The Algebra of Unbounded Continuous Functions on a Stonean Space and Unbounded Operators

Fotios C. Paliogiannis

1. Introduction

The investigation of commutative operator algebras by means of function space techniques is due to M. H. Stone [7]. The notion of a space such that the closure of every open set G, $\operatorname{clos}(G)$, is open (thus, $\operatorname{clos}(G)$ is *clopen*) was introduced by Stone in [8]. Such spaces are called *extremely disconnected*. Extremely disconnected spaces are also characterized as those topological spaces X for which (i) the interior of a closed subset F of X, $\operatorname{int}(F)$, is clopen, or (ii) disjoint open subsets of X have disjoint closures. A compact Hausdorff extremely disconnected space X is also known as a *Stonean space.* If $\mathcal{A}$ is an abelian von Neumann algebra then $\mathcal{A}$ is isomorphic with $C(X)$, where X is a Stonean space (see [5, Thm. 5.2.1]).

In [4] (and [5]), Kadison studies a class of unbounded continuous complex-valued (real-valued) functions on an extremely disconnected space X (called *normal functions* and *self-adjoint functions* and denoted by $N(X)$ and $S(X)$, respectively), and he proves that $N(X)$ is an algebra [4, Thm. 2.11]. Starting with an abelian von Neumann algebra $\mathcal{A}$, Kadison introduces $N(\mathcal{A})$, the algebra of (normal) operators affiliated with $\mathcal{A}$ and $S(\mathcal{A})$, the algebra of self-adjoint operators affiliated with $\mathcal{A}$ [4, Thm. 3.3], extending the isomorphism of $\mathcal{A}$ with $C(X)$ to a *-isomoprhism of $N(\mathcal{A})$ onto $N(X)$ [4, Thm. 4.1]. In this direction, one is enabled to obtain the spectral theorem for self-adjoint and normal operators (see also [2]).

In this article, we present a closely related approach to the study of $N(X)$, $S(X)$, and the spectral theorem for unbounded self-adjoint operators. We begin in Section 2 with a theorem (Theorem 2.1) on continuous extensions from open dense subsets of extremely disconnected spaces (see also [3, p. 96]). Theorem 2.1 leads to a substantial simplification of the proof that $N(X)$ is an algebra, and it plays a key role in our development. We continue, in Section 3, with a discussion on the spectral analysis of a function in $S(X)$, and we give an alternative proof of the fact that $S(X)$ is a boundedly complete lattice. In Section 4 we prove the spectral theorem and characterizations of the spectrum and the spectral projections for unbounded self-adjoint operators.

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2. $N(X)$ and $S(X)$

THEOREM 2.1. *Let X be an extremely disconnected space, and let Y be a compact Hausdorff space. Suppose that U is an open dense subset of X. If $f: U \to Y$ is a continuous function, then f has a unique continuous extension $\tilde{f}$ on X.*

Proof. The uniqueness is clear, since if two continuous functions agree on a dense subset then they agree everywhere.

We prove the existence. For each $y \in Y$, let $A_y = \bigcap_{G \in N_y} \overline{f^{-1}(G)}$ (closure denotes the closure in X), where $N_y = \{ G : G \text{ is open in } Y,\ y \in G \}$. Clearly, A_y is a closed (possibly empty) subset of X for all $y \in Y$.

Now, if $x \in X$ then there is a net $\{x_d\}$ in U such that $x_d \to x$. Since Y is compact, the net $\{f(x_d)\}$ has a cluster point, say y, in Y. We claim that $x \in A_y$.

If $x \notin A_y$, then there is an open set G in Y such that $y \in G$ and $x \notin \overline{f^{-1}(G)}$. Hence there exists a d_0 such that, for $d \geq d_0$, $x_d \notin \overline{f^{-1}(G)}$. In particular, for $d \geq d_0$ we have $f(x_d) \notin G$. Since G is a neighborhood of y and y is a cluster point for $\{f(x_d)\}$, this is a contradiction. Thus, $x \in A_y$.

We define $\tilde{f}(x) = y$ for $x \in A_y$. This is well-defined; for if $y_1 \neq y_2$ then $A_{y_1} \cap A_{y_2} = \emptyset$. To see this, suppose $y_1 \neq y_2$. Then there exist open disjoint sets G_1, G_2 in Y with $y_1 \in G_1$ and $y_2 \in G_2$. Hence, $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are disjoint and open in U. Therefore, they are disjoint and open in X. Since X is extremely disconnected, their closures are disjoint as well. Thus, $A_{y_1} \cap A_{y_2} = \emptyset$.

To see that $\tilde{f}(x) = f(x)$ for all $x \in U$, let D be any directed set and take $x_d = x$ for all $d \in D$. Then $f(x_d) = f(x)$ and $f(x_d) \to f(x)$, so $x \in A_{f(x)}$. Hence, $\tilde{f}(x) = f(x)$.

It remains to show the continuity of $\tilde{f}$. Let F be any closed subset of Y and $N_F = \{ G : G \text{ is open in } Y \text{ and } F \subseteq G \}$. We claim that $\tilde{f}^{-1}(F) = \bigcap_{G \in N_F} \overline{f^{-1}(G)}$ (which immediately gives the continuity of $\tilde{f}$). In fact, if $x \in \tilde{f}^{-1}(F)$ then $\tilde{f}(x) = y \in F$. Hence $x \in A_y \subseteq \bigcap_{G \in N_F} \overline{f^{-1}(G)}$.

Conversely, if $x \notin \tilde{f}^{-1}(F)$ then $\tilde{f}(x) = y \notin F$. Choose G_1, G_2 disjoint open sets in Y such that $y \in G_1$ and $F \subseteq G_2$. The same argument as before gives that $\overline{f^{-1}(G_1)}$ and $\overline{f^{-1}(G_2)}$ are disjoint. Therefore, $A_y \cap \bigcap_{G \in N_F} \overline{f^{-1}(G)} = \emptyset$. Since $x \in A_y$, we have $x \notin \bigcap_{G \in N_F} \overline{f^{-1}(G)}$. □

Let $\dot{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ denote the one-point compactification of the complex plane $\mathbf{C}$ and $\ddot{\mathbf{R}} = [-\infty, +\infty]$ the two-point compactification of the real line $\mathbf{R}$.

DEFINITION. Let X be a Stonean space. A continuous function $f: X \to \dot{\mathbf{C}}$, such that $U_f = \{ x : f(x) \neq \infty \}$ is (open) dense in X, is called a *normal function* on X. We denote by $N(X)$ the set of normal functions on X.

A continuous function $f: X \to \ddot{\mathbf{R}}$, such that $U_f = \{ x : -\infty < f(x) < +\infty \}$ is (open) dense in X, is called a *self-adjoint function* on X. We denote by $S(X)$ the set of self-adjoint functions on X.

Let $f \in N(X)$. We define f^* to be the unique element of $N(X)$ that extends $\bar{f}$ defined on U_f.

PROPOSITION 2.2. *$N(X)$ is a *-algebra containing $C(X)$, and $S(X)$ is the subalgebra of self-adjoint elements of $N(X)$.*

Proof. By definition of f^*, $f^* \in N(X)$ whenever $f \in N(X)$.

To see that $N(X)$ is an algebra, suppose that f and g are in $N(X)$. Write $U_f = \{x : f(x) \neq \infty\}$ and $U_g = \{x : g(x) \neq \infty\}$. Then $U_f \cap U_g$ is open and dense in X, and both $f+g$ and fg are defined and continuous on $U_f \cap U_g$. By Theorem 2.1, $f+g$ and fg both have unique continuous extensions on X, $f \dot{+} g$ and $f \cdot g$, respectively. Now it is easy to see that, with these operations ($\dot{+}$ and $\cdot$), $N(X)$ becomes an algebra with the constant function 1 as unit and $C(X)$ as a subalgebra.

It is also easy to see that, for $f \in N(X)$, $f = f^*$ iff there is a unique $g \in S(X)$ such that $f = \theta \circ g$, where $\theta : \ddot{\mathbf{R}} \to \dot{\mathbf{C}}$ is defined by

$$\theta(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in \mathbf{R}, \\ \infty & \text{if } \lambda = \pm\infty. \end{cases}$$

Thus $S(X)$ is the subalgebra of self-adjoint elements ($f^* = f$) of $N(X)$. $\square$

Note that f is invertible in $N(X)$ precisely when $1/f$ makes sense on a dense open set of X, iff $\operatorname{int}\{x : f(x) = 0\} = \emptyset$. Note also that, if $f \in N(X)$ and e is a projection in $C(X)$—that is, $e = \mathcal{X}_G$ (the characteristic function of G) with G a clopen set in X—then

$$f \cdot e(x) = \begin{cases} f(x) & \text{if } x \in G, \\ 0 & \text{if } x \notin G. \end{cases}$$

3. The Spectral Analysis of a Self-Adjoint Function

For real-valued functions f, g in $C(X)$, we will write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. Let $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection of real-valued functions in $C(X)$. We denote by $\bigvee_{\alpha\in\Omega} f_\alpha$ the l.u.b.$\{f_\alpha : \alpha \in \Omega\}$, that is, $\bigvee_{\alpha\in\Omega} f_\alpha = f$ is such that $f_a \leq f$ for all $\alpha \in \Omega$, and if $f_\alpha \leq g$ for all $\alpha \in \Omega$ then $f \leq g$. Similarly, $\bigwedge_{\alpha\in\Omega} f_\alpha$ denotes the g.l.b.$\{f_\alpha : \alpha \in \Omega\}$.

DEFINITION. Let $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ be a collection of projections in $C(X)$ and $G_\lambda = \{x \in X : e_\lambda(x) = 1\}$. The family $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ is called a *resolution* of the identity in $C(X)$ if

(i) $\bigvee_{\lambda\in\mathbf{R}} e_\lambda = 1 \iff \operatorname{clos}\big(\bigcup_{\lambda\in\mathbf{R}} G_\lambda\big) = X$,
(ii) $\bigwedge_{\lambda\in\mathbf{R}} e_\lambda = 0 \iff \operatorname{int}\big(\bigcap_{\lambda\in\mathbf{R}} G_\lambda\big) = \emptyset$,
(iii) $\bigwedge_{\mu>\lambda} e_\mu = e_\lambda \iff \operatorname{int}\big(\bigcap_{\mu>\lambda} G_\mu\big) = G_\lambda$ for all $\lambda \in \mathbf{R}$.

Clearly, condition (iii) implies that $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ is monotonic in λ.

PROPOSITION 3.1. *There is a bijective correspondence between $S(X)$ and the collection of all resolutions of the identity in $C(X)$.*

Proof. Given any $\varphi \in S(X)$, if $G_\lambda = \operatorname{int}\{\, x : \varphi(x) \le \lambda \,\}$ for $\lambda \in \mathbf{R}$, then $e_\lambda = \mathcal{X}_{G_\lambda}$ defines a resolution of the identity in $C(X)$.

Moreover, $G_\lambda = \operatorname{int}\{\, x : \varphi(x) \le \lambda \,\}$ is equivalent to $\{\, x : \varphi(x) < \lambda \,\} \subseteq G_\lambda \subseteq \{\, x : \varphi(x) \le \lambda \,\}$, which in turn is equivalent to $\varphi \cdot (1 - e_\lambda) \ge \lambda(1 - e_\lambda)$ and $\varphi \cdot e_\lambda \le \lambda e_\lambda$ for all $\lambda \in \mathbf{R}$.

Conversely, let $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ be any resolution of the identity in $C(X)$, and let $G_\lambda = \{\, x : e_\lambda(x) = 1 \}$. Define the function $\varphi \colon X \to \ddot{\mathbf{R}}$ by

$$\varphi(x) = \begin{cases} +\infty & \text{on } \left(\bigcup_\lambda G_\lambda\right)^c, \\ -\infty & \text{on } \bigcap_\lambda G_\lambda, \\ \sup\{\, \lambda : x \notin G_\lambda \,\} = \inf\{\, \lambda : x \in G_\lambda \,\} & \text{on } \bigcup_\lambda G_\lambda \sim \bigcap_\lambda G_\lambda. \end{cases}$$

It is easy to see that φ as defined satisfies the condition $\{\, x : \varphi(x) < \lambda \,\} \subseteq G_\lambda \subseteq \{\, x : \varphi(x) \le \lambda \,\}$ for all $\lambda \in \mathbf{R}$, and this condition determines φ uniquely.

We now show that φ is continous. Suppose $\varphi(x_0) = +\infty$. Let R be any positive real number. Choose any $\lambda > R$. Then $x_0 \in (G_\lambda)^c$ and for all $x \in (G_\lambda)^c$ we have $\varphi(x) \ge \lambda > R$. Thus, $U = (G_\lambda)^c$ is an open neighborhood of x_0, and $\varphi(U) \subseteq (R, +\infty]$. Similarly, suppose $\varphi(x_0) = -\infty$. Given any $R > 0$, choose $\lambda < -R$. Then $U = G_\lambda$ is an open neighborhood of x_0 and $\varphi(U) \subseteq [-\infty, R)$.

Now suppose that $\varphi(x_0)$ is finite. Let $\alpha, \beta \in \mathbf{R}$ such that $\alpha < \varphi(x_0) < \beta$. Choose $\lambda, \mu \in \mathbf{R}$ such that $\alpha < \lambda < \varphi(x_0) < \mu < \beta$. Then $U = G_\mu \cap (G_\lambda)^c$ is an open neighborhood of x_0, and $\varphi(U) \subseteq [\lambda, \mu] \subseteq (\alpha, \beta)$.

To see that $\varphi \in S(X)$, note that $\left(\bigcup_\lambda G_\lambda\right)^c$ and $\bigcap_\lambda G_\lambda$ both have empty interior. Hence, $U_\varphi = \bigcup_\lambda G_\lambda \sim \bigcap_\lambda G_\lambda$ is dense in X. □

THEOREM 3.2. *Let $\varphi \in S(X)$ and with $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ be the resolution of the identity in $C(X)$ defined by φ. For $J = (\alpha, \beta]$ with $-\infty < \alpha < \beta < +\infty$, let $f_J = e_\beta - e_\alpha$. If $\Pi = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ with $\alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \beta$ is any partition of $[\alpha, \beta]$, if $\xi_j \in [\lambda_{j-1}, \lambda_j]$ for $j = 1, 2, \ldots, n$, and if $\|\Pi\| = \max_{j=1,2,\ldots,n}(\lambda_j - \lambda_{j-1})$, then*

$$\left\| \varphi \cdot f_J - \sum_{j=1}^{n} \xi_J (e_{\lambda_j} - e_{\lambda_{j-1}}) \right\|_{C(X)} \le \|\Pi\|;$$

that is, $\varphi \cdot f_J = \int_\alpha^\beta \lambda \, de_\lambda$ (in the Riemann–Stieltjes sense).

Proof. Set $\varphi_{\Pi,\xi} = \sum_{j=1}^n \xi_j (e_{\lambda_j} - e_{\lambda_{j-1}})$. For $x \in G_\beta \sim G_\alpha$ we have $e_{\lambda_0}(x) = e_\alpha(x) = 0$ and $e_{\lambda_n}(x) = e_\beta(x) = 1$. Hence there exists a unique $j = 1, 2, \ldots, n$ such that $e_{\lambda_{j-1}}(x) = 0$ and $e_{\lambda_j}(x) = 1$. Then $\varphi_{\Pi,\xi}(x) = \xi_j$ and $\varphi \cdot f_J(x) = \varphi(x) \in [\lambda_{j-1}, \lambda_j]$, so

$$|\varphi \cdot f_J(x) - \varphi_{\Pi,\xi}(x)| = |\varphi(x) - \xi_j| < (\lambda_j - \lambda_{j-1}) \le \|\Pi\|.$$

For $x \in G_\alpha$ we have $\varphi \cdot f_J(x) = 0$ and $e_{\lambda_0}(x) = e_\alpha(x) = 1$. Hence, $\varphi_{\Pi,\xi}(x) = 0$ and the estimate trivially holds. For $x \notin G_\beta$ we have $\varphi \cdot f_J(x) = 0$ and $e_{\lambda_n}(x) = e_\beta(x) = 0$, and again the estimate trivially holds. □

REMARK 3.3. Note that φ, the element of $S(X)$ associated with $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ is the unique element of $S(X)$ satisfying $\varphi \cdot f_J = \int_\alpha^\beta \lambda \, de_\lambda$. In fact, if $\psi \in S(X)$ with

$\psi \cdot f_J = \varphi \cdot f_J$ for all $J = (\alpha, \beta]$, then $\psi = \varphi$ on $\bigcup_J \{ x : f_J(x) \neq 0 \} = \bigcup_\lambda G_\lambda \sim \bigcap_\lambda G_\lambda$, which is dense in X. Therefore, $\psi = \varphi$ everywhere.

DEFINITION. Let $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ be a resolution of the identity in $C(X)$. We define $e_{\lambda^-} = \bigvee_{\mu<\lambda} e_\mu$ and $G_{\lambda^-} = \{ x \in X : e_{\lambda^-}(x) = 1 \}$.

Note that e_{λ^-} is a projection and that $e_{\lambda^-} \leq e_\lambda$. Moreover, $G_{\lambda^-} = \operatorname{clos}\big(\bigcup_{\mu<\lambda} G_\mu\big)$.

For each $f \in N(X)$, the *spectrum* of f is defined to be the set

$$\sigma(f) = \{ \lambda \in \mathbf{C} : f - \lambda 1 \text{ is not invertible in } N(X) \}.$$

PROPOSITION 3.4. *Let φ be a self-adjoint function and let $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ be its resolution of the identity. Then $\lambda \in \sigma(\varphi)$ iff $e_\lambda - e_{\lambda^-} \neq 0$.*

Proof. Given $\varphi \in S(X)$, define $G^0_\lambda = \operatorname{int}\{ x : \varphi(x) = \lambda \}$ for all $\lambda \in \mathbf{R}$. We prove that $e_\lambda - e_{\lambda^-} = \mathcal{X}_{G^0_\lambda}$.

First observe that G^0_λ and $\bigcup_{\mu<\lambda} G_\mu$ are disjoint open sets. Since X is extremely disconnected, their closures are disjoint as well; that is, $G^0_\lambda \cap G_{\lambda^-} = \emptyset$. Moreover, $G_{\lambda^-} \subseteq G_\lambda$ and $G^0_\lambda \subseteq G_\lambda$, so $G_{\lambda^-} \cup G^0_\lambda \subseteq G_\lambda$.

To get equality, suppose $x \in G_\lambda \sim G^0_\lambda$. Then, since G_λ is open, there exists a net $x_d \to x$ such that $x_d \in G_\lambda$ and $x_d \notin \{ x : \varphi(x) = \lambda \}$, so $\varphi(x_d) < \lambda$ for all d; but then $x_d \in \bigcup_{\mu<\lambda} G_\mu$. Hence $x_d \in \operatorname{clos}\big(\bigcup_{\mu<\lambda} G_\mu\big)$. Thus, for all $\lambda \in \mathbf{R}$, $G_\lambda = G_{\lambda^-} \cup G^0_\lambda$ and $e_\lambda - e_{\lambda^-} = \mathcal{X}_{G^0_\lambda}$. □

There is a natural partial ordering in $S(X)$ that may be defined as follows: $\varphi \geq \psi$ when $\varphi \dot{-} \psi \geq 0$, that is, $\varphi(x) \geq \psi(x)$ for all $x \in U_\varphi \cap U_\psi$. This partial ordering induces a lattice structure on $S(X)$, for if $\varphi, \psi \in S(X)$ then the functions $\varphi \vee \psi = \frac{1}{2}(\varphi \dot{+} \psi) \dot{+} \frac{1}{2}|\varphi \dot{-} \psi|$ and $\varphi \wedge \psi = \frac{1}{2}(\varphi \dot{+} \psi) \dot{-} \frac{1}{2}|\varphi \dot{-} \psi|$ are, respectively, the least upper and greatest lower bounds of φ and ψ in $S(X)$.

LEMMA 3.5. *Let $\varphi, \psi \in S(X)$, and let $\{e_\lambda\}_{\lambda \in \mathbf{R}}, \{f_\lambda\}_{\lambda \in \mathbf{R}}$ be their respective resolutions of the identity. Then $\varphi \leq \psi$ iff $f_\lambda \leq e_\lambda$ iff $H_\lambda \subseteq G_\lambda$ for all $\lambda \in \mathbf{R}$, where $H_\lambda = \operatorname{int}\{ x : \psi(x) \leq \lambda \}$ and $G_\lambda = \operatorname{int}\{ x : \varphi(x) \leq \lambda \}$.*

The proof is obvious.

In the following theorem we prove that $S(X)$ has the least upper bound property—or, in Kadison's terminology, that $S(X)$ is a boundedly complete lattice.

THEOREM 3.6. *If X is Stonean, then $S(X)$ has the least upper bound property.*

Proof. Suppose $\mathcal{F} = \{\varphi_\alpha\}_{\alpha \in \Omega}$ is a nonempty subset of $S(X)$ with an upper bound (say, ψ_0) in $S(X)$. We prove that there is φ_0 in $S(X)$ such that φ_0 is the least upper bound of $\mathcal{F} = \{\varphi_\alpha\}_{\alpha \in \Omega}$.

Define $G_\lambda = \operatorname{int}\big(\bigcap_{\alpha \in \Omega}\{ x : \varphi_\alpha(x) \leq \lambda \}\big)$ and let $e_\lambda = \mathcal{X}_{G_\lambda}$ for all $\lambda \in \mathbf{R}$. We claim that the family $\{e_\lambda\}_{\lambda \in \mathbf{R}}$ is a resolution of the identity in $C(X)$.

Let $\lambda = \psi_0(x) < +\infty$. Then $\varphi_\alpha(x) \leq \lambda$ for all $\alpha \in \Omega$ and hence

$$\{ x : \psi_0(x) < +\infty \} \subseteq G_\lambda.$$

Since $\{ x : \psi_0(x) < +\infty \}$ is dense in X, we have $\operatorname{clos}\big(\bigcup_{\lambda \in \mathbf{R}} G_\lambda\big) = X$.

Next note that $\bigcap_{\lambda\in\mathbf{R}} G_\lambda \subseteq \bigcap_{\alpha\in\Omega}\{\, x : \varphi_\alpha(x) = -\infty \,\}$. Thus, $\operatorname{int}\big(\bigcap_{\lambda\in\mathbf{R}} G_\lambda\big) \subseteq \operatorname{int}\big(\bigcap_{\alpha\in\Omega}\{\, x : \varphi_\alpha(x) = -\infty \,\}\big) = \emptyset$.

To complete the proof of our claim, let λ and μ be real numbers with $\mu > \lambda$. Clearly, $G_\lambda \subseteq G_\mu$ and so $G_\lambda \subseteq \operatorname{int}\big(\bigcap_{\mu>\lambda} G_\mu\big)$. On the other hand, if $x \in \bigcap_{\mu>\lambda} G_\mu$ then, for all $\alpha \in \Omega$ and all $\mu > \lambda$, we have $\varphi_\alpha(x) \le \mu$. Hence $\varphi_\alpha(x) \le \lambda$ for all $\alpha \in \Omega$. Therefore, $\bigcap_{\mu>\lambda} G_\mu \subseteq \bigcap_{\alpha\in\Omega}\{\, x : \varphi_\alpha(x) \le \lambda \,\}$ and so $G_\lambda = \operatorname{int}\big(\bigcap_{\mu>\lambda} G_\mu\big)$.

Now let φ_0 be the (unique) function in $S(X)$ that corresponds to the resolution of the identity $\{e_\lambda\}_{\lambda\in\mathbf{R}}$. Note now that $G_\lambda = \operatorname{int}\{\, x : \varphi_0(x) \le \lambda \,\}$. From Lemma 3.5, we have $\varphi_\alpha \le \varphi_0$ for all $\alpha \in \Omega$.

Moreover, if $\psi \in S(X)$ is such that $\varphi_\alpha \le \psi$ for all $\alpha \in \Omega$ and if $H_\lambda = \operatorname{int}\{\, x : \psi(x) \le \lambda \,\}$, then $H_\lambda \subseteq \operatorname{int}\{\, x : \varphi_\alpha(x) \le \lambda \,\}$ for all $\alpha \in \Omega$. It follows that $H_\lambda \subseteq G_\lambda$ and, from Lemma 3.5 again, we get $\varphi_0 \le \psi$. This completes the proof. □

4. The Spectral Theorem

Let $B(H)$ be the algebra of bounded linear operators on a Hilbert space H, and let $\mathrm{Op}(H)$ be the set of unbounded densely defined linear operators on H. We recall that, for $A, B \in \mathrm{Op}(H)$, B is called an *extension* of A, denoted by $A \subset B$, if $D(A) \subseteq D(B)$ and $Ax = Bx$ for all $x \in D(A)$.

Let $A \in \mathrm{Op}(H)$ be a closed operator, and let $T \in B(H)$. We say that T *commutes* with A if $TA \subset AT$; that is, if $x \in D(A)$ then $Tx \in D(A)$ and $TAx = ATx$. We denote by $\{A\}'$ the set of all operators in $B(H)$ that commute with the operator A in the foregoing sense:

$$\{A\}' = \{\, T \in B(H) : TA \subset AT \,\}.$$

It is easy to see that $\{A\}'$ is a subalgebra of $B(H)$ that is closed in the strong operator topology (s.o.t.). Note also that $T \in \{A\}'$ iff $T^* \in \{A^*\}'$. Thus, $\{A\}' \cap \{A^*\}'$ is a von Neumann algebra. We write $\{A\}'' = \{\{A\}'\}'$ for the commutant of $\{A\}'$.

DEFINITION. Let $\mathcal{A}$ be a von Neumann algebra of operators on H, and let $A \in \mathrm{Op}(H)$ be a closed operator. We say that A is *affiliated* with $\mathcal{A}$, denoted $A\eta\mathcal{A}$, when $\mathcal{A}' \subset \{A\}'$. We denote by $S(\mathcal{A})$ the family of self-adjoint operators affiliated with the algebra $\mathcal{A}$.

Note that $A\eta\mathcal{A}$ iff $\mathcal{A}' \subset \{A\}' \cap \{A^*\}'$ iff $\{\{A\}' \cap \{A^*\}'\}' \subset \mathcal{A}$. Note also that $W^*(A) = \{\{A\}' \cap \{A^*\}'\}'$ is the smallest von Neumann algebra with which A is affiliated, and is referred to as the von Neumann algebra *generated by* A. Clearly, if A is self-adjoint ($A = A^*$), then

$$A\eta\mathcal{A} \quad \text{iff} \quad W^*(A) = \{A\}'' \subset \mathcal{A}.$$

At this point, we recall some facts from the basic theory of self-adjoint operators. Let $\sigma(A)$ denote the spectrum of a self-adjoint operator A. Then $\sigma(A) \subseteq \mathbf{R}$ and $V = (iI - A)^{-1}$ is a bounded operator with adjoint $V^* = (-iI - A)^{-1}$ (see [1, p. 318]). It is easy to see that V is a normal operator in $B(H)$. In fact,

$V^*V = VV^* = (V^* - V)/2i$. Moreover, $\{A\}' = \{V\}'$. Thus, $W^*(A) = \{A\}'' = \{V\}''$, where $\{V\}''$ is the abelian von Neumann algebra generated by V (i.e., the s.o.t.-closure of the set of polynomials in V).

Let $\mathcal{A}$ be an abelian von Neumann algebra, and let $X = X_{\mathcal{A}}$ be the Gelfand space (or maximal ideal space) of $\mathcal{A}$. From the Gelfand–Naimark representation theorem for abelian C^*-algebras, the Gelfand map $\Gamma : \mathcal{A} \to C(X)$ (where $\Gamma(A) = \hat{A}$ is the Gelfand transform of A for $A \in \mathcal{A}$) is an isometric *-isomorphism from $\mathcal{A}$ onto $C(X)$. As we noted in the introduction, $N(\mathcal{A})$ is a (commutative) *-algebra and the isomorphism Γ extends to a *-isomorphism of $N(\mathcal{A})$ with $N(X)$. Although the algebraic properties of $N(\mathcal{A})$ and the extension of Γ will not be used in the sequel, we shall also extend Γ to a bijection of $S(\mathcal{A})$ with $S(X)$.

THEOREM 4.1. *Let A be a self-adjoint operator, and let $\mathcal{A}$ be any abelian von Neumann algebra such that $A\eta\mathcal{A}$. Let $X = X_{\mathcal{A}}$. Then there exists a unique $\varphi \in S(X)$ such that $(AB)\hat{} = \varphi \cdot \hat{B}$, whenever $B \in \mathcal{A}$ and $AB \in \mathcal{A}$. We write $\dot{\Gamma}(A) = \dot{\hat{A}} = \varphi$.*

Proof. Let $V = (iI - A)^{-1}$. Since $A\eta\mathcal{A}$ and $\{V\}'' = \{A\}''$, we have that $V \in \mathcal{A}$. Let $v = \hat{V} \in C(X)$ be the Gelfand transform of V.

Note that $AV = -(iI - A)V + iV = -I + iV \in \mathcal{A}$. Hence $(AV)\hat{} = -1 + iv$. If F is the projection onto $\operatorname{Ker}(V)$, then F is the largest projection in $\mathcal{A}$ such that $VF = 0$. Therefore, $\hat{F} = \mathcal{X}_G$, where G is the largest clopen set contained in $\{x : v(x) = 0\}$, that is, $G = \operatorname{int}\{x : v(x) = 0\}$. Since V is one-to-one, $F = 0$ and so $G = \emptyset$. Thus, $1/v$ exists in $N(X)$.

Define $\varphi = -1/v + i$. Note that $\varphi^* - \varphi = -2i + (\bar{v} - v)/\bar{v}v$. Furthermore, since $V^*V = VV^* = (V^* - V)/2i$, it follows that $\varphi^* = \varphi$. Hence, $\varphi \in S(X)$.

Note also that, on $\{x : v(x) \neq 0\}$, an open dense subset of X, $\varphi v = -1 + iv = (AV)\hat{}$. Thus, by Theorem 2.1, φv has a unique continuous extension, $\varphi \cdot v = (AV)\hat{}$, on X.

Now, if $C = AB$ with $B, C \in \mathcal{A}$, then

$$VC = VAB = -V(iI - A)B + iVB \subset -B + iVB.$$

Since $VC \in B(H)$, it follows that $VC = -B + iVB$.

Let $b, c \in C(X)$ be such that $b = \hat{B}$ and $c = \hat{C}$. Then $vc = (-1 + iv)b$. This implies $c = (-1/v + i)b$ on $\{x : v(x) \neq 0\}$. Therefore, $c = \varphi \cdot b$; that is, $(AB)\hat{} = \varphi \cdot \hat{B}$.

To see that the restriction of $\dot{\Gamma}$ in $\mathcal{A}$ is Γ, let A be a bounded self-adjoint operator in $\mathcal{A}$ whose Gelfand transform $\hat{A} = a$. Then $v = 1/(i - a)$ and $\dot{\Gamma}(A) = \varphi = \alpha = \Gamma(A)$. $\square$

LEMMA 4.2. *Let A be a self-adjoint operator, and let $\mathcal{A}$ be any abelian von Neumann algebra such that $A\eta\mathcal{A}$. Let $\varphi = \dot{\hat{A}}$. Suppose $B \in \mathcal{A}$ and $\operatorname{supp}(\hat{B}) \subseteq U_\varphi = \{x : -\infty < \varphi(x) < +\infty\}$. Then $AB \in \mathcal{A}$.*

Proof. Let $V = (iI - A)^{-1}$, $v = \hat{V}$, and $b = \hat{B}$. Set $G = \operatorname{supp}(b) = \operatorname{clos}\{x : b(x) \neq 0\}$ (the support of b). Then G is a clopen set and $G \subseteq \{x : v(x) \neq 0\}$. Now, if $e = \mathcal{X}_G$ and $E \in \mathcal{A}$ with $\hat{E} = e$, then $EB = B$.

Define $c = e \cdot 1/v$. If $C \in \mathcal{A}$ with $\hat{C} = c$, then $c \in C(X)$ and $VC = E$. Thus, $AB = AEB = AVCB \in \mathcal{A}$. □

DEFINITION. A *resolution* of the identity in $B(H)$ is a family of projections $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ in $B(H)$ satisfying:

(i) $\bigvee_{\lambda\in\mathbf{R}} E_\lambda = I$,
(ii) $\bigwedge_{\lambda\in\mathbf{R}} E_\lambda = 0$, and
(iii) $\bigwedge_{\mu>\lambda} E_\mu = E_\lambda$ for all $\lambda \in \mathbf{R}$.

We are now ready to prove the spectral theorem for unbounded self-adjoint operators.

THEOREM 4.3 (Spectral Theorem). *Let A be an unbounded self-adjoint operator on H. Then there exists a unique resolution of the identity $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ in $B(H)$ such that*:

(i) *for any interval $J = (\alpha, \beta]$, if $F_J = E_\beta - E_\alpha$ then AF_J is a bounded self-adjoint operator on H and $AF_J = \int_\alpha^\beta \lambda\, dE_\lambda$;*
(ii) *$x \in D(A)$ iff the net $\{AF_J x\}_{J\in\mathcal{J}}$ converges and in fact*

$$Ax = \lim AF_J x = \lim\left(\int_\alpha^\beta \lambda\, dE_\lambda\right)x = \left(\int_{-\infty}^\infty \lambda\, dE_\lambda\right)x.$$

Moreover, $E_\lambda \in \{A\}''$, and $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is called the spectral family *of A.*

Proof. (i) Let $\mathcal{A} = \{A\}''$ and $X = X_{\mathcal{A}}$. If $\varphi = \dot{\hat{A}}$ and $e_\lambda = \mathcal{X}_{G_\lambda}$, where $G_\lambda = \operatorname{int}\{x : \varphi(x) \le \lambda\}$, then $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ is a resolution of the identity in $C(X)$. For $J = (\alpha, \beta]$, let $f_J = e_\beta - e_\alpha$. From Theorem 3.2, $\varphi \cdot f_J \in C(X)$ and $\varphi \cdot f_J = \int_\alpha^\beta \lambda\, de_\lambda$. Now take $E_\lambda \in \mathcal{A}$ such that $\hat{E}_\lambda = e_\lambda$, and let $\hat{F}_J = f_J$. Then $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is a resolution of the identity in $B(H)$ and $F_J = E_\beta - E_\alpha$.

Note that $\operatorname{supp}(f_J) = G_\beta \sim G_\alpha \subseteq U_\varphi$. Hence, by Lemma 4.2, $AF_J \in \mathcal{A}$. Moreover, $(AF_J)\hat{} = \varphi \cdot f_J$ is real-valued, hence AF_J is self-adjoint. At the same time, since the Gelfand map is an isometry, $AF_J = \int_\alpha^\beta \lambda\, dE_\lambda$.

(ii) Note that $F_J \in \mathcal{A} \subseteq \mathcal{A}' = \{A\}'$ and so $F_J A \subset AF_J$ for all J. Let $\mathcal{J}$ be the directed set of half-open intervals $J = (\alpha, \beta]$ in $\mathbf{R}$ ordered by inclusion. Since $\bigcup_{J\in\mathcal{J}}\{x : f_J(x) \neq 0\}$ is dense in X, it follows that $\bigvee_{J\in\mathcal{J}} f_J = 1$. Hence, $\bigvee_{J\in\mathcal{J}} F_J = I$. Therefore, $F_J \uparrow I$ in the strong operator topology.

Now, if $x \in D(A)$, then $AF_J x = F_J Ax \to Ax$. Conversely, suppose $AF_J x \to y$. Then, since A is closed and $F_J x \to x$, we have $x \in D(A)$ and $Ax = y$.

It remains to prove the uniqueness of the spectral family. Suppose $\{E'_\lambda\}_{\lambda\in\mathbf{R}}$ is another resolution of the identity satisfying (i) and (ii). Let $\mathcal{B}$ be the abelian von Neumann algebra generated by $\{E'_\lambda\}_{\lambda\in\mathbf{R}}$.

Let $F'_J = E'_\beta - E'_\alpha \in \mathcal{B}$. By (i), AF'_J is the limit in the uniform operator topology (hence, s.o.t.) of a net of operators in $\mathcal{B}$, so $AF'_J \in \mathcal{B}$. If $B \in \mathcal{B}'$ and $x \in D(A)$, then by (ii) we have $A(BF'_J x) = BAF'_J x \to BAx$. At the same time $BF'_J x \to Bx$. Since A is closed, we conclude that $B \in \{A\}'$. Thus, $\mathcal{A} \subseteq \mathcal{B}$.

Now, if Y is the Gelfand space of $\mathcal{B}$ and $e'_\lambda = (E'_\lambda)\hat{}$, then $\varphi \cdot f'_J = \int_\alpha^\beta \lambda\, de'_\lambda$ and $\varphi \cdot f_J = \int_\alpha^\beta \lambda\, de_\lambda$ in Y. Because such a representation is unique, it follows that $e'_\lambda = e_\lambda$ for all λ. Therefore, $E'_\lambda = E_\lambda$ for all $\lambda \in \mathbf{R}$. □

REMARK 4.4. For any $x \in H$, let μ_x be the unique Borel measure on $\mathbf{R}$ satisfying $\mu_x(J) = (F_J x, x) = \|F_J x\|^2 = (E_\beta x, x) - (E_\alpha x, x)$. Part (ii) of the spectral theorem can be rewritten equivalently as follows:

$$x \in D(A) \quad \text{iff} \int_{-\infty}^{\infty} \lambda^2\, d\mu_x(\lambda) < \infty.$$

For this, first note that $F_J F_K = F_{J\cap K}$ for $J, K \in \mathcal{J}$ (since $E_\lambda E_\mu = E_{\min(\lambda,\mu)}$). Furthermore, we have

$$\begin{aligned}\|AF_J x\|^2 &= \lim_{n\to\infty} \Big\| \sum_{j=1}^n \xi_j (E_{\lambda_j} - E_{\lambda_{j-1}})x \Big\|^2 = \lim_{n\to\infty} \Big\| \sum_{j=1}^n \xi_j F_{J_j} x \Big\|^2 \\ &= \lim_{n\to\infty} \sum_{j=1}^n \xi_j^2 \|F_{J_j} x\|^2 = \lim_{n\to\infty} \sum_{j=1}^n \xi_j^2 \mu_x(J_j) \\ &= \int_J \lambda^2\, d\mu_x(\lambda) \to \int_{-\infty}^{\infty} \lambda^2\, d\mu_x(\lambda).\end{aligned}$$

Now, if $x \in D(A)$ then $AF_J x \to Ax$. So $\|AF_J x\|^2 \to \|Ax\|^2$. Therefore,

$$\int_{-\infty}^{\infty} \lambda^2\, d\mu_x(\lambda) = \|Ax\|^2 < \infty.$$

Conversely, suppose that $\int_{-\infty}^{\infty} \lambda^2\, d\mu_x(\lambda) < \infty$. Let $\varepsilon > 0$ and $\gamma < \alpha < \beta < \delta$ be any real numbers. If $J = (\alpha, \beta]$, $K = (\gamma, \delta]$, $L = (\gamma, \alpha]$, and $M = (\beta, \delta]$, then by choosing J large enough we have

$$\|AF_K x - AF_J x\|^2 = \|AF_L x\|^2 + \|AF_M x\|^2 \le \int_{R-J} \lambda^2\, d\mu_x(\lambda) < \varepsilon.$$

Hence, the net $\{AF_J x\}_{J\in\mathcal{J}}$ is Cauchy and so converges in H.

Note also that, for $x \in D(A)$, the representation $(Ax, x) = \int_{-\infty}^{\infty} \lambda\, d\mu_x(\lambda)$ is valid. By the polarization identity, $(Ax, y) = \int_{-\infty}^{\infty} \lambda\, d\mu_{x,y}(\lambda)$ for $x \in D(A)$ and $y \in H$, where $\mu_{x,y}(J) = (F_J x, y)$. This is the classical form of the spectral decomposition of a self-adjoint operator (see e.g. [6, Thm. 13.30]).

For the proof of the following lemma, see [5, Lemma 5.6.1] (and replace the sequence by a net).

LEMMA 4.5. *If $\{F_d\}$ is an increasing net of projections on the Hilbert space H such that $\bigvee_d F_d = I$, and if A_0 is a linear operator with dense domain $\bigcup_d F_d(H) = (D_0)$ such that $A_0 F_d$ is a bounded self-adjoint operator on H, then A_0 is closable and its closure is the unique self-adjoint operator satisfying $AF_d = A_0 F_d$ for all d.*

PROPOSITION 4.6. *If $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is a resolution of the identity in $B(H)$ and $\mathcal{A}$ is an abelian von Neumann algebra containing $\{E_\lambda\}_{\lambda\in\mathbf{R}}$, then there is a self-adjoint operator A in $S(\mathcal{A})$ whose spectral family is $\{E_\lambda\}_{\lambda\in\mathbf{R}}$.*

Moreover, if $X = X_{\mathcal{A}}$ then the mapping $\dot{\Gamma}\colon S(\mathcal{A}) \to S(X)$ is a bijection.

Proof. Let $e_\lambda = \hat{E}_\lambda$ for all $\lambda \in \mathbf{R}$. Then $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ is a resolution of the identity in $C(X)$. Let $\varphi \in S(X)$ be the (unique) self-adjoint function associated to $\{e_\lambda\}_{\lambda\in\mathbf{R}}$ (Proposition 3.1). Then $\varphi \cdot f_J \in C(X)$ and $\varphi \cdot f_J = \int_\alpha^\beta \lambda\, de_\lambda$, with $J = (\alpha, \beta]$ and $f_J = e_\beta - e_\alpha$.

If, for each $J \in \mathcal{J}$, F_J and A_J are the operators in $\mathcal{A}$ whose Gelfand transforms (in $C(X)$) are f_J and $\varphi \cdot f_J$, respectively, then $\{F_J\}_{J\in\mathcal{J}}$ is an increasing net of projections such that $\bigvee_{J\in\mathcal{J}} F_J = I$, and A_J is a bounded self-adjoint operator.

Define an operator A_0 with domain $D_0 = \bigcup_{J\in\mathcal{J}} F_J(H)$ by $A_0x = A_Jx$ if $x \in F_J(H)$. We claim that A_0 is well-defined; for this, note first that since $(\varphi\cdot f_J) f_K = \varphi \cdot f_{J\cap K}$, it follows that $A_JF_K = A_{J\cap K}$. Now, if x is also in $F_K(H)$, then $x = F_Jx$, $x = F_Kx$, and $A_Jx = A_JF_Kx = A_{J\cap K}x = A_KF_Jx = A_Kx$.

From Lemma 4.5, A_0 is closable and its closure A $(= \bar{A}_0)$ is self-adjoint. Since $A_0 \subset A$ and $A_0F_J = A_J$ is everywhere defined, we have $AF_J = A_0F_J = A_J$. Therefore, $AF_J = \int_\alpha^\beta \lambda\, dE_\lambda$.

Next, we show that $F_JA \subset AF_J$ for all J. Suppose $x \in D(A)$. Then there is a sequence $\{x_n\}$ in D_0 such that $x_n \to x$ and $A_0x_n \to Ax$ (since $A = \bar{A}_0$). Hence, $F_JA_0x_n \to F_JAx$. Note that, for each n, there exists a K such that $x_n = F_Kx_n$. Hence,

$$F_JA_0x_n = F_JA_0F_Kx_n = F_JA_Kx_n = A_KF_Jx_n = A_0F_Jx_n.$$

Now, $F_Jx_n \to F_Jx$ and $AF_Jx_n = A_0F_Jx_n = F_JA_0x_n \to F_JAx$. Since A is closed, $F_Jx \in D(A)$ and $AF_Jx = F_JAx$. The same argument as in the proof of Theorem 4.3 gives that $x \in D(A)$ iff the net $\{AF_Jx\}_{J\in\mathcal{J}}$ converges (and $AF_Jx \to Ax$).

To see that A is affiliated with $\mathcal{A}$, suppose that T is in $\mathcal{A}'$ and that $x \in D(A)$. Then $TF_Jx \to Tx$ (since $F_Jx \to x$) and $ATF_Jx = AF_JTx = TAF_Jx \to TAx$. Since A is closed, $Tx \in D(A)$ and $ATx = TAx$. Thus, $\mathcal{A}' \subset \{A\}'$.

It is now clear that $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is the spectral family of A (by uniqueness, as in Theorem 4.3). Moreover, since A is in $S(\mathcal{A})$, $\dot{\Gamma}(A)$ makes sense and $(AF_J)\hat{} = \dot{\Gamma}(A)\cdot f_J$. Therefore, $\dot{\Gamma}(A)\cdot f_J = \varphi\cdot f_J$ for all J (since $(AF_J)\hat{} = \varphi\cdot f_J$). Thus, $\dot{\Gamma}(A) = \varphi$. (Invoking Theorem 3.2, this also proves that $\dot{\Gamma}\colon S(\mathcal{A}) \to S(X)$ is a bijection.) □

COROLLARY 4.7. *If A is a self-adjoint operator and $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is its spectral family, then $W^*(A) = \{A\}'' = \{E_\lambda : \lambda\in\mathbf{R}\}''$.*

Proof. Take $\mathcal{A} = \{E_\lambda : \lambda \in \mathbf{R}\}''$ in Proposition 4.6. Then A is affiliated with $\mathcal{A}$. Therefore, $\{A\}'' \subseteq \mathcal{A}$. On the other hand, since $E_\lambda \in \{A\}''$, we have $\{E_\lambda : \lambda\in\mathbf{R}\}'' \subseteq \{A\}''$. □

PROPOSITION 4.8. *Let A be a self-adjoint operator, and let $\mathcal{A}$ be any abelian von Neumann algebra such that $A\eta\mathcal{A}$. Let $\varphi = \dot{\hat{A}}$. Then $\sigma(A) = \varphi(U_\varphi)$.*

Proof. If $\lambda \notin \sigma(A)$, then $B = (\lambda I - A)^{-1} \in \mathcal{A}$ and $I = (\lambda I - A)B$. Taking Gelfand transforms, this gives $1 = (\lambda - \varphi) \cdot b$ with $b = \hat{B}$. Thus, $\lambda \notin \varphi(U_\varphi)$.

Conversely, if $\lambda \notin \varphi(U_\varphi)$ then $b \equiv (1/(\lambda - \varphi)) \in C(X)$. Take $B \in \mathcal{A}$ such that $\hat{B} = b$. Since $\operatorname{supp}(b) \subseteq U_\varphi$, $AB \in \mathcal{A}$. Now $b \cdot (\lambda - \varphi) = (\lambda - \varphi) \cdot b = 1$ and so $B(\lambda I - A) \subset I = (\lambda I - A)B$. Thus, $\lambda \notin \sigma(A)$. □

The spectral family $\{E_\lambda\}_{\lambda \in \mathbf{R}}$ of a self-adjoint operator completely determines the spectrum of the operator.

THEOREM 4.9. *Let $A \in \mathrm{Op}(H)$ be a self-adjoint operator, $\{E_\lambda\}_{\lambda \in \mathbf{R}}$ its resolution of the identity, and $\alpha, \beta \in \mathbf{R}$ with $\alpha < \beta$. Then $\sigma(A) \cap (\alpha, \beta) = \emptyset$ iff $E_{\beta^-} - E_\alpha = 0$, where $E_{\beta^-} = \bigvee_{\mu < \beta} E_\mu$.*

Proof. Let $\mathcal{A} = \{A\}''$, $X = X_{\mathcal{A}}$, $\varphi = \dot{\hat{A}}$, and $e_\lambda = \hat{E}_\lambda$. Recall that $\sigma(A) = \varphi(U_\varphi)$. Suppose that $\varphi(U_\varphi) \cap (\alpha, \beta) = \emptyset$. Then $\{x : \varphi(x) < \beta\} = \{x : \varphi(x) = \alpha\}$. This implies that $\{x : \varphi(x) < \beta\} = \operatorname{int}\{x : \varphi(x) = \alpha\} = G_\alpha$ and so $\operatorname{clos}\{x : \varphi(x) < \beta\} = G_\alpha$.

Let $g = \mathcal{X}_{\operatorname{clos}\{x : \varphi(x) < \beta\}}$. Then we have $g \geq e_\mu$ for all $\mu < \beta$ and thus

$$g \geq \bigvee_{\mu < \beta} e_\mu = e_{\beta^-}.$$

If $\psi \in S(X)$ is such that $e_\mu \leq \psi$ for all $\mu < \beta$, then $G_\mu \subseteq \{x : \psi(x) \geq 1\}$. Hence

$$\{x : \varphi(x) < \beta\} = \bigcup_{\mu < \beta} \{x : \varphi(x) < \beta\} \subseteq \bigcup_{\mu < \beta} G_\mu \subseteq \{x : \psi(x) \geq 1\}.$$

Therefore, $\operatorname{clos}\{x : \varphi(x) < \beta\} \subseteq \{x : \psi(x) \geq 1\}$; in other words, $g \leq \psi$. Thus, $e_{\beta^-} = g = \mathcal{X}_{\operatorname{clos}\{x : \varphi(x) < \beta\}} = \mathcal{X}_{G_\alpha} = e_\alpha$; that is, $E_{\beta^-} - E_\alpha = 0$.

Conversely, let $E_{\beta^-} - E_\alpha = 0$ or (equivalently) $e_{\beta^-} - e_\alpha = 0$. Suppose there is an $x \in X$ such that $\alpha < \varphi(x) < \beta$. Then there exist real numbers λ, μ with $\alpha < \lambda < \mu < \beta$ such that $e_\lambda(x) = 0$ and $e_\mu(x) = 1$. It follows that $e_\alpha(x) = 0$ and $e_{\beta^-}(x) = \left(\bigvee_{\mu < \beta} e_\mu\right)(x) = 1$. Therefore, $e_{\beta^-}(x) - e_\alpha(x) = 1$, which is a contradiction. □

THEOREM 4.10. *Let $A \in \mathrm{Op}(H)$ be a self-adjoint operator and $\{E_\lambda\}_{\lambda \in \mathbf{R}}$ its resolution of the identity. Then $\operatorname{Ker}(\lambda I - A) = \operatorname{Range}(E_\lambda - E_{\lambda^-})$ for all $\lambda \in \mathbf{R}$. (Thus, λ is an eigenvalue of A iff $E_\lambda - E_{\lambda^-} \neq 0$.)*

Proof. Let P be the projection onto $\operatorname{Ker}(\lambda I - A)$; then $AP = \lambda P$. Since $PA \subset (AP)^* = \lambda P^* = \lambda P$, it follows that $P \in \{A\}'$. Thus, P commutes with the spectral projections E_λ of A.

Take now $\mathcal{A}$ to be the abelian von Neumann algebra generated by A, P, and $\{E_\lambda\}_{\lambda \in \mathbf{R}}$, that is, $\mathcal{A} = \{A, P, E_\lambda\}''$. Let $X = X_{\mathcal{A}}$, $\varphi = \dot{\hat{A}}$, $e_\lambda = \hat{E}_\lambda$, and $p = \hat{P}$. Note that P is the largest projection in $\mathcal{A}$ such that $(\lambda I - A)P = 0$ and, as a result, $\{x : p(x) = 1\} = \operatorname{int}\{x : \varphi(x) = \lambda\} = G^0_\lambda$. Hence, from Proposition 3.4, $p = e_\lambda - e_{\lambda^-}$. Thus, $P = E_\lambda - E_{\lambda^-}$. □

In the remainder of this section we shall characterize the spectral family of a self-adjoint operator. We begin with some preliminaries.

Let A be any operator. A *bounding sequence* for A is a nondecreasing sequence $\{F_n\}$ of projections such that $\bigvee_{n=1}^{\infty} F_n = I$ and $F_n A \subset AF_n$, with $AF_n \in B(H)$ for all n. Note that, for a self-adjoint operator A, we can construct a bounding sequence $\{F_n\}$ for A from its spectral family $\{E_\lambda\}_{\lambda\in\mathbf{R}}$. In fact, $F_n = F_{J_n}$ where $J_n = (-n, n]$, $n = 1, 2, \ldots$.

LEMMA 4.11. *If $\{F_n\}$ is a bounding sequence for a closable operator A, then $\{F_n\}$ is also a bounding sequence for $\bar{A}$ (the closure of A) and $\bar{A}F_n = AF_n$ for all n.*

Proof. Let $x \in D(\bar{A})$. Fix m and choose $x_n \in D(A)$ such that $x_n \to x$ and $Ax_n \to \bar{A}x$. Then $F_m x_n \to F_m x$ and $\bar{A}F_m x_n = AF_m x_n = F_m A x_n \to F_m \bar{A} x$. Therefore $F_m \bar{A} \subset \bar{A} F_m$. Since $AF_m \subseteq \bar{A}F_m$ and $AF_m \in B(H)$, it follows that $\bar{A}F_m = AF_m$. □

A *core* for a closed linear operator A is a dense linear subspace D_0 of the domain of A such that $A = \overline{A|_{D_0}}$. That is, given any $x \in D(A)$, there exist $\{x_n\} \in D_0$ such that $x_n \to x$ and $Ax_n \to Ax$. Note that if A is a closed operator and $\{F_n\}$ is a bounding sequence for A, then $D_0 = \bigcup_{n=1}^{\infty} \mathrm{Range}(F_n)$ is a core for A.

A self-adjoint operator A is said to be *positive* ($A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in D(A)$ or, equivalently, if $\sigma(A) \subseteq [0, +\infty)$ (see [6, Thm. 13.31]). From Proposition 4.8 we see that the mapping $\dot{\Gamma}\colon S(\mathcal{A}) \to S(X)$ is order-preserving. Note also that, for $A, B \in S(\mathcal{A})$, $AB = A^*B^* \subset (BA)^*$ and so AB is closable. We shall denote the closure of AB by $A \cdot B$.

THEOREM 4.12. *Let $A \in \mathrm{Op}(H)$ be a self-adjoint operator and $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ a resolution of the indentity in $B(H)$. Then $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is the spectral family of A iff $AE_\lambda \leq \lambda E_\lambda$ and $A(I - E_\lambda) \geq \lambda(I - E_\lambda)$ for all $\lambda \in \mathbf{R}$.*

Proof. Let $\mathcal{A} = \{A\}''$, $X = X_{\mathcal{A}}$, $\varphi = \dot{\hat{A}}$, $e_\lambda = \hat{E}_\lambda$, $F_J = E_\beta - E_\alpha$, and $\hat{F}_J = f_J$. Then $\varphi \cdot e_\lambda \leq \lambda e_\lambda$ and $\varphi \cdot (1 - e_\lambda) \geq \lambda(1 - e_\lambda)$. Since $\dot{\Gamma}\colon S(\mathcal{A}) \to S(X)$ is order-preserving, all we need show is that $AE_\lambda \in S(\mathcal{A})$, $\dot{\Gamma}(AE_\lambda) = \varphi \cdot e_\lambda$, and $\dot{\Gamma}(A(I - E_\lambda)) = \varphi \cdot (1 - e_\lambda)$.

First note that AE_λ is closed. Also, since $E_\lambda \in \mathcal{A}$, we have $E_\lambda A \subset AE_\lambda$. Therefore, $(AE_\lambda)^* \subset AE_\lambda$. On the other hand, $E_\lambda A = E_\lambda^* A^* \subset (AE_\lambda)^*$. Hence, $E_\lambda \cdot A \subset (AE_\lambda)^*$.

We show that $E_\lambda \cdot A = AE_\lambda$. For this, first note that $F_n = F_{J_n}$, where $J_n = (-n, n]$ for $n = 1, 2, \ldots$ is a bounding sequence for both AE_λ and $E_\lambda A$. By Lemma 4.11, this is also the case for $E_\lambda \cdot A$ and $E_\lambda \cdot AF_n = E_\lambda AF_n$. Now

$$AE_\lambda F_n = AF_n E_\lambda F_n = E_\lambda F_n AF_n = E_\lambda AF_n = E_\lambda \cdot AF_n,$$

that is, AE_λ and $E_\lambda \cdot A$ agree on their common core $D_0 = \bigcup_{n=1}^{\infty} \mathrm{Range}(F_n)$. Hence, $E_\lambda \cdot A = AE_\lambda$ and so $(AE_\lambda)^* = AE_\lambda$. If $T \in \mathcal{A}'$ then, since $A \in S(\mathcal{A})$, $TAE_\lambda \subset ATE_\lambda = AE_\lambda T$. Thus, $AE_\lambda \in S(\mathcal{A})$.

Now, since F_J and E_λ commute and $AF_JE_\lambda \in \mathcal{A}$, we have $\Gamma(AF_JE_\lambda) = \Gamma(AE_\lambda F_J)$; that is, $\varphi \cdot f_J e_\lambda = \dot{\Gamma}(AE_\lambda) \cdot f_J$. Thus, $\dot{\Gamma}(AE_\lambda) = \varphi \cdot e_\lambda$. Similarly, $\dot{\Gamma}(A(I - E_\lambda)) = \varphi \cdot (1 - e_\lambda)$. Conversely, suppose $AE_\lambda \leq \lambda E_\lambda$ and $A(I - E_\lambda) \geq \lambda(I - E_\lambda)$ for all $\lambda \in \mathbf{R}$. From $AE_\lambda \leq \lambda E_\lambda$ we have that AE_λ is self-adjoint. Hence, $E_\lambda A \subset AE_\lambda$.

If $\{P_\mu\}_{\mu \in \mathbf{R}}$ is the spectral family of A, then $E_\lambda P_\mu = P_\mu E_\lambda$ for all λ, μ. Take $\mathcal{A} = \{E_\lambda, P_\mu\}''$, the abelian von Neumann algebra generated by E_λ and P_μ. Note that A is affiliated with $\mathcal{A}$.

Let $X = X_{\mathcal{A}}$, $\varphi = \dot{\hat{A}}$, $e_\lambda = \hat{E}_\lambda$, $\hat{P}_\mu = p_\mu$, and $F_J = P_\beta - P_\alpha$. As before, we have $\dot{\Gamma}(AE_\lambda) = \varphi \cdot e_\lambda$ and $\dot{\Gamma}(A(I - E_\lambda)) = \varphi \cdot (1 - e_\lambda)$. The hypothesis implies that $\varphi \cdot e_\lambda \leq \lambda e_\lambda$ and $\varphi \cdot (1 - e_\lambda) \geq \lambda(1 - e_\lambda)$.

Now, if $X_\lambda = \{x : e_\lambda(x) = 1\}$ then $X_\lambda = \operatorname{int}\{x : \varphi(x) \leq \lambda\} = \{x : p_\lambda(x) = 1\}$. Therefore, $E_\lambda = P_\lambda$ for all $\lambda \in \mathbf{R}$. □

THEOREM 4.13. *Let $A \in \mathrm{Op}(H)$ be a self-adjoint operator and $\{E_\lambda\}_{\lambda \in \mathbf{R}}$ a resolution of the identity in $B(H)$. Then $\{E_\lambda\}_{\lambda \in \mathbf{R}}$ is the spectral family of A iff:*

(i) *$F_J A \subset AF_J$ for any interval $J = (\alpha, \beta]$ where $F_J = E_\beta - E_\alpha$—that is, F_J reduces A;*
(ii) *the relation $x \in \mathrm{Range}(F_J)$ implies $\alpha\|x\|^2 \leq (Ax, x) \leq \beta\|x\|^2$.*

Proof. If $\{E_\lambda\}_{\lambda \in \mathbf{R}}$ is the spectral family of A then, as noted in the proof of the spectral theorem, $F_J A \subset AF_J$ for all J. Furthermore, if $x \in \mathrm{Range}(F_J)$ then $Ax = AF_J x = \left(\int_\alpha^\beta \lambda \, dE_\lambda\right)x$. Hence $(Ax, x) = \int_\alpha^\beta \lambda \, d\mu_x(\lambda)$, which clearly implies (ii).

Conversely, suppose that conditions (i) and (ii) hold. First note that $\alpha F_J \leq AF_J \leq \beta F_J$ means that AF_J is a bounded self-adjoint operator. Let $D_0 = \bigcup_J \mathrm{Range}(F_J)$. Then D_0 is a core for A. Since AF_JE_λ is self-adjoint, we have $E_\lambda AF_J \subset AF_JE_\lambda = AE_\lambda F_J$. It follows that $E_\lambda A \subset AE_\lambda$.

Now let $\lambda \in \mathbf{R}$. Choose $\kappa \in \mathbf{R}$ such that $\kappa < \lambda$ and let $F_J = E_\lambda - E_\kappa$. If $x \in \mathrm{Range}(F_J)$ then we have

$$(AE_\lambda x, x) = (AF_J x, x) = (Ax, x) \leq \lambda\|x\|^2 \leq \lambda(E_\lambda x, x).$$

Thus, $AE_\lambda \leq \lambda E_\lambda$.

Similarly, given $\lambda \in \mathbf{R}$, choose $\mu \in \mathbf{R}$ such that $\mu > \lambda$ and let $F_J = E_\mu - E_\lambda$. If $x \in \mathrm{Range}(F_J)$ then we have

$$(A(I - E_\lambda)x, x) = (AF_J x, x) = (Ax, x) \geq \lambda\|x\|^2 = \lambda(\|E_\mu x\|^2 - \|E_\lambda x\|^2).$$

Letting $\mu \to +\infty$, we get $(A(I - E_\lambda)x, x) \geq \lambda((I - E_\lambda)x, x)$. Thus, $A(I - E_\lambda) \geq \lambda(I - E_\lambda)$.

Since D_0 is a core for A, both inequalities hold for any $x \in D(A)$. By Theorem 4.12, the proof is complete. □

EXAMPLE 4.14. Let $(S, \mathcal{S}, \mu)$ be a σ-finite measure space and let $g : S \to \mathbf{R}$ be a measureable function finite a.e. on S. The multiplication operator M_g with $D(M_g) = \{f \in L^2(S) : gf \in L^2(S)\}$ and $M_g(f) = gf$ for $f \in D(M_g)$ is a self-adjoint operator. Let $E_\lambda = M_{\phi_\lambda}$, where $\phi_\lambda = \mathcal{X}_{\{g \leq \lambda\}}$ for $\lambda \in \mathbf{R}$. One can see that $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is a resolution of the identity in $B(L^2(S))$. Now, since $g\phi_\lambda \leq \lambda\phi_\lambda$ and $g(1-\phi_\lambda) \geq \lambda(1-\phi_\lambda)$ a.e., it follows that $M_g E_\lambda \leq \lambda E_\lambda$ and $M_g(I - E_\lambda) \geq \lambda(I - E_\lambda)$ for all $\lambda \in \mathbf{R}$. Therefore, from Theorem 4.12, $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is the spectral family of M_g.

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Graduate School and University Center
City University of New York
New York, NY 10036