Some Banach Space Properties of the Duals of the Disk Algebra and H^∞

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1. Introduction

The program of knowing the Banach space properties of the space H^∞ of bounded analytic functions on the disk and their duals has been greatly stimulated since the appearance of Pełczyński's notes [Pe2] about Banach spaces of analytic functions. Specifically, Pełczynski asked about some classical properties such as weak sequential completeness, the Dunford–Pettis property, property (V), or being a Grothendieck space. Most of this program has been realized in a series of papers by Bourgain [B1; B3; B4]. Namely, Bourgain has shown that H^∞ and all its duals have the Dunford–Pettis property, that all the odd duals of H^∞ are weakly sequentially complete and have property (V^*), and that H^∞ is a Grothendieck space and has property (V). Therefore, the following question is open: Are the even duals of H^∞ Grothendieck spaces, and do they have property (V)? In this paper, we give a positive answer to this question. In fact, we prove several stronger results which we detail below.

Properties (V) and (V^*) were introduced by Pełczyński in [Pe1]. Following [Pe1], a Banach space X is said to have property (V) (resp. property (V^*)) if every subset $W \subset X^*$ (resp. $W \subset X$) satisfying

$$\lim_n \sup_{x^* \in W} |\langle x^*, x_n\rangle| = 0 \quad \left(\text{resp. } \lim_n \sup_{x \in W} |\langle x_n^*, x\rangle| = 0\right)$$

for every weakly unconditionally Cauchy series $\sum_n x_n$ in X (resp. $\sum_n x_n^*$ in X^*) is weakly relatively compact in X^* (resp. in X). On the other hand, a Banach space X is said to be a Grothendieck space if every sequence in X^* that is weak-* convergent is also weakly convergent in X^*.

We also recall that if X is a Banach space, I is an index set, and $\mathcal{U}$ is an ultrafilter on I, then the ultrapower $(X)_{\mathcal{U}}$ is defined as the quotient of the Banach space

$$\ell_\infty(I, X) := \{ (x_i)_{i\in I} \subset X : \sup\{ \|x_i\| : i \in I \} < \infty \}$$

by its closed subspace

$$N_{\mathcal{U}} := \{ (x_i)_{i\in I} \in \ell_\infty(I, X) : \lim_{i\to\mathcal{U}} \|x_i\| = 0 \}.$$

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A detailed study of ultrapowers of Banach spaces can be found in Heinrich [H] and Stern [St]. It is well known that ultrapowers are used as a convenient device for the *localization* of infinite-dimensional properties, and it is said that X has local property $(\mathcal{P})$ if each ultrapower $(X)_{\mathcal{U}}$ has property $(\mathcal{P})$. Our central result of this paper says that the disk algebra A has the local property (V). This improves previous results due to Delbaen [De] and Kisliakov [K]. From this fact we deduce that H^∞ also has local property (V), a result that extends Bourgain's theorem [B3, Thm. 1]. We also show that, for every ultrafilter $\mathcal{U}$, the dual of the ultrapower $(L^1/H_0^1)_{\mathcal{U}}$ has property (V). Bearing in mind that, for a Banach space X, one has

$$X^* \text{ has property (V)} \Rightarrow X \text{ has property (V}^*\text{)},$$

our result improves another result of Bourgain's which says that $(L^1/H_0^1)_{\mathcal{U}}$ has property (V^*). It is worth mentioning that there are examples that show the foregoing implication is not an equivalence [SS1]. Finally, we obtain that the odd duals of A and H^∞ are Grothendieck spaces, answering in the positive the question stated at the beginning.

The rest of the paper is divided into two sections. In the first we present an abstract condition about subalgebras of a certain $C(K)$ that allows us to deduce that these subalgebras have property weak (V), a weakening of the property (V) that was introduced by E. Saab and P. Saab [SS2]. In the second section, we obtain our results about ultrapowers and duals of A and H^∞. Some of our proofs use and refine certain ideas from Bourgain's paper [B3]. Our notation and terminology are standard: $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$ and $\partial\mathbb{D}$ the boundary of $\mathbb{D}$; $\boldsymbol{\lambda}$ is the normalized Lebesgue measure on $\partial\mathbb{D}$; and $X \oplus_p Y$ denotes the ℓ_p-sum of the Banach spaces X and Y $(1 \le p \le \infty)$. Our references for Banach space theory are [Di], [DS], and [La]. Pełczińsky's notes [Pe2] and [W] contain all the facts we shall need about Banach spaces of analytic functions.

2. Property Weak (V)

In their study of the unconditionally convergent operators defined over Banach spaces of continuous vector-valued functions, Saab and Saab [SS2] introduced property weak (V). A Banach space X has property weak (V) if every subset $W \subset X^*$ satisfying

$$\lim_n \sup_{x^* \in W} |\langle x^*, x_n\rangle| = 0,$$

for every weakly unconditionally Cauchy (wuC) series $\sum_n x_n$ in X is weakly conditionally compact in X^*. It is clear, using Rosenthal's theorem, that X has the property (V) if and only if X has property weak (V) and X^* is weakly sequentially complete. On the other hand, Saab and Saab proved that X has the property weak (V) if and only if, for every Banach space Y, each unconditionally convergent operator $T: X \to Y$ (i.e., T sends wuC series into unconditionally convergent series) has a weakly precompact adjoint $T^*: Y^* \to X^*$ (i.e., $T^*(B_{Y^*})$ is weakly conditionally compact). In our first result, we present some other characterizations of property weak (V) that we shall need later.

PROPOSITION 2.1. *The following assertions are equivalent.*

(1) *X has property weak* (V).
(2) *For every Banach space Y and every operator $T: X \to Y$, either T fixes a copy of c_0 or $T^*: Y^* \to X^*$ is weakly precompact.*
(3) *For each sequence (x_n^*) in X^* that is equivalent to the unit basis of ℓ_1, there exist $\varepsilon > 0$, a subsequence $(x_{n_k}^*) \subset (x_n^*)$, and a sequence (x_k) in X such that $\sum x_k$ is wuC and $\langle x_{n_k}^*, x_k\rangle \geq \varepsilon$ for all $k \in \mathbb{N}$.*
(4) *For each sequence (x_n^*) in X^* that is equivalent to the unit basis of ℓ_1, there exist $\varepsilon > 0$, a sequence (λ_n) in the scalar field $\mathbb{K}$, a sequence (x_n) in X, and a sequence (F_n) of pairwise disjoint finite subsets of $\mathbb{N}$ such that $\sum_{i\in F_n}|\lambda_i| \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$, $\sum x_n$ is wuC, and*

$$\left\langle \sum_{i\in F_n} \lambda_i x_i^*, x_n \right\rangle \geq \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Proof. (1) $\Leftrightarrow$ (2) This follows directly from [Di, Chap. V, Exer. 8].

(2) $\Rightarrow$ (3) If X has property weak (V) and (x_n^*) is a sequence in X^* equivalent to an ℓ_1-sequence, then we can define

$$T: X \to \ell_\infty, \qquad x \mapsto T(x) = (\langle x_n^*, x\rangle)_n.$$

Then $T^*: \ell_\infty^* \to X^*$ is a non-weakly conditionally compact operator, so T fixes a copy of c_0. That is, there exists a sequence (x_k) in X such that (x_k) and (Tx_k) are equivalent to the unit basis of c_0. Therefore, $\sum x_k$ is wuC and $\|Tx_k\| > \varepsilon$ for some $\varepsilon > 0$ and all $k \in \mathbb{N}$. Looking at the definition of T and using a standard argument, we may obtain a subsequence $(x_{n_k}^*)$ of (x_n^*) that satisfies $\sup_k \langle x_{n_k}^*, x_k\rangle \geq \varepsilon$.

It remains only to prove (4) $\Rightarrow$ (2) (since (3) $\Rightarrow$ (4) is obvious). Let $T: X \to Y$ be an operator such that T^* is non-weakly conditionally compact. Then the subset of X^* defined by

$$B = \{ T^*(y^*) : y^* \in Y^* \text{ and } \|y^*\| \leq 1\}$$

is non-weakly conditionally compact. Appealing to Rosenthal's theorem, take a sequence $(T^*(y_n^*))$ in B that is equivalent to the unit basis of ℓ_1. By assumption, we find an $\varepsilon > 0$, a constant $M > 0$, a sequence $(\lambda_n) \subset \mathbb{K}$, a sequence (F_n) of pairwise disjoint finite subsets of $\mathbb{N}$, and a sequence (x_n) in X such that $\sup_n \sum_{i\in F_n}|\lambda_i| \leq M$, $\sum x_n$ is wuC, and

$$\left\langle \sum_{i\in F_n} \lambda_i y_i^*, T(x_n) \right\rangle = \left\langle \sum_{i\in F_n} \lambda_i T^*(y_i^*), x_n \right\rangle \geq \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Therefore, we have two wuC series $\sum x_n$ and $\sum T(x_n)$ with $\inf_n \|T(x_n)\| > 0$ and $\inf_n \|x_n\| > 0$. Finally, by the Bessaga–Pełczyńki selection principle and [Di, Chap. V, Cor. 7], we may obtain an increasing sequence of natural numbers (n_k) such that (x_{n_k}) and $(T(x_{n_k}))$ are basic sequences that are equivalent to the unit basis of c_0. Hence, T fixes a copy of c_0 and we get (2). □

REMARKS. In Proposition 2.1 and whenever X is a dual space, we can change statement (2) as follows: For every Banach space Y and every operator $T: X \to Y$, either T fixes a copy of ℓ_∞ or $T^*: Y^* \to X^*$ is weakly precompact. To see this, note that the series $\sum \alpha_n x_n$ is weak-* convergent for every $(\alpha_n) \in \ell_\infty$, where (x_n) is the sequence obtained in (4) $\Rightarrow$ (2). This allows us to define an obvious operator from ℓ_∞ to X. Now, bearing in mind [R], we can deduce that T fixes a copy of ℓ_∞.

The following theorem gives a sufficient condition for a subalgebra of $C(K)$ to have the property weak (V). The proposition uses several ideas of the proof of Lemma 3 from [B3]. We fix the following notation: If X is a Banach algebra, $\psi \in X$, and $\Phi \in X^*$, then the functional $\psi\Phi \in X^*$ is defined by

$$\langle \psi\Phi, \varphi\rangle := \langle \Phi, \psi\varphi\rangle \quad \text{for all } \varphi \in X.$$

THEOREM 2.2. *Let K be a compact Hausdorff space, and let X be a (closed) subalgebra of $C(K)$ containing the unit $1_{C(K)}$ such that $X^* = Z \oplus_1 Y$. Assume there are elements 1_{Z^*} in the unit sphere of Z^* and 1_{Y^*} in the unit sphere of Y^*, with $1_{Z^*} + 1_{Y^*} = 1_{C(K)}$ (this equality must be viewed in the bidual of $C(K)$). Then X has property weak (V) if the following condition holds: There exist $\tau > 0$, $\varkappa > 0$, and a sequence $(\beta(n))_n$ with $\beta(n)/n \xrightarrow{n} 0$ such that, if $\Phi_1, \dots, \Phi_n$ are elements in the unit ball of Z (resp. Y) with*

$$\left\| \sum_{m=1}^n a_m \Phi_m \right\| \ge (1-\tau) \sum_{m=1}^n |a_m| \quad (a_1, \dots, a_n \in \mathbb{C}),$$

then there exist φ_m, ψ_m in Z^* (resp. in Y^*) $(m = 1, \dots, n)$ satisfying

(I) *$\|a\varphi_m + b\psi_m\| \le 1$ for all $m = 1, \dots, n$ and $a, b \in \mathbb{D}$;*
(II) *$\left\|\sum_{m=1}^n a_m(1_{Z^*} - \psi_m)\right\| \le \beta(n)$ (resp. $\left\|\sum_{m=1}^n a_m(1_{Y^*} - \psi_m)\right\| \le \beta(n)$) for all $a_1, \dots, a_n \in \mathbb{D}$; and*
(III) *$\langle \Phi_m, \varphi_m\rangle \ge \varkappa$ for all $m = 1, \dots, n$.*

Proof. We want to use Proposition 2.1(4), so let (Γ_n) be a sequence in X^* equivalent to the unit basis of ℓ_1. Without loss of generality, we may assume that $\|\Gamma_n\| \le 1$ $(n \in \mathbb{N})$. Now, take a sequence (Γ_n^Z) in Z and a sequence (Γ_n^Y) in Y such that $\Gamma_n = \Gamma_n^Z + \Gamma_n^Y$. Passing to a subsequence if necessary and bearing in mind Rosenthal's theorem, it is clear that we may (and do) assume that either (Γ_n^Z) or (Γ_n^Y) is equivalent to the unit basis of ℓ_1. Suppose, for instance, that (Γ_n^Z) is equivalent to an ℓ_1-sequence (in the other case, the proof follows until the end with a completely similar argument). Then $\|\Gamma_n^Z\| \le 1$ and there is a $\delta > 0$ such that

$$\delta \sum_{m=1}^n |a_m| \le \left\| \sum_{m=1}^n a_m \Gamma_m^Z \right\| \le \sum_{m=1}^n |a_m| \quad (a_1, \dots, a_n \in \mathbb{C},\ n \in \mathbb{N}).$$

Take a positive number $c_n > 0$ such that $c_n \|\Gamma_n^Z\| = 1$. Applying the James regularization principle for ℓ_1-sequences [J] (see also [Pfi, Thm. 2.1.4]), we can find a sequence (λ_n) in $\mathbb{K}$ and a sequence (F_n) of pairwise disjoint finite subsets of $\mathbb{N}$ such that

$$\sup_n \sum_{i\in F_n} |\lambda_i| \le \frac{1}{\delta}$$

and

$$(1-\tau)\sum_{m=1}^{n} |a_m| \le \Big\| \sum_{m=1}^{n} a_m \Phi_m^Z \Big\| \le \sum_{m=1}^{n} |a_m| \quad (a_1,\dots,a_n \in \mathbb{C},\ n\in\mathbb{N}),$$

where $\Phi_n^Z = \sum_{i\in F_n} \lambda_i c_i \Gamma_i^Z \in Z$. Now define, for each $n\in\mathbb{N}$,

$$\Phi_n^Y := \sum_{i\in F_n} \lambda_i c_i \Gamma_i^Y \in Y \quad \text{and} \quad \Phi_n := \Phi_n^Y + \Phi_n^Z = \sum_{i\in F_n} \lambda_i c_i \Gamma_i \in X^*.$$

Since $1 \le c_n \le \delta^{-1}$, we see that $\|\Phi_n\| \le 1/\delta^2$ $(n\in\mathbb{N})$. Moreover, by appealing to Hahn–Banach theorem, for each $n\in\mathbb{N}$ we can obtain a regular countably additive measure μ_n over K (i.e., an element of $M(K)$) such that μ_n is an extension of Φ_n to the whole of $C(K)$, with $\|\mu_n\| = \|\Phi_n\|$ and

$$\langle \Phi_n, \varphi\rangle = \int_\Omega \varphi\, d\mu_n \quad (\varphi \in X).$$

Take a sequence of positive numbers (ε_n) and a decreasing sequence (δ_n) of real numbers in the interval $[0,1]$ such that

$$\prod_{n=1}^{\infty}(1+\delta_n)\sum_{n=1}^{\infty}\varepsilon_n \le \frac{\varkappa}{4}.$$

Since $(\beta(n)/n)$ is a null sequence, we can obtain a strictly increasing sequence of positive integers $(N_n)_n$ such that $2\beta(N_n) < \varepsilon_n \delta^2 N_n$. Then we make the following construction.

Defining $D_0 := \mathbb{N}$ and fixing the first N_1 elements $\Phi_1^Z,\dots,\Phi_{N_1}^Z$, the assumption gives us $\varphi_m^Z, \psi_m^Z \in Z^* \subseteq X^{**}$ $(1\le m\le N_1)$ verifying

(I) $\|a\varphi_m^Z + b\psi_m^Z\| \le 1$ for all $m=1,\dots,N_1$ and $a,b\in\mathbb{D}$;
(II) $\big\|\sum_{m=1}^{N_1} a_m(1_{Z^*} - \psi_m^Z)\big\| \le \beta(N_1)$ for all $a_1,\dots,a_{N_1}\in\mathbb{D}$; and
(III) $\langle \Phi_m^Z, \varphi_m^Z\rangle \ge \varkappa$ for all $m=1,\dots,N_1$.

Set $\widehat{\psi_m} := \psi_m^Z + 1_{Y^*} \in X^{**}$ $(m=1,\dots,N_1)$. Since $X^{**} = Z^* \oplus_\infty Y^*$, we have

(I′) $\|a\varphi_m^Z + b\widehat{\psi_m}\| = \max\{\|a\varphi_m^Z + b\psi_m^Z\|, \|b1_{Y^*}\|\} \le 1$ for all $m=1,\dots,N_1$ and $a,b\in\mathbb{D}$;
(II′) $\big\|\sum_{m=1}^{N_1} a_m(1_{C(K)} - \widehat{\psi_m})\big\| = \big\|\sum_{m=1}^{N_1} a_m(1_{Z^*} - \psi_m^Z)\big\| \le \beta(N_1)$ for all $a_1,\dots,a_{N_1}\in\mathbb{D}$; and
(III′) $\langle \Phi_m, \varphi_m^Z\rangle = \langle \Phi_m^Z + \Phi_m^Y, \varphi_m^Z\rangle = \langle \Phi_m^Z, \varphi_m^Z\rangle \ge \varkappa$ for all $m=1,\dots,N_1$.

Applying the principle of local reflexivity, yields a continuous linear operator $T\colon H\to X$, where H is the subspace of X^{**} spanned by $\varphi_m^Z, \widehat{\psi_m}$ $(m=1,\dots,N_1)$ and $1_{C(K)}$, into X such that

$$\|T(f)\| \le (1+\delta_{N_1})\|f\| \quad \text{for all} \ \ f\in H,$$

$$\langle \Phi_m, T(f)\rangle = \langle f, \Phi_m\rangle \quad \text{for all} \ \ f\in H \ \text{ and } \ m=1,\dots,N_1,$$

$$T(1_{C(K)}) = 1_{C(K)}.$$

Define $\varphi_m := T(\varphi_m^Z) \in X$ and $\psi_m := T(\widehat{\psi_m}) \in X$ $(m = 1, \dots, N_1)$. Then we have:

(IV) $\|a\varphi_m + b\psi_m\| \le 1 + \delta_{N_1}$ for all $m = 1, \dots, N_1$ and $a, b \in \mathbb{D}$;

(V) $\left\|\sum_{m=1}^{N_1} a_m(1_{C(K)} - \psi_m)\right\| \le (1+\delta_{N_1})\beta(N_1) \le 2\beta(N_1)$ for all $a_1, \dots, a_{N_1} \in \mathbb{D}$; and

(VI) $\langle \Phi_m, \varphi_m\rangle \ge \varkappa$ for all $m = 1, \dots, N_1$.

Note that (V) implies $\sum_{m=1}^{N_1}|1 - \psi_m(t)| \le 2\beta(N_1)$ for all $t \in K$. Fix $\theta > 0$ and let $\mu \in M(K)$. Then there exist functions $h_1, \dots, h_{N_1} \in C(K)$ with $\|h_m\| \le 1$ such that

$$\|\mu - \psi_m\mu\| \le \int_\Omega |(1 - \psi_m)h_m|\, d|\mu| + \frac{\theta}{N_1} \quad (m = 1, \dots, N_1),$$

where $|\mu|$ denotes the total variation of μ. Therefore,

$$\begin{aligned}
\sum_{m=1}^{N_1}\|\mu - \psi_m\mu\| &\le \sum_{m=1}^{N_1}\left(\int_\Omega |1 - \psi_m||h_m|\, d|\mu| + \frac{\theta}{N_1}\right)\\
&\le \sum_{m=1}^{N_1}\int_\Omega |(1 - \psi_m)|\, d|\mu| + \theta\\
&\le 2\beta(N_1)\|\mu\| + \theta.
\end{aligned}$$

Since θ was arbitrary, we can assume that

$$\sum_{m=1}^{N_1}\|\mu - \psi_m\mu\| \le 2\beta(N_1)\|\mu\|.$$

In particular, given $p \in D_0$, we have

$$\begin{aligned}
\sum_{m=1}^{N_1}\|\Phi_p - \psi_m\Phi_p\| &\le \sum_{m=1}^{N_1}\|\mu_p - \psi_m\mu_p\|\\
&\le 2\beta(N_1)\|\mu_p\|\\
&= 2\beta(N_1)\|\Phi_p\|.
\end{aligned}$$

Therefore, there exists an $m(p) \in \{1, \dots, N_1\}$ such that

$$\|\Phi_p - \psi_{m(p)}\Phi_p\| \le \frac{2\beta(N_1)\|\Phi_p\|}{N_1} \le \frac{2\beta(N_1)}{\delta^2 N_1} \le \varepsilon_1.$$

This shows us that there is necessarily an $m_1 \in \{1, \dots, N_1\}$ for which

$$\|\Phi_p - \psi_{m_1}\Phi_p\| \le \varepsilon_1$$

holds for all p in an infinite subset D_1 of D_0, with $\min D_1 > N_1$. Hence,

$$\|a\varphi_{m_1} + b\psi_{m_1}\| \le 1 + \delta_{N_1} \le 1 + \delta_{m_1} \quad \text{for all } \ a, b \in \mathbb{D},$$

$$\langle \Phi_{m_1}, \varphi_{m_1}\rangle \ge \varkappa,$$

$$\|\Phi_p - \psi_{m_1}\Phi_p\| \le \varepsilon_1 \quad \text{for all } \ p \in D_1.$$

It is clear that the foregoing procedure can now be applied to D_1. Thus, starting again with the first N_2 elements of D_1, we can find $m_2 \in D_1$ $(m_2 > m_1)$, functions $\varphi_{m_2}, \psi_{m_2}$ in X, and an infinite subset D_2 of D_1 (with min $D_2 > N_2$) satisfying

$$\|a\varphi_{m_2} + b\psi_{m_2}\| \leq 1 + \delta_{m_2} \quad \text{for all } \ a, b \in \mathbb{D},$$

$$\langle \Phi_{m_2}, \varphi_{m_2} \rangle \geq \varkappa,$$

$$\|\Phi_p - \psi_{m_2}\Phi_p\| \leq \varepsilon_2 \quad \text{for all } \ p \in D_2.$$

Continuing in this way, we can construct a subsequence (Φ_{m_n}) of (Φ_n), two sequences (φ_{m_n}) and (ψ_{m_n}) in X, and a sequence (D_n) of infinite subsets of $\mathbb{N}$ (with $D_{n+1} \subset D_n$ and $m_{n+1} \in D_n$) such that, for each $n \in \mathbb{N}$, we have

(VII) $\|a\varphi_{m_n} + b\psi_{m_n}\| \leq 1 + \delta_{m_n}$ for all $a, b \in \mathbb{D}$,
(VIII) $\|\Phi_p - \psi_{m_n}\Phi_p\| \leq \varepsilon_n$ for all $p \in D_n$, and
(IX) $\langle \Phi_{m_n}, \varphi_{m_n} \rangle \geq \varkappa$.

Now consider the following elements of X:

$$\eta_1 := \varphi_{m_1},$$

$$\eta_n := \psi_{m_1} \cdots \psi_{m_{n-1}}\varphi_{m_n} \quad (n \geq 2).$$

We will show by induction that, given k different positive integers

$$p_1, \ldots, p_k \subset \{m_n : n \in \mathbb{N}\},$$

we have

$$\sum_{r=1}^{k} |\psi_{p_1} \cdots \psi_{p_{r-1}}\varphi_{p_r}| \leq \prod_{r=1}^{k}(1 + \delta_{p_r}).$$

For $k = 1$ and by (VII), we see that $|\varphi_{p_1}| \leq 1 + \delta_{p_1}$. Suppose that the statement is true for all the positive integers smaller than or equal to k. Then

$$\begin{aligned}\sum_{r=1}^{k+1} |\psi_{p_1} \cdots \psi_{p_{r-1}}\varphi_{p_r}| &= |\varphi_{p_1}| + |\psi_{p_1}\varphi_{p_2}| + \cdots + |\psi_{p_1} \cdots \psi_{p_k}\varphi_{p_{k+1}}| \\ &\leq |\varphi_{p_1}| + |\psi_{p_1}| \left(\sum_{r=2}^{k+1} |\psi_{p_2} \cdots \psi_{p_{r-1}}\varphi_{p_r}| \right) \\ &\leq |\varphi_{p_1}| + |\psi_{p_1}| \left(\prod_{r=2}^{k+1}(1 + \delta_{p_r}) \right) \\ &\leq \prod_{r=1}^{k+1}(1 + \delta_{p_r}).\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} |\eta_n| \leq \prod_{n=1}^{\infty}(1 + \delta_{m_n}) \leq \prod_{n=1}^{\infty}(1 + \delta_n) < \infty.$$

This means that the series $\sum \eta_n$ is wuC in $C(K)$ and, since X is a subspace of $C(K)$, it is indeed a wuC series in X.

On the other hand, if $n \geq 2$ then we have that $m_n \in D_{n-1}$ and

$$\begin{aligned}
&\|\Phi_{m_n} - (\psi_{m_1} \cdots \psi_{m_{n-1}})\Phi_{m_n}\| \\
&\quad \leq \|\Phi_{m_n} - \psi_{m_1}\Phi_{m_n}\| + \|\psi_{m_1}\Phi_{m_n} - (\psi_{m_1} \cdots \psi_{m_{n-1}})\Phi_{m_n}\| \\
&\quad \leq \varepsilon_1 + \|\psi_{m_1}\| \|\Phi_{m_n} - (\psi_{m_2} \cdots \psi_{m_{n-1}})\Phi_{m_n}\| \\
&\quad \leq \varepsilon_1 + (1+\delta_{m_1})\|\Phi_{m_n} - (\psi_{m_2} \cdots \psi_{m_{n-1}})\Phi_{m_n}\| \\
&\quad \vdots \\
&\quad \leq \varepsilon_1 + (1+\delta_{m_1})\varepsilon_2 + \cdots + \prod_{j=1}^{n-2}(1+\delta_{m_j})\varepsilon_{n-1} \\
&\quad \leq \prod_{n=1}^{\infty}(1+\delta_n) \sum_{j=1}^{n-1} \varepsilon_j \leq \frac{\varkappa}{4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\langle \Phi_{m_n}, \eta_n \rangle| &= |\langle (\psi_{m_1} \cdots \psi_{m_{n-1}})\Phi_{m_n}, \varphi_{m_n} \rangle| \\
&\geq |\langle \Phi_{m_n}, \varphi_{m_n} \rangle| - \frac{\varkappa}{4}\|\varphi_{m_n}\| \geq \varkappa - \frac{\varkappa}{4}2 = \frac{\varkappa}{2}.
\end{aligned}$$

The proof is completed by applying Proposition 2.1(4) to the sequences (Φ_{m_n}) and (η_n). $\square$

3. Property (V) and Ultrapowers of A and H^∞

In this section we shall prove that A has the local property (V) and obtain several consequences of this fact. Apart from Theorem 2.2, we also need some useful lemmas. It is important to point out that each ultrapower $(A)_{\mathcal{U}}$ is a Banach algebra. In fact, the following is known [H, Prop. 3.1]: If X is a Banach algebra and $\mathcal{U}$ is an ultrafilter, then the multiplication

$$(x_i)_{\mathcal{U}} \cdot (y_i)_{\mathcal{U}} := (x_i y_i)_{\mathcal{U}}, \quad (x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}} \in (X)_{\mathcal{U}},$$

is well-defined and induces a Banach algebra structure on the ultrapower $(X)_{\mathcal{U}}$. From this, we see that if X has a unit 1_X then $(X)_{\mathcal{U}}$ also has a unit—namely, $(1_X)_{\mathcal{U}}$. Therefore, if $1_{C(\partial\mathbb{D})}$ denotes the function that assigns the value 1 to every element of $\partial\mathbb{D}$, then $1_{C(\partial\mathbb{D})}$ is in A and H^∞ and $(A)_{\mathcal{U}}$ and $(H^\infty)_{\mathcal{U}}$ are Banach algebras with unit $(1_{C(\partial\mathbb{D})})_{\mathcal{U}}$.

The first lemma is due to Dor [Do]. Although he states it for $[0,1]$ with the Lebesgue measure, a detailed analysis of the proof shows that it works also for every finite measure space.

LEMMA 3.1. *Let (Ω, Σ, μ) be a probability space, let $0 < \theta \leq 1$, and let $f_1, \dots, f_n$ be functions in the unit ball of $L^1(\mu, \mathbb{C})$ such that*

$$\left\| \sum_{m=1}^{n} a_m f_m \right\| \geq \theta \sum_{m=1}^{n} |a_m| \quad (a_1, \dots, a_n \in \mathbb{C}).$$

Then there exist pairwise disjoint measurable subsets $A_1, \ldots, A_n \in \Sigma$ *satisfying*

$$\int_{A_m} |f_m|\, d\mu \geq \theta^2 \quad (m = 1, \ldots, n).$$

Using Dor's lemma, we can give a result for the space $L^1(\mu, \mathbb{C})$ that is similar to the one given by Bourgain for the space L^1/H_0^1.

LEMMA 3.2. *Let* (Ω, Σ, μ) *be a probability space, let* $0 < \tau < 1$, *and let* $f_1, \ldots, f_n$ *be functions in the unit ball of* $L^1(\mu, \mathbb{C})$ *such that*

$$\left\| \sum_{m=1}^{n} a_m f_m \right\| \geq (1 - \tau) \sum_{m=1}^{n} |a_m| \quad (a_1, \ldots, a_n \in \mathbb{C}).$$

Then there exist $\varphi_m \in L^\infty(\mu, \mathbb{C})$ *and* $\psi_m \in L^\infty(\mu, \mathbb{R})$ $(1 \leq m \leq n)$ *satisfying*

(I) $\|a\varphi_m + b\psi_m\| \leq 1$ *for all* $m = 1, \ldots, n$ *and* $a, b \in \mathbb{D}$;
(II) $\left\| \sum_{m=1}^{n} a_m(\boldsymbol{\chi}_\Omega - \psi_m) \right\| \leq 1$ *for all* $a_1, \ldots, a_n \in \mathbb{D}$; *and*
(III) $\langle f_m, \varphi_m \rangle \geq (1 - \tau)^2$ *for all* $m = 1, \ldots, n$.

Proof. Apply Lemma 3.1 to obtain pairwise disjoint measurable subsets $A_1, \ldots, A_n \in \Sigma$ satisfying

$$\int_{A_m} |f_m|\, d\mu \geq (1 - \tau)^2 \quad (m = 1, \ldots, n).$$

For each $m = 1, \ldots, n$, define,

$$r_m(\omega) := \begin{cases} \frac{|f_m(\omega)|}{f_m(\omega)} & \text{if } f_m(\omega) \neq 0, \\ 0 & \text{if } f_m(\omega) = 0. \end{cases}$$

It is clear that r_m belongs to the unit sphere of $L^\infty(\mu, \mathbb{C})$. On the other hand, consider the functions

$$\varphi_m := r_m \chi_{A_m} \in L^\infty(\mu, \mathbb{C}), \quad \psi_m := \chi_{\Omega \setminus A_m} \in L^\infty(\mu, \mathbb{R}) \quad (m = 1, \ldots, n).$$

It can be shown that these functions verify (I), (II), and (III):

$$\|a\varphi_m + b\psi_m\| = \operatorname{ess\,sup}\{\, |a r_m(\omega)\chi_{A_m}(\omega) + b - b\chi_{A_m}(\omega)| : \omega \in \Omega \,\} \leq 1;$$

$$\left\| \sum_{m=1}^{n} a_m(\chi_\Omega - \psi_m) \right\| = \operatorname{ess\,sup}\left\{ \left| \sum_{m=1}^{n} a_m \chi_{A_m}(\omega) \right| : \omega \in \Omega \right\} \leq 1;$$

$$\langle f_m, \varphi_m \rangle = \int_{A_m} r_m f_m\, d\mu = \int_{A_m} |f_m|\, d\mu \geq (1 - \tau)^2. \qquad \square$$

For the third lemma, we recall some facts about the dual of $C(\partial\mathbb{D})$. If V_{sing} denotes the Banach space of singular measures with respect to the normalized Lebesgue measure $\boldsymbol{\lambda}$ on $\partial\mathbb{D}$, then the following well-known decompositions [Pe2, p. 11] hold:

$$M(\partial\mathbb{D}) = C(\partial\mathbb{D})^* = L^1(\boldsymbol{\lambda}) \oplus_1 V_{\text{sing}}, \qquad A^* = L^1/H_0^1 \oplus_1 V_{\text{sing}}.$$

Moreover [La, pp. 44, 46], $M(\partial\mathbb{D})$ is the complexification of the Banach space $M(\partial\mathbb{D},\mathbb{R})$ of regular countably additive $\mathbb{R}$-valued measures over $\partial\mathbb{D}$. In the next proof, it will be useful to consider the element of $M(\partial\mathbb{D})^*$ given by

$$\mathbf{1}_{M(\partial\mathbb{D})^*}[h+\sigma] := \int_{\partial\mathbb{D}} h\,d\boldsymbol{\lambda} + \sigma(\partial\mathbb{D}) \quad (h\in L^1(\boldsymbol{\lambda}),\ \sigma\in V_{\text{sing}}),$$

and its restriction to V_{sing}, obviously given by

$$\mathbf{1}_{(V_{\text{sing}})^*}(\sigma) := \sigma(\partial\mathbb{D}) \quad (\sigma\in V_{\text{sing}}).$$

LEMMA 3.3. *Let* $0<\tau<1$ *and* $\sigma_1,\dots,\sigma_n$ *be elements in the unit ball of* V_{sing} *such that*

$$\left\|\sum_{m=1}^n a_m\sigma_m\right\| \ge (1-\tau)\sum_{m=1}^n |a_m| \quad (a_1,\dots,a_n\in\mathbb{C}).$$

Then there exist $\varphi_m,\psi_m\in(V_{\text{sing}})^*$ $(1\le m\le n)$ *satisfying*

(1) $\|a\varphi_m+b\psi_m\|\le 1$ *for all* $m=1,\dots,n$ *and* $a,b\in\mathbb{D}$;
(2) $\left\|\sum_{m=1}^n a_m(\mathbf{1}_{(V_{\text{sing}})^*}-\psi_m)\right\|\le 1$ *for all* $a_1,\dots,a_n\in\mathbb{D}$; *and*
(3) $\langle\sigma_m,\varphi_m\rangle\ge(1-\tau)^2$ *for all* $m=1,\dots,n$.

Proof. Since $M(\partial\mathbb{D},\mathbb{R})$ is an abstract L-space [LT, pp. 111, 113], by Kakutani's theorem [La, p. 135] we have that $M(\partial\mathbb{D},\mathbb{R})$ is linearly isometric and lattice isomorphic to the ℓ_1-sum

$$L_{\mathbb{R}} := \left(\bigoplus_{i\in I} L^1(\mu_i,\mathbb{R})\right)_1,$$

where $(\Omega_i,\Sigma_i,\mu_i)$ is a certain finite measure space for every $i\in I$. Without loss of generality, we may assume that $\Omega_i\cap\Omega_l=\emptyset$ $(i\ne l)$. Denote by T the corresponding isometric lattice isomorphism from $M(\partial\mathbb{D},\mathbb{R})$ to $L_{\mathbb{R}}$. Now, define

$$\tilde T\colon M(\partial\mathbb{D})\to L_{\mathbb{C}} := \left(\bigoplus_{i\in I} L^1(\mu_i,\mathbb{C})\right)_1,$$

$$\mu=\mu_1+i\mu_2\mapsto \tilde T(\mu)=T(\mu_1)+iT(\mu_2).$$

According to [La, Chap. 5, Exer. 10], we know that $\tilde T$ is a linear isometry. On the other hand, denote by S the inclusion of V_{sing} into $M(\partial\mathbb{D})$. Then, we can get an infinite countable subset J of I such that $\tilde TS(\sigma_m)$ vanishes outside J for all $m=1,\dots,n$. Hence, let R be the canonical projection from $L_{\mathbb{C}}$ to

$$\left(\bigoplus_{j\in J} L^1(\mu_j,\mathbb{C})\right)_1.$$

Take a sequence $(\varepsilon_j)_{j\in J}$ of positive numbers such that

$$\sum_{j\in J}\varepsilon_j\mu_j(\Omega_j)=1.$$

Because the sets Ω_j are pairwise disjoint, if we put

$$\Omega := \bigcup_{j\in J} \Omega_j,$$

$$\Sigma := \{ A \subset \Omega : A \cap \Omega_j \in \Sigma_j,\ j \in J \},$$

$$\mu(A) := \sum_{j\in J} \varepsilon_j \mu_j(A \cap \Omega_j) \quad (A \in \Sigma),$$

then we see that (Ω, Σ, μ) is a probability space and that

$$U : \Big(\bigoplus_{j\in J} L^1(\mu_j, \mathbb{C})\Big)_1 \to L^1(\mu, \mathbb{C}), \qquad (f_j)_{j\in J} \mapsto U[(f_j)_{j\in J}],$$

given by

$$U[(f_j)_{j\in J}](\omega) := \frac{1}{\varepsilon_j} f_j(\omega) \quad \text{whenever } \omega \in \Omega_j,$$

is a well-defined linear isometry. Now consider

$$\widehat{\sigma_m} := UR\tilde{T}S(\sigma_m) \in L^1(\mu, \mathbb{C}) \quad (m = 1, \ldots, n).$$

By hypothesis, $\|\sigma_m\| \le 1$ $(1 \le m \le n)$ and so

$$\Big\| \sum_{m=1}^n a_m \sigma_m \Big\| \ge (1-\tau) \sum_{m=1}^n |a_m| \quad (a_1, \ldots, a_n \in \mathbb{C});$$

we also have that $\|\widehat{\sigma_m}\| \le 1$ $(1 \le m \le n)$ and

$$\Big\| \sum_{m=1}^n a_m \widehat{\sigma_m} \Big\| \ge (1-\tau) \sum_{m=1}^n |a_m| \quad (a_1, \ldots, a_n \in \mathbb{C}).$$

Therefore, applying Lemma 3.2, we find $\widehat{\varphi_m} \in L^\infty(\mu, \mathbb{C})$ and $\widehat{\psi_m} \in L^\infty(\mu, \mathbb{R})$ $(1 \le m \le n)$ satisfying

(I) $\|a\widehat{\varphi_m} + b\widehat{\psi_m}\| \le 1$ whenever $m = 1, \ldots, n$ and $a, b \in \mathbb{D}$;
(II) $\big\|\sum_{m=1}^n a_m(\boldsymbol{\chi}_\Omega - \widehat{\psi_m})\big\| \le 1$ whenever $a_1, \ldots, a_n \in \mathbb{D}$; and
(III) $\langle \widehat{\sigma_m}, \widehat{\varphi_m} \rangle \ge (1-\tau)^2$ for all $m = 1, \ldots, n$.

Finally, the functions we are looking for are defined by

$$\varphi_m := S^*(\tilde{T})^* R^* U^*(\widehat{\varphi_m}) \in (V_{\text{sing}})^* \quad (m = 1, \ldots, n),$$

$$\psi_m := S^*(\tilde{T})^*[R^* U^*(\widehat{\psi_m}) + \Gamma(I \setminus J)] \in (V_{\text{sing}})^* \quad (m = 1, \ldots, n),$$

where $\Gamma(I \setminus J) \in \big(\bigoplus_{i\in I} L^\infty(\mu_i, \mathbb{C})\big)_\infty$ with

$$\Gamma(I \setminus J) = (\Gamma_i) \equiv \begin{cases} \Gamma_i = 0 & \text{for } i \in J, \\ \Gamma_i = \chi_{\Omega_i} & \text{for } i \notin J. \end{cases}$$

We now show that the functions φ_m, ψ_m satisfy conditions (1), (2), and (3). Statement (3) is trivial since $\langle \widehat{\sigma_m}, \widehat{\varphi_m} \rangle = \langle \sigma_m, \varphi_m \rangle$ for all $m = 1, \ldots, n$. On the other hand, given $a, b \in \mathbb{D}$, we have

$$\begin{aligned} \|a\varphi_m + b\psi_m\| &\le \|R^* U^*[a\widehat{\varphi_m} + b\widehat{\psi_m}] + b\Gamma(I \setminus J)\| \\ &\le \max\{\|a\widehat{\varphi_m} + b\widehat{\psi_m}\|, |b|\|\Gamma(I \setminus J)\|\} \le 1. \end{aligned}$$

Therefore, condition (1) also holds. For the second condition, we claim that

$$\mathbf{1}_{(V_{\mathrm{sing}})^*} := S^*(\tilde{T})^*[R^*U^*(\chi_\Omega) + \Gamma(I \setminus J)].$$

Hence, given $a_1, \ldots, a_n \in \mathbb{D}$,

$$\begin{aligned}\left\| \sum_{m=1}^{n} a_m(\mathbf{1}_{(V_{\mathrm{sing}})^*} - \psi_m) \right\| &\leq \left\| \sum_{m=1}^{n} a_m[R^*U^*(\boldsymbol{\chi}_\Omega) - R^*U^*(\widehat{\psi_m})] \right\| \\ &\leq \left\| \sum_{m=1}^{n} a_m(\boldsymbol{\chi}_\Omega - \widehat{\psi_m}) \right\| \leq 1.\end{aligned}$$

Proof of the Claim. It is easy to check that

$$f \in L^\infty(\mu, \mathbb{C}) \mapsto U^*(f) = (f|_{\Omega_j})_{j\in J} \in \left(\bigoplus_{j\in J} L^\infty(\mu_j, \mathbb{C}) \right)_\infty .$$

Therefore, $U^*(\chi_\Omega) = (\chi_{\Omega_j})_{j\in J}$. On the other hand, R^* is the canonical injection of $\left(\bigoplus_{j\in J} L^\infty(\mu_j, \mathbb{C})\right)_\infty$ into $\left(\bigoplus_{j\in J} L^\infty(\mu_i, \mathbb{C})\right)_\infty$. Thus,

$$R^*U^*(\chi_\Omega) + \Gamma(I \setminus J) = (\chi_{\Omega_i})_{i\in I}.$$

Bearing in mind that every $f \in L^*_{\mathbb{C}} = \left(\bigoplus_{i\in I} L^\infty(\mu_i, \mathbb{C})\right)_\infty$ can be written as $f = f_1 + if_2$ with $f_1, f_2 \in L^*_{\mathbb{R}} = \left(\bigoplus_{i\in I} L^\infty(\mu_i, \mathbb{R})\right)_\infty$, it is clear that

$$f = f_1 + if_2 \in L^*_{\mathbb{C}} \mapsto (\tilde{T})^*(f) = T^*(f_1) + iT^*(f_2) \in M(\partial\mathbb{D})^*.$$

Note that $(\chi_{\Omega_i})_{i\in I} \in L^*_{\mathbb{R}}$. Since $M(\partial\mathbb{D})^*$ is the complexification of $M(\partial\mathbb{D}, \mathbb{R})^*$ [M, p. 71], we deduce that $(\tilde{T})^*[(\chi_{\Omega_i})_{i\in I}] = T^*[(\chi_{\Omega_i})_{i\in I}]$ in the following sense:

$$(\tilde{T})^*[(\chi_{\Omega_i})_{i\in I}](\mu_1 + i\mu_2) = T^*[(\chi_{\Omega_i})_{i\in I}](\mu_1) + iT^*[(\chi_{\Omega_i})_{i\in I}](\mu_2),$$

where $\mu_1, \mu_2 \in M(\partial\mathbb{D}, \mathbb{R})$.

At this point, we need to recall a concept from the theory of Banach lattices [La, p. 13]: A positive element e of a real Banach lattice X is said to be a strong unit if

$$e \geq |x| \iff \|x\| \leq 1 \quad (x \in X).$$

Of course, a strong unit is unique whenever it exists. It is obvious that $(\chi_{\Omega_i})_{i\in I}$ is the strong unit of $\left(\bigoplus_{i\in I} L^\infty(\mu_i, \mathbb{R})\right)_\infty$. We will show that the positive element of $M(\partial\mathbb{D}, \mathbb{R})^*$ given by

$$\mathbf{1}_{M(\partial\mathbb{D},\mathbb{R})^*}(\mu) := \mu(\partial\mathbb{D}) \quad (\mu \in M(\partial\mathbb{D}, \mathbb{R}))$$

is the strong unit of $M(\partial\mathbb{D},\mathbb{R})^*$. On the one hand, if $|\varphi| \leq \mathbf{1}_{M(\partial\mathbb{D},\mathbb{R})^*}$ ($\varphi \in M(\partial\mathbb{D}, \mathbb{R})^*$), then

$$\|\varphi\| = \||\varphi|\| \leq \|\mathbf{1}_{M(\partial\mathbb{D},\mathbb{R})^*}\| \leq 1.$$

On the other hand, if $\|\varphi\| \leq 1$ ($\varphi \in M(\partial\mathbb{D}, \mathbb{R})^*$) and $\mu \in M(\partial\mathbb{D}, \mathbb{R})$ ($\mu \geq 0$), then

$$\begin{aligned}(\mathbf{1}_{M(\partial\mathbb{D},\mathbb{R})^*} - |\varphi|)(\mu) &= \mu(\partial\mathbb{D}) - |\varphi|(\mu) \geq \mu(\partial\mathbb{D}) - \||\varphi|\|\|\mu\| \\ &= \mu(\partial\mathbb{D}) - \|\varphi\|\|\mu\| \geq \mu(\partial\mathbb{D}) - \|\mu\| = 0.\end{aligned}$$

Therefore, $\mathbf{1}_{M(\partial\mathbb{D},\mathbb{R})^*} \geq |\varphi|$. Bearing in mind that every isometric lattice isomorphism maps the strong unit into the strong unit, we have that

$$T^*[(\chi_{\Omega_i})_{i\in I}] = \mathbf{1}_{M(\partial\mathbb{D},\mathbb{R})^*}.$$

Therefore, given $\mu = \mu_1 + i\mu_2 \in M(\partial\mathbb{D})$ and $\mu_1, \mu_2 \in M(\partial\mathbb{D},\mathbb{R})$,

$$(\tilde{T})^*[(\chi_{\Omega_i})_{i\in I}](\mu) = \mathbf{1}_{M(\partial\mathbb{D},\mathbb{R})^*}(\mu_1) + i\mathbf{1}_{M(\partial\mathbb{D},\mathbb{R})^*}(\mu_2) = \mathbf{1}_{M(\partial\mathbb{D})^*}(\mu).$$

Hence, $(\tilde{T})^*[(\chi_{\Omega_i})_{i\in I}] = \mathbf{1}_{M(\partial\mathbb{D})^*}$ and the proof of the claim will be complete if we show that $S^*(\mathbf{1}_{M(\partial\mathbb{D})^*}) = \mathbf{1}_{(V_{\mathrm{sing}})^*}$. But this is trivial, since the adjoint of an inclusion is always the corresponding restriction. $\square$

In the following lemma and for each ultrapower $(V_{\mathrm{sing}})_{\mathcal{U}}$ of V_{sing}, we use the notation $\mathbf{1}_{[(V_{\mathrm{sing}})_{\mathcal{U}}]^*}$ for the element of $[(V_{\mathrm{sing}})_{\mathcal{U}}]^*$ defined by

$$\langle \mathbf{1}_{[(V_{\mathrm{sing}})_{\mathcal{U}}]^*}, (\sigma_i)_{\mathcal{U}}\rangle := \lim_{i\to\mathcal{U}} \langle \mathbf{1}_{(V_{\mathrm{sing}})^*}, \sigma_i\rangle = \lim_{i\to\mathcal{U}} \sigma_i(\partial\mathbb{D}),$$

where $(\sigma_i)_{\mathcal{U}} \in (V_{\mathrm{sing}})_{\mathcal{U}}$.

LEMMA 3.4. *Let $0 < \tau < 1$ and $\Phi_1, \ldots, \Phi_n$ be elements in the unit ball of $(V_{\mathrm{sing}})_{\mathcal{U}}$ such that*

$$\left\| \sum_{m=1}^{n} a_m \Phi_m \right\| \geq \left(1 - \frac{1}{2}\tau\right) \sum_{m=1}^{n} |a_m| \quad (a_1, \ldots, a_n \in \mathbb{C}).$$

Then there exist $\varphi_m, \psi_m \in [(V_{\mathrm{sing}})_{\mathcal{U}}]^*$ $(1 \leq m \leq n)$ satisfying

(I) $\|a\varphi_m + b\psi_m\| \leq 1$ *whenever* $m = 1, \ldots, n$ *and* $a, b \in \mathbb{D}$;
(II) $\left\|\sum_{m=1}^n a_m\big(\mathbf{1}_{[(V_{\mathrm{sing}})_{\mathcal{U}}]^*} - \psi_m\big)\right\| \leq 1$ *whenever* $a_1, \ldots, a_n \in \mathbb{D}$; *and*
(III) $\langle \Phi_m, \varphi_m\rangle \geq (1-\tau)^2$ *for all* $m = 1, \ldots, n$.

Proof. We may (and do) assume that

$$\|\Phi_m(i)\| = \|\Phi_m\| \leq 1 \quad (m = 1, \ldots, n,\ i \in I).$$

Take $\theta = (1 - \frac{\tau}{2})/(1-\tau) > 1$ and let $\{b_1, \ldots, b_r\}$ be a δ-net in the unit sphere of ℓ_1^n, where $0 < \delta < (1-\tau)\frac{1}{\theta}$. For each $l = 1, \ldots, r$, take an element U_l in the ultrafilter $\mathcal{U}$ such that, for $i \in U_l$,

$$\left\| \sum_{m=1}^{n} b_l(m)\Phi_m(i) \right\| \geq \frac{1}{2\theta}\left(1 - \frac{\tau}{2}\right), \quad b_l = (b_l(1), \ldots, b_l(n)).$$

Define $U = U_1 \cap \cdots \cap U_r \in \mathcal{U}$. Then, given $(a_1, \ldots, a_n) \in \mathbb{C}^n$ with $\sum_{m=1}^n |a_m| = 1$, we can find $l' \in \{1, \ldots, r\}$ such that

$$\sum_{m=1}^{n} |b_{l'}(m) - a_m| \leq \delta.$$

Hence, for each $i \in U$,

$$\left\| \sum_{m=1}^{n} a_m \Phi_m(i) \right\| \geq \frac{1}{2\theta}\left(1 - \frac{\tau}{2}\right) - \sum_{m=1}^{n} |b_{l'}(m) - a_m| \|\Phi_m(i)\| \\ \geq \frac{1}{\theta}\left(1 - \frac{\tau}{2}\right).$$

We thus have that

$$\left\| \sum_{m=1}^{n} a_m \Phi_m(i) \right\| \geq \frac{1}{\theta}\left(1 - \frac{\tau}{2}\right) \sum_{m=1}^{n} |a_m| = (1-\tau) \sum_{m=1}^{n} |a_m|$$

whenever $a_1, \dots, a_n \in \mathbb{C}$ and $i \in U$. Applying Lemma 3.3, we obtain

$$\varphi_m(i), \psi_m(i) \in (V_{\text{sing}})^* \quad (1 \leq m \leq n,\ i \in U)$$

satisfying

(1) $\|a\varphi_m(i) + b\psi_m(i)\| \leq 1$ whenever $m = 1, \dots, n$, $i \in U$, and $a, b \in \mathbb{D}$;
(2) $\left\| \sum_{m=1}^{n} a_m (\mathbf{1}_{(V_{\text{sing}})^*} - \psi_m(i)) \right\| \leq 1$ whenever $a_1, \dots, a_n \in \mathbb{D}$ and $i \in U$; and
(3) $\langle \Phi_m(i), \varphi_m(i) \rangle \geq (1-\tau)^2$ for all $m = 1, \dots, n$ and $i \in U$.

Consider the elements φ_m, ψ_m of $[(V_{\text{sing}})_{\mathcal{U}}]^*$ given by

$$\langle \varphi_m, \xi \rangle := \lim_{i \to \mathcal{U}} \langle \xi_i, \varphi_m(i) \rangle \quad \text{and} \quad \langle \psi_m, \xi \rangle := \lim_{i \to \mathcal{U}} \langle \xi_i, \psi_m(i) \rangle$$

for each $\xi = (\xi_i)_{\mathcal{U}} \in (V_{\text{sing}})_{\mathcal{U}}$, where $\varphi_m(i) = \psi_m(i) = 0$ for $i \notin U$. Finally, taking limits in the ultrafilter $\mathcal{U}$, it is clear that φ_m and ψ_m satisfy conditions (I), (II), and (III). □

The next result is due to Bourgain [B3, Lemma 2]. In a manner similar to the foregoing, for each ultrapower $(L^1/H_0^1)_{\mathcal{U}}$ of L^1/H_0^1, the element of $[(L^1/H_0^1)_{\mathcal{U}}]^*$ denoted by $\mathbf{1}_{[(L^1/H_0^1)_{\mathcal{U}}]^*}$ is defined by

$$\left\langle \mathbf{1}_{[(L^1/H_0^1)_{\mathcal{U}}]^*}, (f_i)_{\mathcal{U}} \right\rangle := \lim_{i \to \mathcal{U}} \int_{\partial \mathbb{D}} f_i \, d\boldsymbol{\lambda} \quad ((f_i)_{\mathcal{U}} \in (L^1/H_0^1)_{\mathcal{U}}).$$

LEMMA 3.5. *Given an ultrapower* $(L^1/H_0^1)_{\mathcal{U}}$, *there exist* $\tau_{\mathcal{U}} > 0$, $\varkappa_{\mathcal{U}} > 0$, *and a sequence* $(\beta_{\mathcal{U}}(n))_n$ *with* $\beta_{\mathcal{U}}(n)/n \xrightarrow{n} 0$ *such that, whenever* $\Phi_1, \dots, \Phi_n$ *are elements in the unit ball of* $(L^1/H_0^1)_{\mathcal{U}}$ *with*

$$\left\| \sum_{m=1}^{n} a_m \Phi_m \right\| \geq (1 - \tau_{\mathcal{U}}) \sum_{m=1}^{n} |a_m| \quad (a_1, \dots, a_n \in \mathbb{C}),$$

there exist $\varphi_m, \psi_m \in [(L^1/H_0^1)_{\mathcal{U}}]^*$ $(1 \leq m \leq n)$ *satisfying*

(1) $\|a\varphi_m + b\psi_m\| \leq 1$ *whenever* $m = 1, \dots, n$ *and* $a, b \in \mathbb{D}$;
(2) $\left\| \sum_{m=1}^{n} a_m \left(\mathbf{1}_{[(L^1/H_0^1)_{\mathcal{U}}]^*} - \psi_m\right) \right\| \leq \beta_{\mathcal{U}}(n)$ *whenever* $a_1, \dots, a_n \in \mathbb{D}$; *and*
(3) $\langle \Phi_m, \varphi_m \rangle \geq \varkappa_{\mathcal{U}}$ *for all* $m = 1, \dots, n$.

We are now ready to prove the main result of this paper, which improves the result of Delbaen [De, theorem of p. 292] and Kisliakov [Ki, Thm. 1].

THEOREM 3.6. *The disk algebra A has local property (V); that is, every ultrapower of A has the property (V).*

Proof. Let $(A)_{\mathcal{U}}$ be an arbitrary ultrapower of A. First of all, we shall embed $[(A)_{\mathcal{U}}]^*$ into some ultrapower of A^* following [H, Cor. 7.6]. For the proof of our theorem, we need to know in detail the definition of that ultrapower, so let us start by describing it. Define the index set J to be the collection of all (M, N, ε), with M a finite-dimensional subspace of $[(A)_{\mathcal{U}}]^*$, N a finite-dimensional subspace of $(A)_{\mathcal{U}}$, and $\varepsilon > 0$. Let $\mathcal{V}$ be an ultrafilter dominating a certain order filter defined over J. Then, for each $(M_j, N_j, \varepsilon_j) \in J$, there is a mapping $T_j : M_j \to (A^*)_{\mathcal{U}}$, an $(1+\varepsilon_j)$-isomorphism onto its image, such that

$$\langle T_j(\Phi), (f_i^*)_{\mathcal{U}} \rangle = \langle \Phi, (f_i^*)_{\mathcal{U}} \rangle \quad (\Phi \in M_j,\ (f_i^*)_{\mathcal{U}} \in N_j)$$

and

$$T_j(\Pi((f_i^*)_{\mathcal{U}})) = (f_i^*)_{\mathcal{U}} \quad ((f_i^*)_{\mathcal{U}} \in (A^*)_{\mathcal{U}} \cap M_j),$$

where Π is the canonical embedding of $(A^*)_{\mathcal{U}}$ into $[(A)_{\mathcal{U}}]^*$. Moreover, the map

$$T : [(A)_{\mathcal{U}}]^* \to ((A^*)_{\mathcal{U}})_{\mathcal{V}}, \qquad \Phi \mapsto T(\Phi) = (F_j)_{\mathcal{V}},$$

given by

$$F_j = \begin{cases} T_j(\Phi) & \text{if } \Phi \in M_j, \\ 0 & \text{otherwise,} \end{cases}$$

is a linear isometry, and the map

$$Q : ((A^*)_{\mathcal{U}})_{\mathcal{V}} \to [(A)_{\mathcal{U}}]^*,$$

$$Q[((f_{i,j}^*)_{\mathcal{U}})_{\mathcal{V}}]((g_i)_{\mathcal{U}}) := \lim_{j\to\mathcal{V}} \left(\lim_{i\to\mathcal{U}} \langle f_{i,j}^*, g_i \rangle \right) \quad ((g_i)_{\mathcal{U}} \in (A)_{\mathcal{U}})$$

is surjective and satisfies that QT is the identity in $[(A)_{\mathcal{U}}]^*$. In particular [H, Cor. 7.6], $[(A)_{\mathcal{U}}]^*$ is isometrically isomorphic to a norm-1 complemented subspace of the ultrapower $(A^*)_{\mathcal{U}\times\mathcal{V}}$.

According to [SS2, p. 529], we must show that $(A)_{\mathcal{U}}$ has the property weak (V) and that $[(A)_{\mathcal{U}}]^*$ is weakly sequentially complete. It is clear that we can check this last property in $(A^*)_{\mathcal{U}\times\mathcal{V}}$. Obviously,

$$(A^*)_{\mathcal{U}\times\mathcal{V}} = (L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}} \oplus_1 (V_{\text{sing}})_{\mathcal{U}\times\mathcal{V}}.$$

Now, $(L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}}$ is weakly sequentially complete by [B4, Thm. 5.3]. On the other hand, since V_{sing} is a norm-1 complemented subspace of $M(\partial\mathbb{D})$ and this last space is isometric to $L^1(\mu', \mathbb{C})$ (see the proof of Lemma 3.3), we get that V_{sing} is isometric to some $L^1(\mu, \mathbb{C})$ [La, Sec. 17, Thm. 3]. Applying [H, Thm. 3.3], we deduce that $(V_{\text{sing}})_{\mathcal{U}\times\mathcal{V}}$ is also isometric to some other $L^1(\mu'', \mathbb{C})$ and therefore $(V_{\text{sing}})_{\mathcal{U}\times\mathcal{V}}$ is weakly sequentially complete.

Hence the proof will be finished if we can show that $(A)_{\mathcal{U}}$ has property weak (V). Of course, we want to apply Theorem 2.2. By [H, Thm. 3.3], $(C(\partial\mathbb{D}))_{\mathcal{U}}$ is isometric to $C(K)$ for some compact Hausdorff space K with an isometry preserving the algebraic structure and, in particular, the unit. Therefore, we can identify

$(A)_{\mathcal{U}}$ with a certain subalgebra of $C(K)$ containing the corresponding unit $1_{C(K)}$, which we can identify with $(1_{C(\partial\mathbb{D})})_{\mathcal{U}}$. Bearing in mind that

$$((A^*)_{\mathcal{U}})_{\mathcal{V}} = ((L^1/H_0^1)_{\mathcal{U}})_{\mathcal{V}} \oplus_1 ((V_{\text{sing}})_{\mathcal{U}})_{\mathcal{V}},$$

define

$$Z := \{ \Phi \in [(A)_{\mathcal{U}}]^* : T(\Phi) \in ((L^1/H_0^1)_{\mathcal{U}})_{\mathcal{V}} \},$$

$$Y := \{ \Phi \in [(A)_{\mathcal{U}}]^* : T(\Phi) \in ((V_{\text{sing}})_{\mathcal{U}})_{\mathcal{V}} \}.$$

Since T is an isometry, we deduce that

$$[(A)_{\mathcal{U}}]^* = Z \oplus_1 Y.$$

Denote by S_1 (resp. S_2) the canonical isometry from the iterated ultrapowers $((L^1/H_0^1)_{\mathcal{U}})_{\mathcal{V}}$ (resp. $((V_{\text{sing}})_{\mathcal{U}})_{\mathcal{V}}$) into $[(L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}}]$ (resp. $[(V_{\text{sing}})_{\mathcal{U}\times\mathcal{V}}]$). Now consider the elements of $[(A)_{\mathcal{U}}]^*$ given by

$$1_{Z^*} := T^* S_1^*\big(\mathbf{1}_{[(L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}}]^*}\big) \in Z^*,$$

$$1_{Y^*} := T^* S_2^*\big(\mathbf{1}_{[(V_{\text{sing}})_{\mathcal{U}\times\mathcal{V}}]^*}\big) \in Y^*.$$

It is clear that $\|1_{Y^*}\| = \|1_{Z^*}\| = 1$. To use Theorem 2.2, we must also verify the equality

$$1_{Z^*} + 1_{Y^*} = (1_{C(\partial\mathbb{D})})_{\mathcal{U}}.$$

Toward this end, take $\Phi \in ((A)_{\mathcal{U}})^*$ with $\Phi = \Phi^Z + \Phi^Y$, where $\Phi^Z \in Z$ and $\Phi^Y \in Y$. On the one hand,

$$\begin{aligned}
\langle 1_{Z^*} + 1_{Y^*}, \Phi\rangle &= \langle 1_{Z^*}, \Phi^Z\rangle + \langle 1_{Y^*}, \Phi^Y\rangle \\
&= \Big\langle S_1^*\big(\mathbf{1}_{[(L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}}]^*}\big), T(\Phi^Z)\Big\rangle + \Big\langle S_2^*\big[\mathbf{1}_{[(V_{\text{sing}})_{\mathcal{U}\times\mathcal{V}}]^*}\big], T(\Phi^Y)\Big\rangle \\
&= \lim_{j\to\mathcal{V}} \Big\langle \mathbf{1}_{[(L^1/H_0^1)_{\mathcal{U}}]^*}, T_j(\Phi^Z)\Big\rangle + \lim_{j\to\mathcal{V}} \Big\langle \mathbf{1}_{[(V_{\text{sing}})_{\mathcal{U}}]^*}, T_j(\Phi^Y)\Big\rangle \\
&= \lim_{j\to\mathcal{V}} \lim_{i\to\mathcal{U}} \int_{\partial\mathbb{D}} (T_j(\Phi^Z))_i \, d\boldsymbol{\lambda} + \lim_{j\to\mathcal{V}} \lim_{i\to\mathcal{U}} (T_j(\Phi^Y))_i(\partial\mathbb{D}) \\
&= \lim_{j\to\mathcal{V}} \lim_{i\to\mathcal{U}} \bigg[\int_{\partial\mathbb{D}} (T_j(\Phi^Z))_i \, d\boldsymbol{\lambda} + (T_j(\Phi^Y))_i(\partial\mathbb{D}) \bigg].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle (1_{C(\partial\mathbb{D})})_{\mathcal{U}}, \Phi\rangle &= \langle (1_{C(\partial\mathbb{D})})_{\mathcal{U}}, Q(T(\Phi))\rangle \\
&= \lim_{j\to\mathcal{V}} \lim_{i\to\mathcal{U}} \langle (T_j(\Phi^Z))_i + (T_j(\Phi^Y))_i, 1_{C(\partial\mathbb{D})}\rangle \\
&= \lim_{j\to\mathcal{V}} \lim_{i\to\mathcal{U}} \bigg[\int_{\partial\mathbb{D}} (T_j(\Phi^Z))_i \, d\boldsymbol{\lambda} + (T_j(\Phi^Y))_i(\partial\mathbb{D}) \bigg].
\end{aligned}$$

To apply Theorem 2.2, consider

(i) τ the minimum of $\frac{1}{4}$ and the positive number $\tau_{\mathcal{U}\times\mathcal{V}}$ given by Lemma 3.5 for the ultrapower $(L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}}$,

(ii) $\varkappa$ the minimum of $(1-2\tau)^2$ and the number $\varkappa_{\mathcal{U}\times\mathcal{V}}$ given by Lemma 3.5 for the ultrapower $(L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}}$,

(iii) $\beta(n)$ the maximum of 1 and $\beta_{\mathcal{U}\times\mathcal{V}}(n)$ given by Lemma 3.5 for the ultrapower $(L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}}$ $(n \in \mathbb{N})$.

We shall deal with Z, but the Y-case is identical using Lemma 3.4 instead of Lemma 3.5 and S_2 instead of S_1.

Take $\Phi_1, \dots, \Phi_n$ elements in the unit ball of Z, with

$$\left\| \sum_{m=1}^{n} a_m \Phi_m \right\| \geq (1-\tau) \sum_{m=1}^{n} |a_m| \quad (a_1, \dots, a_n \in \mathbb{C}).$$

It is clear that the elements $S_1 T(\Phi_m)$ $(m = 1, \dots, n)$ belong to the unit ball of $(L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}}$ and that

$$\left\| \sum_{m=1}^{n} a_m S_1 T(\Phi_m) \right\| \geq (1-\tau_{\mathcal{U}\times\mathcal{V}}) \sum_{m=1}^{n} |a_m| \quad (a_1, \dots, a_n \in \mathbb{C}).$$

Therefore, applying Lemma 3.5, there exist $\widehat{\varphi_m}, \widehat{\psi_m} \in ((L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}})^*$ for all $m = 1, \dots, n$ satisfying

(I) $\|a\widehat{\varphi_m} + b\widehat{\psi_m}\| \leq 1$ whenever $m = 1, \dots, n$ and $a, b \in \mathbb{D}$;

(II) $\left\| \sum_{m=1}^{n} a_m \left(\mathbf{1}_{((L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}})^*} - \widehat{\psi_m}\right) \right\| \leq \beta_{\mathcal{U}\times\mathcal{V}}(n) \leq \beta(n)$ whenever $a_1, \dots, a_n \in \mathbb{D}$; and

(III) $\langle S_1 T(\Phi_m), \widehat{\varphi_m} \rangle \geq \varkappa_{\mathcal{U}\times\mathcal{V}} \geq \varkappa$ for all $m = 1, \dots, n$.

Put

$$\varphi_m := T^* S_1^*(\widehat{\varphi_m}), \quad \psi_m := T^* S_1^*(\widehat{\psi_m}) \quad (m = 1, \dots, n).$$

Note that

$$\langle \varphi_m, \Phi \rangle = \langle \widehat{\varphi_m}, S_1 T(\Phi) \rangle = 0 \quad (\Phi \in Y)$$

because $\widehat{\varphi_m} \in ((L^1/H_0^1)_{\mathcal{U}\times\mathcal{V}})^*$. Therefore, $\varphi_m \in Z^*$. In a completely similar way, we have that $\psi_m \in Z^*$. Finally, we see that

(IV) $\|a\varphi_m + b\psi_m\| \leq 1$ whenever $m = 1, \dots, n$ and $a, b \in \mathbb{D}$;

(V) $\|a\varphi_m + b\psi_m\| \leq 1$ whenever $m = 1, \dots, n$ and $a, b \in \mathbb{D}$; and

(VI) $\langle \Phi_m, \varphi_m \rangle \geq \varkappa$ for all $m = 1, \dots, n$. □

Using [H, Prop. 6.7] and the remarks of Section 2, and recalling that a dual Banach space with property (V) is a Grothendieck space [Di, Chap. VII, Exer. 12], we can give the following corollaries of Theorem 3.6.

Corollary 3.7. *A has property (V) and all its even duals are Grothendieck spaces. In fact, if $n \geq 1$ and $T: A^{(2n)} \to Y$ is an operator, then either T is weakly compact or T fixes a copy of ℓ_∞.*

The following result extends the one given by Bourgain [B3, Thm. 1].

Corollary 3.8. *H^∞ has local property (V); that is, every ultrapower of H^∞ has the property (V).*

Proof. Note that every ultrapower of H^∞ is isometric to a complemented subspace of an ultrapower of A^{**}, which, in turn, is isometric to a complemented subspace of some ultrapower of A [H, Prop. 6.7]. □

COROLLARY 3.9. *H^∞ and all its even duals are Grothendieck spaces. In fact, if $n \geq 0$ and $T\colon (H^\infty)^{(2n)} \to Y$ is an operator, then either T is weakly compact or T fixes a copy of ℓ_∞.*

Recalling that if X^* has property (V) then X has property (V*), we have the following result.

COROLLARY 3.10. *All the odd duals of H^∞ and A have property (V*).*

Finally, we improve the result of Bourgain [B1] which says that every ultrapower of L^1/H^1_0 has property (V*).

COROLLARY 3.11. *The dual of every ultrapower of L^1/H^1_0 is a Grothendieck space.*

Proof. Note that the dual of every ultrapower of L^1/H^1_0, is isometric to a complemented subspace of some ultrapower of H^∞ [H, Cor. 7.6]. □

REMARKS. In view of the proof of Theorem 3.6, we know that $(V_{\text{sing}})^*$ (and in fact any ultrapower of this space) is an $\mathcal{L}^\infty$-space and moreover is a Grothendieck space. It is worth mentioning that there are $\mathcal{L}^\infty$-spaces that have the Schur property and so are not Grothendieck spaces [B2].

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