each n, $K_n \subset L_n$ and $Vx_n \subset L_n \setminus L_{n-1}$, where $L_0 = \phi$.

(v) If G is compact, then $L_0^\infty(G) = L^\infty(G)$ and $\beta^1 = \mu = \|\cdot\|_1$-topology.

2. THEOREM. *The dual of* $(L^1(G), \beta^1)$ (*with the strong topology*) *can be identified with* $L_0^\infty(G)$ (*with* $\|\cdot\|_\infty$) *and thus the second dual of* $(L^1(G), \beta^1)$ *can be identified with* $L_0^\infty(G)^*$.

Proof. Let $B = \{\phi \in L^1(G) : \|\phi\|_1 \leq 1\}$. Then B is β^1-bounded. Hence every β^1-continuous linear functional on $L^1(G)$ is bounded on B and thus is continuous on $(L^1(G), \|\cdot\|_1)$. Each such functional is therefore given by an element of $L^\infty(G)$. We show that such an f is in $L_0^\infty(G)$. Since f is β^1-continuous, there is a $((K_n), (a_n)) \in \mathcal{S} \times \mathcal{R}$ such that

$$\left|\int \phi(x) f(x)\, d\lambda(x)\right| \leq 1 \quad \text{for each } \phi \text{ in } U((K_n), (a_n)).$$

Also, there exists a $g \in L^\infty(G)$ with $\|g\|_\infty \leq 1$ and $gf = |f|$.

Let $j \in \mathbb{N}$. Let A be a Borel subset of $G\setminus K_j$ with $0 < \lambda A < \infty$ and $\alpha \geq 0$ such that $|f|\chi_A \geq \alpha\chi_A$. Let $\phi = a_{j+1}(\lambda A)^{-1}\chi_A g$. Then $\phi \in U((K_n), (a_n))$, and so

$$1 \geq \left|\int \phi(x) f(x)\, d\lambda(x)\right| = \int a_{j+1}(\lambda A)^{-1}(\chi_A|f|)(x)\, d\lambda(x) \geq a_{j+1}\alpha.$$

Therefore, $\alpha \leq 1/a_{j+1}$ and so $\|f\|_{G\setminus K_j} \leq 1/a_{j+1}$. Since $a_j \to \infty$, we also have $\|f\|_{G\setminus K_j} \to 0$ as $j \to \infty$; hence, $f \in L_0^\infty(G)$.

Now let $f \in L_\infty^0(G)$. Then there exists a $(K_n) \in \mathcal{S}$ such that $\|f\|_{G\setminus K_n} \to 0$ as $n \to \infty$. Put $K_0 = \phi$ and, for $n \in \mathbb{N}$, set $b_n = \|f\|_{G\setminus K_{n-1}}$ and $\beta_n = \sqrt{b_n}$. Let $(a_n) \in \mathcal{R}$ be such that $a_n\beta_n \leq 1$ for each n. Let $\phi \in U((K_n), (a_n))$. For $n \in \mathbb{N}$, let $r_n = \|\phi\chi_{K_n\setminus K_{n-1}}\|_1$ and $s_n = \sum_{1\leq j\leq n} r_j$. Put $s_0 = 0$. Then, for $p \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{1\leq n\leq p+1} b_n r_n &= \sum_{1\leq n\leq p} (b_n - b_{n+1})s_n + b_{p+1}s_{p+1}\\
&= \sum_{1\leq n\leq p} (\beta_n - \beta_{n+1})(\beta_n + \beta_{n+1})s_n + \beta_{p+1}^2 s_{p+1}\\
&\leq \sum_{1\leq n\leq p} (\beta_n - \beta_{n+1})2\beta_n a_n + \beta_{p+1}^2 a_{p+1}\\
&\leq \sum_{1\leq n\leq p} 2(\beta_n - \beta_{n+1}) + \beta_{p+1}.
\end{aligned}$$

As a result, $\left|\int \phi(x) f(x)\, d\lambda(x)\right| \leq \sum_{n=1}^\infty b_n r_n \leq 2[\|f\|_\infty]^{1/2}$ and so f is β^1-continuous.

We next show that B absorbs all β^1-bounded subsets of $L^1(G)$. Suppose not. Then there is a β^1-bounded subset X of $L^1(G)$ such that $X \not\subset \rho B$ for each $\rho > 0$. Hence, for each $n \in \mathbb{N}$, there is a $\phi_n \in X$ with $\|\phi_n\|_1 > n$ and thus a $C_n \in \mathcal{K}$ with $\|\phi_n\chi_{C_n}\|_1 > n$. We can have a sequence K_n in $\mathcal{S}$ with $C_n \subset K_n$ for each n. Put $a_n = \sqrt{n}$ for n in $\mathbb{N}$. Then there is a $\rho > 0$ such that $X \subset \rho U((K_n), (a_n))$. Therefore, for each n,

$$\|\phi_n\chi_{K_n}\|_1 \leq \rho a_n = \rho\sqrt{n}.$$

But $\|\phi_n\chi_{K_n}\|_1 \geq \|\phi_n\chi_{C_n}\|_1 > 1$ and thus $n < \rho\sqrt{n}$ for each n—this gives us a contradiction. Hence B absorbs every β^1-bounded subset of $L^1(G)$.

Consequently, the strong topology τ_b on $(L^1(G), \beta^1)^*$ identified with $L_0^\infty(G)$ is the topology given by the norm defined by

$$\|f\| = \sup\left\{ \left| \int f(x)\phi(x)\, d\lambda(x) \right| : \phi \in B \right\} = \|f\|_\infty.$$

Hence the second dual of $(L^1(G), \beta^1)$ is $L_0^\infty(G)^*$. □

3. THEOREM. *Let G be infinite. Then there are uncountably many locally convex topologies τ on $L^1(G)$ such that $L_0^\infty(G)$ (with $\|\cdot\|_\infty$) is the strong dual of $(L^1(G), \tau)$ and thus $L_0^\infty(G)^*$ is the second dual of $(L^1(G), \tau)$.*

Proof. By Remark 1(ii) there is a $(K_n) \in \mathcal{S}$ with $\lambda(K_n \backslash K_{n-1}) > 0$ for each n, where $K_0 = \phi$. Let $(a_n) \in \mathcal{R}$ and put $V = U((K_n), (a_n))$. Then V contains the space generated by an f in $L^1(G)$ if and only if $f = 0$ on $\bigcup_n K_n$. Since $\{ \chi_{K_n \backslash K_{n-1}} : n \in \mathbb{N} \}$ is a linearly independent set, the space $F_1 = \{ f \in L^1(G) : f = 0$ on each $K_n \}$ has infinite codimension in $L^1(G)$. Every σ-neighborhood of zero contains a subspace of $L^1(G)$ of finite codimension, so V cannot be a σ-neighborhood of zero and thus $\sigma < \beta^1$. Hence, by [11], there exist infinitely many locally convex topologies τ lying between σ and β^1; in fact, using [9], we have uncountably many such topologies τ. Each one of them has $(L_0^\infty(G), \|\cdot\|_\infty)$ as its strong dual. □

4. REMARKS. (i) For any topology τ with $(L^1(G), \tau)^* = L_0^\infty(G)$ (in particular, if $\sigma \le \tau \le \beta^1$), the set of continuous (nonzero) multiplicative linear functionals on $(L^1(G), \tau)$ is the set of continuous characters of G or empty according as G is compact or noncompact. This follows immediately from [7, Cor. (23.7)], since every multiplicative linear functional on $L^1(G)$ is $\|\cdot\|_1$-continuous and since a character is in $L_0^\infty(G)$ if and only if G is compact.

(ii) Gulick [6] considered a locally convex algebra with hypocontinuous multiplication and constructed its second dual with Arens product. We recall that multiplication in a locally convex algebra E is said to be *hypocontinuous* if, given a neighborhood U of zero in E and a bounded subset C of E, there exists a neighborhood V of zero in E satisfying $(VC) \cup (CV) \subset U$. Interestingly, the Arens product on $L_0^\infty(G)^*$ has already been constructed by Lau and Pym in [10, Prop. 2.7] and the discussion that follows. Take $G = \mathbb{R}$, let $K_n = [-n, n]$ for each n, and take any $(a_n), (b_n) \in \mathcal{R}$ and $r \in \mathbb{N}$. We then see that $\phi_r = b_r \chi_{(r-1,r]} \in U((K_n), (b_n))$ and $\psi_r = \chi_{[-r,-r+1)} \in B$, but

$$\|(\phi_r * \psi_r)\chi_{[-1,1]}\|_1 = b_r.$$

Hence $U((K_n), (b_n)) * B \not\subset U((K_n), (a_n))$. Thus multiplication in $(L_{00}^1(\mathbb{R}), \beta^1)$ (and *a fortiori* in $(L^1(\mathbb{R}, \beta^1))$ is not hypocontinuous. We shall strenghten this result in Theorem 5.

(iii) We are not yet able to see if $(L^1(G), \beta^1)$ has separately continuous multiplication. However, a dense subalgebra—namely, $(L_{00}^1(G), \beta^1)$—has separately continuous multiplication and is thus a locally convex algebra. Further, $(L^1(G), \beta^1)$ is a locally convex module over $(L_{00}^1(G), \beta^1)$. To see this, it is enough to note that, for $f \in L_{00}^1(G)$ with f vanishing outside a compact subset L of G and for $g \in L^1(G)$ and K in $\mathcal{K}$, we have that KL^{-1} and $L^{-1}K$ are in $\mathcal{K}$,

$$\|(f * g)\chi_K\|_1 \leq \|f\|_1 \|g\chi_{L^{-1}K}\|_1,$$

and

$$\|(g * f)\chi_K\|_1 \leq \|f\|_1 \|g\chi_{KL^{-1}}\|_1.$$

5. THEOREM. *Let G be unimodular.*

(a) *$(L^1(G), \sigma)$ and $(L^1(G), \mu)$ are both locally convex algebras.*

(b) *If G is infinite then multiplication in $(L^1(G), \sigma)$ is not hypocontinuous.*

(c) *If G is not compact then multiplication in $(L^1(G), \mu)$ is not hypocontinuous.*

(d) *If G is not compact then multiplication considered as a bilinear map on $(L^1_{00}(G), \beta^1) \times (L^1_{00}(G), \beta^1)$ to $(L^1_{00}(G), \sigma)$ is not hypocontinuous*; a fortiori, *multiplication is hypocontinuous neither in $(L^1(G), \beta^1)$ nor in $(L^1(G), \sigma)$.*

Proof. By [7, Cor. (20.14), item (20.19)], for $f \in L^1(G)$ and $g \in L^\infty(G)$ we have that $f * g$ and $g * f$ are in $L^\infty(G)$, $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$, and $\|g * f\|_\infty \leq \|f\|_1 \|g\|_\infty$. Let $f, g \in L^1(G)$ and $h \in L_0^\infty(G) = (L^1(G), \sigma)^*$, and let g_1 be given by $g_1(x) = g(x^{-1})$ for x in G. Then $g_1 \in L^1(G)$. Hence $h * g_1$ and $g_1 * h$ are both in $L^\infty(G)$. Also, $h(f * g) = (h * g_1)(f)$ and $h(g * f) = (g_1 * h)(f)$.

(a) To prove that multiplication by g is continuous on $(L^1(G), \sigma)$ to itself, it is enough to show that $h * g_1$ and $g_1 * h$ are both in $L_0^\infty(G)$. Let $\varepsilon > 0$ be arbitrary. Then there is a compact subset K of G such that $\|g_1\chi_{G\setminus K}\|_1 < \varepsilon$ and $\|h\chi_{G\setminus K}\|_\infty < \varepsilon$. We thus have

$$\begin{aligned}
\|(h * g_1)\chi_{G\setminus K^2}\|_\infty &= \|(h\chi_K * g_1\chi_K + h\chi_K * g_1\chi_{G\setminus K} + h\chi_{G\setminus K} * g_1)\chi_{G\setminus K^2}\|_\infty \\
&= \|(h\chi_K * g_1\chi_{G\setminus K} + h\chi_{G\setminus K} * g_1)\chi_{G\setminus K^2}\|_\infty \\
&\leq \|h\chi_K * g_1\chi_{G\setminus K}\|_\infty + \|h\chi_{G\setminus K} * g_1\|_\infty \\
&\leq \|h\chi_K\|_\infty \|g_1\chi_{G\setminus K}\|_1 + \|h\chi_{G\setminus K}\|_\infty \|g_1\|_1 \\
&\leq \|h\|_\infty \varepsilon + \varepsilon \|g_1\|_1 \\
&= \varepsilon(\|h\|_\infty + \|g\|_1).
\end{aligned}$$

Similarly, $\|(g_1 * h)\chi_{G\setminus K^2}\|_\infty \leq \varepsilon(\|h\|_\infty + \|g\|_1)$, so both $h * g_1$ and $g_1 * h$ are in $L_0^\infty(G)$.

Further, to prove that multiplication by g is continuous on $(L^1(G), \mu)$ to itself, it is enough to show that, for a balanced convex σ^*-compact subset A of $L_0^\infty(G)$, both $A * g_1$ and $g_1 * A$ are balanced convex σ^*-compact subsets of $L_0^\infty(G)$. They are clearly balanced convex subsets of $L_0^\infty(G)$. We start with a net $(h_\alpha) * g_1$ in $A * g_1$. Then (h_α) has a subnet (ψ_β) in A that converges to a ψ in A in the σ^*-topology. Thus, for an f in $L^1(G)$, $(\psi_\beta * g_1)(f) = \psi_\beta(f * g)$ converges to $\psi(f * g) = (\psi * g_1)(f)$. Hence $(h_\alpha * g_1)$ has a subnet (viz. $(\psi_\beta * g_1)$) convergent to $\psi * g_1$ in $A * g_1$ in the σ^*-topology. This shows that $A * g_1$ is σ^*-compact. Similarly, we can show this fact for $g_1 * A$.

(b) Let (if possible) multiplication in $(L^1(G), \sigma)$ be hypocontinuous. Let $h \in L_{00}^\infty(G)$. By the hypocontinuity of multiplication in $(L^1(G), \sigma)$, we have an n-tuple $\{f_j\}_{j=1}^n$ in $L_0^\infty(G) = (L^1(G), \sigma)^*$ such that, putting $V = \left\{ f \in L^1(G) : \left|\int f(x) f_j(x)\, d\lambda(x)\right| < 1, 1 \leq j \leq n \right\}$, we have

$$V * B \subset \left\{ f \in L^1(G) : \left| \int f(x)h(x)\,d\lambda(x) \right| < 1 \right\}.$$

So $\bigcap_{j=1}^n N(f_j) * L^1(G) \subset N(h)$, where, for $\phi \in L_0^\infty(G)$, $N(\phi)$ denotes the null space of ϕ, that is,

$$\left\{ f \in L^1(G) : \int_G f(x)\phi(x)\,d\lambda(x) = 0 \right\}.$$

Let $g \in L^1(G)$ and $g_1(x) = g(x^{-1})$ for x in G. For f in $\bigcap_{j=1}^n N(f_j)$, $0 = h(f * g) = (h * g_1)(f)$ and so $f \in N(h * g_1)$. Therefore, by duality theory in locally convex spaces, $h * g_1$ is in the linear span F of $\{ f_j : 1 \le j \le n \}$. Thus $h * L^1(G) \subset F$. In particular, $h * L^1(G)$ is finite-dimensional.

The proof will be complete if we produce an h not having this property. If G is discrete then $h = \chi_{\{e\}}$ works fine. Suppose G is not discrete, and let $x \neq e$ be an element of G. Then there is a compact symmetric neighborhood K_0 of e such that $K_0 \cap xK_0 = \emptyset$. Let $K = K_0 \cup \{x\}$. Since G is not discrete, x is a boundary point of K. Let $\mathcal{U} = \{ U : U$ is an open symmetric neighborhood of e with $U \subset K_0 \}$. For $U \in \mathcal{U}$ let $K_U = K\bar{U}$ and $V_U = xU \cap (G \backslash K)$. Then V_U is a non-empty open subset of G and thus $\lambda(V_U) > 0$. Hence $\lambda(K_U) \ge \lambda K + \lambda V_U > \lambda K$ and $\lambda K_U \le \lambda K^2 < \infty$ for all U. Further, $\{ K_U : U \in \mathcal{U} \}$ forms a neighborhood base for K. Thus, by regularity of λ, $\lambda K_U \to \lambda K$ and so there is a decreasing sequence (U_n) in $\mathcal{U}$ with λK_{U_n} all distinct and $\lambda K_{U_n} \to \lambda K$. In particular, $\lambda(K_{U_n} \backslash K_{U_{n+1}}) > 0$ for each n.

Let $h = \chi_K$ and $f_n = \chi_{\bar{U}_n}$. Then $h \in L_{00}^\infty(G)$ and each f_n is in $L_{00}^\infty(G)$. Since $\operatorname{Supp} h * f_n = K_{U_n}$, we have that $\{ h * f_n : n \in \mathbb{N} \}$ is a linearly independent set. Hence $h * L^1(G)$ is not finite-dimensional, completing the proof of part (b).

(c) Let (if possible) multiplication in $(L^1(G), \mu)$ be hypocontinuous, and let (A_n), (B_n), (C_n), and V be as in Remark 1(iii). For $n \in \mathbb{N}$, let $g_n = \chi_{B_n}$ and $h_n = \chi_{C_n}$. Then the $\sigma(L^\infty(G), L^1(G))$-closed envelope H of $\{ h_n : n \in \mathbb{N} \}$ is the set $\left\{ \sum_{n=1}^\infty a_n h_n : a_n \in \mathbb{C} \text{ for each } n \text{ and } \sum_{n=1}^\infty |a_n| \le 1 \right\}$, and so $H \subset L_0^\infty(G)$.

By Alaoglu's theorem, the unit ball D of $(L^\infty(G), \|\cdot\|_\infty)$ is $\sigma(L^\infty(G), L^1(G))$-compact. Since $H \subset D$ is $\sigma(L^\infty(G), L^1(G))$-closed we have that H is a σ^*-compact subset of $L_0^\infty(G)$. Therefore,

$$W = H^0 = \left\{ f \in L^1(G) : \left| \int f(x)h(x)\,d\lambda(x) \right| \le 1 \text{ for } h \text{ in } H \right\}$$

is a μ-neigbourhood of zero in $L^1(G)$. By hypocontinuity of multiplication in $(L^1(G), \mu)$, there is a σ^*-compact balanced convex subset E of $L_0^\infty(G)$ with $E^0 * B \subset H^0$. This gives $E^0 \subset (H * B)^0$, which in turn gives that $H * B \subset E$; thus $(H * B)$ is a relatively compact subset of $(L_0^\infty(G), \sigma^*)$. The sequence (ψ_n) given by $\psi_n = h_n * g_n$ therefore has a subnet σ^*-convergent to a ψ in $L_0^\infty(G)$. But $\psi_n(x) = \lambda(xB_n^{-1} \cap C_n) = \lambda V$ for x in A_n (n in $\mathbb{N}$). Hence $\psi(x) = \lambda V$ for x in $\bigcup_n A_n$. Since $\lambda\big(\bigcup_n A_n\big) = \infty$, we have that $\psi \notin L_0^\infty(G)$. This contradiction completes the proof of (c).

(d) Consider any $((K_n), (a_n)) \in \mathcal{S} \times \mathcal{R}$ and a compact symmetric neighborhood V of e in G with $\lambda V \le 1$. Let (x_n) and (L_n) be as in Remark 1(iv). For

$n \in \mathbb{N}$, we put $\phi_r = a_r\chi_{Vx_r}$ and $\psi_r = \chi_{x_r^{-1}V}$. Then each ϕ_r is in $U((L_n), (a_n)) \subset U((K_n), (a_n))$, and each ψ_r is in B. But $\|(\phi_r * \psi_r)\chi_{V^2}\|_1 = a_r(\lambda(V))^2$, so

$$U((K_n), (a_n)) * B \not\subset \left\{ f \in L^1(G) : \left| \int f(x)\chi_{V^2}(x)\,d\lambda(x) \right| < 1 \right\}.$$

This finishes the proof. □

6. Remarks. (i) For the case of G compact abelian, Theorem 5(b) follows from [3, Thm. 1] applied to the Banach algebra $(L^1(G), \|\cdot\|_1)$ because its dual in this case is $L_0^\infty(G) = L^\infty(G)$. On the other hand, taking G to be noncompact, Theorem 5(b) provides a large set of examples to show that condition (ii) in [3, Thm. 2] is not necessary for the conclusion to be true.

(ii) Since $(L^1(G), \sigma)$ has a bounded bornivore B, it is a boundedly generated space. So [2] can be used to advantage. For instance, it gives a corollary to Theorem 5 as: If G is infinite and unimodular then $(L^1(G), \sigma)$ is not A-convex.

(iii) Unimodularity is not needed for Theorem 5(b) because our proof can be easily modified by considering g in $L_{00}^\infty(G)$ only, instead of in the whole of $L^1(G)$. The proof can then be augmented to show that $(L^1(G), \sigma)$ is not A-convex.

Our next theorem comes as an answer to the following question (posed by the referee): Does Arens regularity of $L_0^\infty(G)^*$ imply G is finite?

7. Theorem.

(i) *$L_0^\infty(G)^*$ is Arens regular if and only if G is finite.*

(ii) *Let τ be any locally convex topology on $L^1(G)$ lying between τ and β^1. Then $(L^1(G), \tau)$ is Arens regular if and only if G is discrete.*

Proof. (i) By [4, Cor. 6.3], if $L_0^\infty(G)^*$ is Arens regular then this implies that the subalgebra $L^1(G)$ is also Arens regular. By the now-classical result from [4] and [13], G is finite. The reverse implication is clear.

(ii) As proved in [10, Thm. 2.11(v)], the topological center of $L_0^\infty(G)^*$ is $L^1(G)$. Thus $(L^1(G), \tau)$ is Arens regular if and only if $L^1(G) = L_0^\infty(G)^*$. This follows when G is discrete, as has been noted in [10, p. 452]. For the converse, as in [10, Sec. 2] let π be the natural projection on $L^1(G)^{**}$ to $\mathrm{LUC}(G)^*$, where $\mathrm{LUC}(G)$ is the subspace of $L^\infty(G)$ consisting of functions that are bounded and uniformly continuous in the left uniformity of G. For $H \in L^1(G)^{**} = L^\infty(G)^*$, $\pi(H)$ is the restriction of H to $\mathrm{LUC}(G)$.

Further, it has been noted in [10] that π is the identity on $L^1(G)$ and, by [10, Thm. 2.8], $\pi L_0^\infty(G)^* = M(G)$. Hence $(L^1(G), \tau)$ is Arens regular implies that $L^1(G) = M(G)$, which in turn gives that G is discrete. □

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Department of Mathematics
University of Delhi South Campus
New Delhi 110021
India

ajitis@csec.ernet.in

Current address
Institute of Biomathematics and Biometry
GSF National Research Centre for Environment and Health
D-85758 Neuherberg
Germany

singh@gsf.de