Cantor Sets, Binary Trees, and Lipschitz Circle Homeomorphisms

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1. Introduction

Let A_1 and A_2 be disjoint compact subintervals of $[0, 1)$, and let L be the smallest compact interval containing $A_1 \cup A_2$. Let $S\colon A_1 \cup A_2 \to L$ be a mapping such that the restrictions $S|_{A_i}$ are affine surjections onto L for each i. Then we define the affine Cantor set K_S by

$$K_S \equiv \{ x \in L : S^i(x) \subset A_1 \cup A_2 \text{ for all } i \geq 1 \}.$$

We call this a *two-branched affine Cantor set*. If we replace the restriction that S be locally affine with the requirement that $|S'| > 1$, than K_S is called a two-branched *hyperbolic Cantor set*. (This is sometimes also called a dynamically defined Cantor set, a self-similar Cantor set, or a "cookie cutter".) A k-branched affine Cantor set or hyperbolic Cantor set is defined similarly for any $k \geq 2$.

A different kind of Cantor set arises as follows. Let f be an orientation-preserving homeomorphism of the circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$. Poincaré showed that, if f has no periodic orbits, then either:

(1) every orbit is dense in the circle and f is topologically conjugate to the irrational rotation $R_\alpha(x) = x + \alpha$, where α is the rotation number of f; or
(2) no orbit is dense and every orbit accumulates on a unique Cantor set Γ_f (in this case, the homeomorphism is called a *Denjoy counterexample* because of Denjoy's theorem).

The Cantor set Γ_f is minimal for f, meaning that it is compact, non-empty, f-invariant, and has no compact non-empty f-invariant subsets.

Intuitively, the Cantor sets Γ_f are fundamentally different from the self-similar Cantor sets K_S described previously. To make precise the sense in which this is true, we introduce the following terminology. For $r \geq 0$, denote by $\mathcal{C}(r)$ the class of C^r*-minimal* sets; that is,

$$\mathcal{C}(r) = \{ C \subset \mathbf{S}^1 : C \text{ is a minimal Cantor set for some } C^r \text{ diffeomorphism of } \mathbf{S}^1 \}.$$

Since any two Cantor sets in $\mathbf{S}^1$ are ambiently homeomorphic, it is easy to see that $\mathcal{C}(0)$ includes every Cantor set. On the other hand, $\mathcal{C}(r)$ is empty for $r \geq 2$ owing to Denjoy's theorem.

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DENJOY'S THEOREM [1]. *If f is a C^1 diffeomorphism of $\mathbf{S}^1$ without periodic points, and if the derivative Df has bounded variation, then f is topologically conjugate to an irrational rotation.*

Herman [3] produced Denjoy counterexamples of class $C^{1+\alpha}$ for all $\alpha < 1$, so $\mathcal{C}(r)$ is non-empty for all $r < 2$. Clearly $\mathcal{C}(r) \subset \mathcal{C}(s)$ if $s < r$. The intuitive notion mentioned before can now be stated as follows.

THEOREM A [5]. *$\mathcal{C}(1)$ contains no affine Cantor sets. Moreover, if the generating map S is C^2-sufficiently close to affine, then $K_S \notin \mathcal{C}(1)$.*

See also [4] and [6] for other results about $\mathcal{C}(r)$.

In this paper we show that at least some affine Cantor sets do belong to $\mathcal{C}(r)$ for all $r < 1$. In fact we will prove the following theorem.

THEOREM 1. *If S is a two-branched affine Cantor set and $|A_1| = |A_2|$, then there is a bi-Lipschitz (i.e., Lipschitz with Lipschitz inverse) homeomorphism f of the circle such that $\Gamma_f = K_S$.*

The rotation number of the f is the golden mean.

In particular, the usual middle-thirds Cantor set (scaled down to fit inside the fundamental domain $[0, 1)$ of $\mathbf{R}/\mathbf{Z}$) is the minimal set for some bi-Lipschitz circle homeomorphism, but not for any C^1 circle diffeomorphism. Note that, when S is defined as in Theorem 1 with $|A_1| = |A_2|$, we call K_S a *linear* Cantor set to distinguish this special case from the more general affine case.

The method of proof of Theorem 1 will be as follows. A hyperbolic Cantor set K has a natural tree structure; such a tree has a natural circular ordering and the resulting order topology. The rotation number of a tree homeomorphism can be defined in the usual way. We construct a tree homeomorphism with rotation number equal to the golden mean. This can be lifted in a natural way to a circle homeomorphism f with $\Gamma_f = K$. The tree homeomorphism can be constructed in such a way that it has bounded distortion in the sense that it changes the "depth" of any node in the tree by a bounded amount. When K is a linear Cantor set, bounded distortion on the tree level lifts to a Lipschitz condition on f.

REMARK. It might have been tempting to think that the reason for Theorem A is an unbounded distortion forced by a conflict between the scaling of gaps of an affine Cantor set and the order property of orbits for an irrational rotation. Theorem 1 shows that this is not the case; instead, the difficulty rests more delicately on the continuity of the derivative of f.

2. Trees

Intuitively, a tree is simply a graph containing no closed loops; it can be specified by giving a list of nodes, together with a description of which nodes are connected by edges. In this paper we will be considering infinite binary trees where every node (except one isolated node) is the parent of two other nodes (its children), and the only edges are those connecting parent and child.

Since our tree rotations will always take nodes to nodes, we will henceforth ignore the edges of the tree and simply consider the nodes, along with the relation of parenthood, as follows. (The whole discussion is determined by the structure of Cantor sets on the circle.)

First, we define the complete circular binary tree T: T is a countable set of elements called *nodes,* with a certain labeling. There are two special nodes, r and r^*, called the root node and the isolated node, respectively. The remaining nodes are in one-to-one correspondence with the set of non-empty finite words from the alphabet $\{0, 1\}$. The node corresponding to $i_1 \dots i_k$ ($k \geq 1$) is denoted $r(i_1 \dots i_k)$.

The node $r(i_1 \dots i_k)$ is called the parent of each of the two nodes $r(i_1 \dots i_k 0)$ and $r(i_1 \dots i_k 1)$; they are its children. Node r is the parent of $r(0)$ and $r(1)$. The isolated node n has no children. Descendant and ancestor are defined in the obvious way. To visualize T as a graph, connect every non-isolated node with each of its children by an edge (see Figure 1).

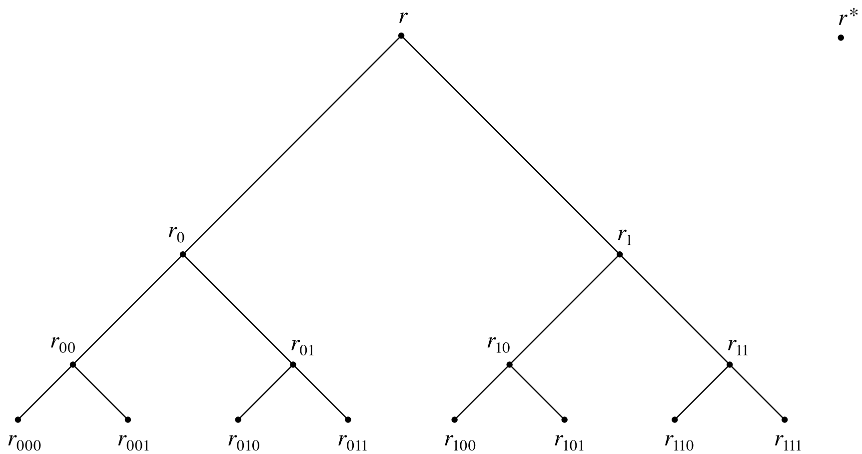

Figure 1 The first four levels of the tree T

If u is the node $r(i_1 \dots i_k)$, then we will use the notation $u(j_1 \dots j_l)$ to denote the node $r(i_1 \dots i_k j_1 \dots j_l)$. The *level* of a node $u \in T$ is defined by $\ell(u) = 0$ if $u = r$ or r^*; $\ell(u) = k$ if $u = r(i_1 \dots i_k)$ for some choice of $i_1 \dots i_k$.

There is also a natural (left-to-right) linear order structure that we can place on T. The simplest way to specify this is to define an injection $\mu \colon T \to (0, 1]$. We let $\mu(r) = 1/2$, $\mu(r^*) = 1$, and

$$\mu(r(i_1, \dots, i_k)) = 2^{-k-1} + \sum_{j=1}^{k} i_j 2^{-j}.$$

The range of μ is the set of dyadic rationals in $(0, 1]$. We pull back the natural circular ordering on $(0, 1]$ to T via μ and denote it by $\prec$. That is, if $u, v, w \in T$ then $u \prec v \prec w$ iff $\mu(u) < \mu(v) < \mu(w) \bmod 1$. This ordering induces the usual order topology on T, for which the open intervals form a basis. In this topology, T is homeomorphic to a countable dense subset of the circle.

There are now effectively two orderings on T: the "left–right" circular ordering and the "vertical" partial order induced by the relation of parenthood. We will write $u \lessdot v$ if v is a descendant of u.

By a *subtree* we mean a subset of T with the induced orderings. A subtree S of T is called an *ideal* (with respect to $\lessdot$) if S contains r and r^* and if, moreover, S contains u whenever (a) S contains v and (b) $u \lessdot v$.

Let g be an order-preserving bijection of T (relative to the circular order $\prec$). Then g is a homeomorphism, and the push-forward $\mu(g)$ defined by $\mu(g)(x) = \mu(g(\mu^{-1}(x)))$ is a homeomorphism of the dyadic rationals in the circle. Therefore $\mu(g)$ extends to an orientation-preserving homeomorphism of the circle, and as such it has a rotation number. This will be our definition of the rotation number of g on T. For this reason, we call any such g a *tree rotation*.

There is a natural one-to-one correspondence τ between T and the collection $\mathcal{I}$ of connected components (intervals) of $S^1 \setminus K_S$. To describe this, we rotate K_S so that its left endpoint is at 0. Let $A_1 = [0, a]$ and $A_2 = [b, c]$, where $0 < a < b < c < 1$. Let $\phi_0 \colon L \to A_1$ and $\phi_1 \colon L \to A_2$ be the two branches of the inverse of S. Then $\tau \colon T \to \mathcal{I}$ is defined by

$$\begin{aligned} \tau(r^*) &= (c, 1), \\ \tau(r) &= (a, b), \\ \tau(r(i_1 \dots i_k)) &= \phi_{i_1} \circ \dots \circ \phi_{i_k}[(a, b)]. \end{aligned}$$

It is easy to check that τ is order-preserving.

Next, we say that the node u *immediately precedes* the node v in a finite ideal S of T if there is no node w of S such that $u \prec w \prec v$.

We need these lemmas for later use.

LEMMA 1. *If u immediately precedes v in a finite ideal S of T, then $\ell(u) = \ell(v)$ implies that $\ell(u) = \ell(v) = 0$ and hence that $u = r$ and $v = r^*$.*

Proof. For contradiction, assume $k := \ell(u) = \ell(v) \neq 0$. Then, for some choice of indices, $u = r(i_1 \dots i_k)$ and $v = r(j_1 \dots j_k)$. Let

$$m = \min\{ n : i_n \neq j_n \} \geq 1,$$

and let

$$w = \begin{cases} r(i_1 \dots i_{m-1}) & \text{if } m > 1, \\ r & \text{if } m = 1. \end{cases}$$

Then $u = w(i_m \dots i_k)$ and $v = w(j_m \dots j_k)$. Since S is an ideal and $u \in S$, this means $w \in S$. Also, w lies between u and v. This contradicts the choice of u and v in S. $\square$

LEMMA 2. *Suppose S is a finite ideal of T, and suppose u immediately precedes v in S. Then there is a unique node $w := S(u, v)$ of $T \setminus S$ such that*

(i) *$S \cup \{w\}$ is an ideal in T,*
(ii) *w is between u and v (i.e., u immediately precedes w and w immediately precedes v in $S \cup \{w\}$), and*
(iii) *$\ell(w) = \max\{\ell(u), \ell(v)\} + 1$.*

Furthermore, any other node of T between u and v has level greater than $\ell(w)$.

Proof. If $u = r$ and $v = r^*$, let $w = u(1)$. Otherwise, by Lemma 1, $\ell(u) \neq \ell(v)$. Let $w = u(1)$ if $\ell(u) > \ell(v)$; otherwise, let $w = v(0)$. Clearly, w satisfies (i), (ii), and (iii). We leave uniqueness and the remaining claim as an exercise. □

LEMMA 3. *Suppose u, v, u', v' are nodes of T satisfying $\ell(u) < \ell(v)$, $\ell(u') > \ell(v')$, and $\max\{|\ell(u) - \ell(u')|, |\ell(v) - \ell(v')|\} \leq 2$. If x, y are nodes with $\ell(x) = \ell(v) + 1$ and $\ell(y) = \ell(u') + 1$, then $|\ell(x) - \ell(y)| \leq 1$.*

Proof. For contradiction, suppose $\ell(x) \geq \ell(y) + 2$. Then

$$\ell(x) \geq \ell(u') + 3 \geq \ell(v') + 4$$

and so $\ell(v) \geq \ell(v') + 3$, a contradiction (similarly if $\ell(y) \geq \ell(x) + 2$). □

LEMMA 4. *Let u, v, u', v' be distinct nodes of a finite ideal S of T such that u is adjacent to v, u' is adjacent to v', and*

$$\max\{|\ell(u) - \ell(u')|,\ |\ell(v) - \ell(v')|\} \leq 2.$$

Then

$$|\ell(S(u, v)) - \ell(S(u', v'))| \leq 2.$$

Proof. By Lemma 1, $\ell(u) \neq \ell(v)$ and $\ell(u') \neq \ell(v')$. There are four possible cases.

Case 1. $\ell(v) > \ell(u)$ and $\ell(v') > \ell(u')$. Then by Lemma 2, $\ell(S(u, v)) = \ell(v) + 1$, $\ell(S(u', v')) = \ell(v') + 1$, and

$$|\ell(S(u, v)) - \ell(S(u', v'))| = |\ell(v) - \ell(v')| \leq 2.$$

Case 2. The treatment of $\ell(v) < \ell(u)$ and $\ell(v') < \ell(u')$ is similar.

The remaining two cases are covered by Lemma 3. □

3. Proof of Theorem 1

Let α denote the golden mean $(\sqrt{5} - 1)/2 = 0.61803\ldots$. Define the irrational rotation $R_\alpha\colon S^1 \to S^1$ by $R_\alpha(x) = x + \alpha \bmod 1$, and define $R\colon \mathbf{Z} \to S^1$ by

$$R(n) = R_\alpha^n(0) = n\alpha \mod 1.$$

The following theorem will be proved in Section 4.

THEOREM 2. *There is a bijection $h\colon \mathbf{Z} \to T$ with the same circular ordering as the bi-infinite sequence $\{ R(n) : n \in Z \}$ and with the property that, for all n,*

$$|\ell(h(n)) - \ell(h(n+1))| \le 2.$$

COROLLARY. *There is an order-preserving homeomorphism $g\colon T \to T$ with rotation number α such that $T = \{ g^n(r) : n \in \mathbf{Z} \}$ and*

$$|\ell(g(u)) - \ell(u)| \le 2$$

for all $u \in T$.

Proof. Define $g\colon T \to T$ by $g = h\sigma h^{-1}$, where h is the function given in Theorem 2 and $\sigma : \mathbf{Z} \to \mathbf{Z}$ is the shift $\sigma(n) = n+1$. Then g is an order-preserving bijection of T; since its orbit has the same ordering as the R-orbit of 0, it must have rotation number α. The rest of the corollary follows immediately. □

Our desired function f will be the extension to S^1 of the natural lift of g to the intervals comprising the complement of K_S in S^1. To make this precise, we use the function τ defined in the previous section. Recall that $\mathcal{I}$ is the collection of connected components of $S^1 \setminus K_S$. For each $I \in \mathcal{I}$, define $f|_I$ to be the unique orientation-preserving affine map taking I onto $\tau \circ g \circ \tau^{-1}(I)$. In this way, f is defined on all of $S^1 \setminus K_S$. Since g is order-preserving on T, it follows that f is also order-preserving. Since $\bigcup \mathcal{I}$ is dense in S^1, f extends to a unique continuous function, also called f, on S^1. By its contruction, f permutes the intervals of $\mathcal{I}$ and is a homeomorphism of the circle; also, $f(K_S) = K_S$.

Let β denote the (constant) derivative of S. Then the length of any interval at level k is simply $(b-a)/\beta^k$. Since g changes the level of any node by at most two, this means that f can expand or contract any $I \in \mathcal{I}$ by at most a factor β^2. That is, on $S^1 \setminus K_S$,

$$1/\beta^2 \le f' \le \beta^2.$$

Since K_S has measure zero and $f(K_S) = K_S$, we have that f is absolutely continuous on S^1. The bound on f' therefore yields a global Lipschitz constant of β^2 for f and f^{-1}.

Because f is a homeomorphism with an invariant Cantor set and irrational rotation number, it must be a Denjoy counterexample and hence $\Gamma_f \subset K_S$. If x is any endpoint of K_S then, since h is surjective, $\{ f^n(x) : n \in \mathbf{Z} \}$ is dense in K_S and so $\Gamma_f = K_S$. This completes the proof of Theorem 1. □

4. Proof of Theorem 2

The proof is by induction. Recall the sequence of denominators of best approximations to the golden mean (the Fibonnaci sequence): $q_0 = 1$, $q_1 = 1$, $q_n = q_{n-1} + q_{n-2}$. (See e.g. Hardy and Wright [2] as a general reference for Diophantine approximation.)

First we need a purely number-theoretic lemma.

LEMMA 5. *Let $n \geq 4$ be a positive integer, and suppose that k, k' are integers satisfying*

$$q_{n-2} \leq k < q_{n-1} \leq k' < q_n.$$

Let

$$Q = \{ i \in \mathbf{Z} : -k \leq i \leq q_n - k - 1 \} \quad \text{and}$$
$$Q' = \{ i \in \mathbf{Z} : -k' \leq i \leq q_{n+1} - k' - 1 \}.$$

Then $Q \subset Q'$. Moreover, $R(Q)$ partitions S^1 into q_n open intervals, and each such interval contains at most one point of $R(Q')$.

Equivalently, the two nearest neighbors in $R(Q')$ of any point of $R(Q' \setminus Q)$ lie in $R(Q)$. In particular, if n is odd then: (a) *for each $i = -k', \ldots, -k - 1$,*

$$R(q_n + i) \prec R(i) \prec R(q_{n-1} + i);$$

and (b) *for each $i = q_n - k, \ldots, q_{n+1} - k' - 1$,*

$$R(i - q_{n-1}) \prec R(i) \prec R(i - q_n).$$

In each case, the three points are nearest neighbors in $R(Q')$.

If n is even then the reverse inequalities hold.

Proof. A standard fact in the theory of Diophantine approximation is the following: For n odd and $x \in S^1$,

$$x + R(q_n) \prec x \prec x + R(q_{n-1})$$

and the interval

$$\{ y \in S^1 : x + R(q_n) \prec y \prec x + R(q_{n-1}) \}$$

contains no other points of the set

$$\{ x + R(i) : i = 1, \ldots, q_{n+1} - 1 \}.$$

For n even the same is true but the inequalities are reversed.

Now fix $i \in \{-k', \ldots, -k - 1\}$. From the definitions of k and k', it is straightforward to check that $i + q_n \in Q$ but $i + q_{n+1} \notin Q'$. This means that the nearest neighbors in $R(Q')$ to $R(i)$ are $R(i + q_n)$ and $R(i + q_{n-1})$, both belonging to $R(Q)$. Also,

$$R(i + q_n) \prec R(i) \prec R(i + q_{n-1}).$$

A similar argument works for $i \in \{q_n - k, \ldots, q_{n+1} - k' - 1\}$, using the fact that $i - q_n \in Q$ and $i - q_{n+1} \notin Q'$. □

In order to prove Theorem 2, we will actually prove the following: For every $n \geq 4$, there is (a) a positive integer k_n such that

$$q_{n-2} \leq k_n \leq q_{n-1} - 1$$

and (b) a function $h_n : Q_n \to T$ such that

(i_n) the values of h_n have the same circular ordering as $\{ R(i) : i \in Q_n \}$,

(ii_n) $|\ell(h_n(i)) - \ell(h_n(i + 1))| \leq 2$ for $i = -k_n, \ldots, q_n - k_n - 2$,

(iii$_n$) $|\ell(h_n(-k_n)) - \ell(h_n(q_n - k_n - 1))| \leq 1$,
(iv$_n$) $h_n(Q_n)$ is an ideal in T, and
(v$_n$) $h_n|_{Q_{n-1}} = h_{n-1}$ $(n > 4)$,

where $Q_n := \{-k_n, -k_n + 1, \dots, 0, \dots, q_n - k_n - 1\}$.

It will follow from the proof that each h_n is injective and the union of the sets $h_n(Q_n)$ is the whole tree T. Then (v) will imply that the h_n define a well-defined function h on $\mathbf{Z}$, and conditions (i) and (ii) will yield the conclusions of the theorem. (Because of (v$_n$), we will henceforth write h instead of h_n for convenience.)

To start, set $k_4 = 2$, $h(0) = r^*$, $h(1) = r$, $h(2) = r(0)$, $h(-1) = r(01)$, and $h(-2) = r(1)$. This defines h on the set $Q_4 = \{-2, -1, 0, 1, 2\}$, and the circular ordering of these points in T is

$$h(2) \prec h(-1) \prec h(1) \prec h(-2) \prec h(0),$$

which coincides with the circular ordering of $\{ R^i(0) : i = -2, \dots, 2, \}$. Conditions (i)–(iv) are easily verified (see Figure 2).

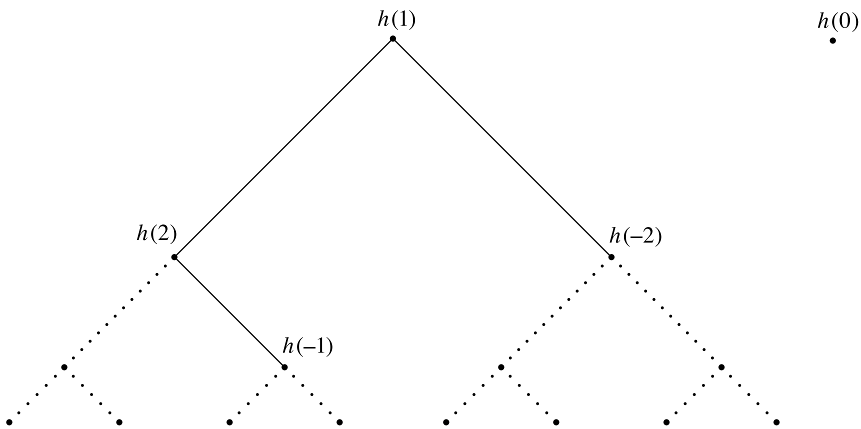

Figure 2 The first five values of h

Now assume by induction that (i$_n$)–(v$_n$) hold. We wish to define k_{n+1} so that

$$q_{n-1} \leq k_{n+1} \leq q_n - 1,$$

and extend h to $Q_{n+1} = \{-k_{n+1}, -k_{n+1} + 1, \dots, 0, \dots, q_{n+1} - k_{n+1} - 1\}$ so that (i$_{n+1}$)–(v$_{n+1}$) hold.

Define k_{n+1} to be largest integer k in $\{q_{n-1}, \dots, q_n - 1\}$ such that $\ell(h(q_n - k)) > \ell(h(q_{n-1} - k))$. (For convenience of notation in the rest of this proof, we will write k for k_{n+1}.) To extend h to the new domain Q_{n+1}, by induction we need only define new values for h on $Q_{n+1} \setminus Q_n = \{-k, \dots, -k_n - 1\} \cup \{q_n - k_n, \dots, q_{n+1} - k - 1\}$.

Notice that $h(Q_n)$ divides the tree T into q_n intervals. Similarly, the set $R(Q_n)$ divides the circle into q_n intervals. Each such interval contains at most one point of the set $R(Q_{n+1} \setminus Q_n)$, by Lemma 5.

For $j \in Q_{n+1} \setminus Q_n$, if $R(j)$ falls into the interval with endpoints $R(j')$ and $R(j'')$ $(j', j'' \in Q_n)$, we then define $h(j)$ to be the unique node (Lemma 2) of $T \setminus h(Q_n)$ between $h(j')$ and $h(j'')$ of least possible level. (This is known as the "standard tree insertion".)

This will guarantee properties (i), (iv), and (v) for $n+1$; we need only verify (ii) and (iii). To do this, we need to be more explicit about where the new values of h lie with respect to the nodes of $h(Q_n)$. Assume that n is odd (otherwise, the argument is similar with inequalities reversed).

By Lemma 5, for $i = q_n - k_n, \dots, q_{n+1} - k - 1$, we have

$$h(i - q_{n-1}) \prec h(i) \prec h(i - q_n) \tag{1}$$

and, for $i = -k, \dots, -k_n - 1$,

$$h(q_n + i) \prec h(i) \prec h(q_{n-1} + i). \tag{2}$$

By Lemma 4, it follows from (ii$_n$) that for

$$i = q_n - k_n, \dots, q_n - k - 2 \quad \text{and} \quad -k, \dots, -k_n - 2$$

we have

$$|\ell(h(i)) - \ell(h(i+1))| \le 2.$$

To complete the verification of (ii$_{n+1}$), it remains to show that

$$|\ell(h(-k_n - 1)) - \ell(h(-k_n))| \le 2 \tag{3}$$

and

$$|\ell(h(q_n - k_n - 1)) - \ell(h(q_n - k_n))| \le 2. \tag{4}$$

We will prove (3), leaving the similar proof of (4) to the reader. The nearest neighbors of $h(-k_n-1)$ in $h(Q_n)$ are $u := h(q_n-k_n-1)$ and $v := h(q_{n-1}-k_n-1)$, by Lemma 5. When the value u was assigned at the previous stage, its nearest neighbors in $h(Q_{n-1})$ were v and $h(q_{n-2}-k_n-1)$, again by Lemma 5. Therefore, by Lemma 2, $\ell(u) > \ell(v)$. This means (again by Lemma 2) that

$$\ell(h(-k_n - 1)) = \ell(u) + 1.$$

But by (iii$_n$), we have

$$|\ell(u) - \ell(h(-k_n))| \le 1.$$

These last two statements imply (3).

Finally, we verify (iii$_{n+1}$). From the order properties (1) and (2), we have

$$\begin{aligned}
&h(0) \prec h(q_n - q_{n-2}) \prec h(-q_{n-2})\\
&h(1) \prec h(q_n - q_{n-2} + 1) \prec h(-q_{n-2} + 1)\\
&\vdots\\
&h(q_n - k - 1) \prec h(q_{n+1} - k - 1) \prec h(-q_{n-2} + q_n - k - 1)\\
&h(q_n - k) \prec h(-k) \prec h(q_{n-1} - k)\\
&\vdots\\
&h(q_n - q_{n-1}) \prec h(-q_{n-1}) \prec h(0).
\end{aligned}$$

It follows from our definition of k that $\ell(h(q_n - k)) > \ell(h(q_{n-1} - k))$ and $\ell(h(q_n - k - 1)) < \ell(h(-q_{n-2} + q_n - k - 1))$. Lemmas 2 and 3 therefore give us property (iii$_{n+1}$). This completes the proof of Theorem 2. $\square$

5. Final Remarks

In the end, the use of trees here is mainly as a convenient device for keeping track of scales in a Cantor set. These results are clearly the tip of a large iceberg. For example, we suspect that similar constructions would work for other rotation numbers—we have only done the simplest case. The problem of handling affine but nonlinear Cantor sets is untouched. One could also imagine using trees other than the complete binary tree to model certain Cantor sets, and some trees could be much better adapted to irrational rotation than the standard binary tree. To retreat to the motivating question of this work, we note that a good geometric intrinsic characterization of Denjoy minimal sets is still unavailable.

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