Uniform Estimates for the Hyperbolic Metric and Euclidean Distance to the Boundary

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Dedicated to Alan F. Beardon, for his interest and insightful discussions

1. Introduction

Throughout this article, D is a proper subdomain of the complex plane $\mathbb{C}$ possessing at least two finite boundary points, usually termed a *hyperbolic* domain. Each such D carries constant negative curvature metrics, and we let λ_D denote the scale factor or density for the maximal constant curvature -1 metric. We call λ_D the *Poincaré hyperbolic metric* for D; it can be defined by

$$\lambda_D(z) = \lambda_{\mathbb{B}}(\zeta)/|p'(\zeta)| = 2/(1-|\zeta|^2)|p'(\zeta)|,$$

where $z = p(\zeta)$ and $p\colon \mathbb{B} \to D$ is any holomorphic covering projection from the unit disk $\mathbb{B} = \{|\zeta| < 1\}$ onto D. See [BP; HM; M1; M2] and their references for basic properties of the Poincaré metric.

An elementary exercise using Schwarz's lemma shows that λ_D satisfies a domain monotonicity property, from which we easily conclude that

$$\lambda_D(z)\,\mathrm{dist}(z, \partial D) \le 2 \tag{1.1}$$

for all points $z \in D$ for any hyperbolic domain D. In the opposite direction, an application of Koebe's one-quarter theorem [P3, 1.4, p. 9] yields

$$\lambda_D(z)\,\mathrm{dist}(z, \partial D) \ge 1/2 \tag{1.2}$$

for all $z \in D$ when D is simply connected. Thus we see from (1.1) and (1.2) that, in simply connected hyperbolic domains D, the Poincaré metric and the Euclidean distance to the boundary ∂D of D are approximately reciprocals; however, for general hyperbolic domains there are no universal lower bounds as in (1.2).

It is well known that equality holds in (1.1) (resp., (1.2)) at some point z if and only if D is a disk centered at z (resp., D is the complement of a ray and z lies on the ray of symmetry). Our purpose here is to investigate when strict inequality holds *uniformly* in (1.1) or (1.2). We exhibit geometric conditions that provide estimates for the quantities

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Received July 29, 1997. Revision received November 17, 1997.

The first author began this research at the University of Texas in Austin; she is grateful for their hospitality. The second author initiated his work while at the Mittag Leffler Institute in Djursholm, Sweden; he happily acknowledges their support in addition to that of the Taft Memorial Fund at UC.

Michigan Math. J. 46 (1999).

$$\sup \lambda\mathrm{d} = \sup_D \lambda\mathrm{d} = \sup_{z\in D} \lambda_D(z)\,\mathrm{dist}(z, \partial D)$$

and

$$\inf \lambda\mathrm{d} = \inf_D \lambda\mathrm{d} = \inf_{z\in D} \lambda_D(z)\,\mathrm{dist}(z, \partial D).$$

More precisely, in this note we characterize the hyperbolic domains for which $\sup \lambda\mathrm{d} < 2$ (see 4.2), and we present estimates for $\inf \lambda\mathrm{d}$ (see 3.3, 3.5, and 3.10) and use them to describe the simply connected hyperbolic domains satisfying $\inf \lambda\mathrm{d} > 1/2$ (see 3.11). We shall discover that whether or not $\sup \lambda\mathrm{d} < 2$ or $\inf \lambda\mathrm{d} > 1/2$ depends in an essential way on the geometry of ∂D.

After studying $\inf \lambda\mathrm{d}$ and $\sup \lambda\mathrm{d}$ in Sections 3 and 4 (respectively), we apply our results in Section 5 to confirm that both $\inf \lambda\mathrm{d} > 1/2$ and $\sup \lambda\mathrm{d} < 2$ hold for nonround quasidisks. In addition, we characterize unbounded convex quasidisks in terms of precise estimates on $\inf \lambda\mathrm{d}$ and $\sup \lambda\mathrm{d}$.

The quantity $1/\mathrm{dist}(z, \partial D)$ is the density for the so-called *quasihyperbolic* metric in D. Hence $\lambda_D(z)\,\mathrm{dist}(z, \partial D)$ can be viewed as the ratio of the hyperbolic and quasihyperbolic metrics at the point z; so $\sup \lambda\mathrm{d}$ and $\inf \lambda\mathrm{d}$ yield sharp upper and lower bounds for the values of this ratio.

2. Preliminaries

We let $B(z; r) = \{\, \zeta : |\zeta - z| < r \,\}$ denote the open disk of radius r centered at the point z. We write $c = c(a, \dots)$ to indicate a constant c that depends only on $a, \dots$; typically, c will depend on various parameters, and we try to make this as clear as possible, often giving explicit values.

We make extensive use of Hejhal's result [He], which describes the behavior of the Poincaré metric with respect to Carathéodory kernel convergence.

2.1. FACT. *Suppose that a sequence of hyperbolic domains $\{D_n\}$ converges to a hyperbolic domain D with respect to a point z_0, in the sense of kernel convergence. Then $\lambda_{D_n}(z_0) \to \lambda_D(z_0)$ as $n \to \infty$.*

In order to certify statements regarding equality of Poincaré metrics, we utilize the following, due to Minda [M1, Cor., p. 63].

2.2. FACT. *Let D and G be hyperbolic subdomains of $\mathbb{C}$ with $D \cap G \neq \emptyset$. Suppose that $\lambda_D(z) \le \lambda_G(z)$ for all z in some neighborhood of a point $z_0 \in D \cap G$. If equality holds at $z = z_0$, then $D = G$.*

We require knowledge of the Poincaré metric in some special domains. To calculate $\lambda_D(z)$, we utilize its conformal invariance: $\lambda_D(z)|dz| = \lambda_{D'}(w)|dw|$ whenever $z \mapsto w$ is a conformal change of variables mapping D onto D'. We denote the infinite wedge with apex angle $\alpha\pi$, $0 < \alpha \le 1$, by

$$\Omega_\alpha = \{\, re^{i\theta} : r > 0,\ |\theta| < \alpha\pi \,\}.$$

An easy calculation, using $w = z^{1/2\alpha}$, yields

$$\lambda_{\Omega_\alpha}(re^{i\theta}) = 1/[2\alpha r \cos(\theta/2\alpha)], \tag{2.3}$$

from which we deduce that, when $0 < \alpha \le 1/2$ and $\beta = 1 - \alpha$,

$$\sup_{\Omega_\alpha} \lambda \mathrm{d} = \sin(\alpha\pi)/2\alpha, \quad \inf_{\Omega_\alpha} \lambda \mathrm{d} = 1 = \sup_{\Omega_\beta} \lambda \mathrm{d}, \quad \inf_{\Omega_\beta} \lambda \mathrm{d} = 1/2\beta.$$

For $\alpha = 0$, Ω_α is replaced by the infinite strip $\Sigma = \{x + iy : |y| < 1\}$. The conformal change of variables $w = \exp((\pi/2)z)$ produces

$$\lambda_\Sigma(x + iy) = \pi/[2\cos((\pi/2)y)], \tag{2.4}$$

and we observe that $\inf_\Sigma \lambda \mathrm{d} = 1$ and $\sup_\Sigma \lambda \mathrm{d} = \pi/2$.

Now we compute the Poincaré metric for the *annular wedge*

$$A(m, \varphi) = \{ re^{i\theta} : e^{-m} < r < 1,\ |\theta| < \varphi \},$$

with angle/radius modulus $2\varphi/m$, where $m > 0$ and $0 < \varphi < \pi$.

2.5. Lemma. *At the "center point" $z_0 = e^{-m/2}$ of $A = A(m, \varphi)$, we have*

$$|z_0|\lambda_A(z_0) = \frac{K}{\varphi}(1 + k) = \frac{K'}{m}(1 + k).$$

Here $0 < k = k(m, \varphi) < 1$ is chosen so that $\varphi/m = K/K'$; $K = K(k)$; and $K' = K'(k) = K(\sqrt{1 - k^2})$.

Proof. Recall that (for $0 < k < 1$) the mapping $w = F(\zeta) = F(\zeta, k)$, given by

$$F(\zeta) = F(\zeta, k) = \int_0^\zeta [(1 - z^2)(1 - k^2 z^2)]^{-1/2}\, dz,$$

defines a conformal homeomorphism of the upper half-plane $H = \{\Im(\zeta) > 0\}$ onto the rectangle $\{|\Re(w)| < K,\ 0 < \Im(w) < K'\}$ and the points $\zeta = -1/k, -1, 1, 1/k$ correspond to $w = -K + iK', -K, K, K + iK'$, respectively, where $K = K(k) = F(1, k)$ and $K' = K(\sqrt{1 - k^2})$ [N, p. 280] (F^{-1} is the Jacobian elliptic sine function).

Now λ_A can be computed using any conformal mapping $f : H \to A$. Letting $k = k(m, \varphi)$ be defined as indicated, we see that a formula for f is

$$z = f(\zeta) = \exp\left(\frac{mi}{K'} F(\zeta)\right).$$

Since $|z|\lambda_A(z) = \lambda_H(\zeta)|f(\zeta)/f'(\zeta)|$ and $\lambda_H(\zeta) = 1/\Im(\zeta)$, we find that

$$|z_0|\lambda_A(z_0) = \frac{K'}{m}(1 + k) = \frac{K}{\varphi}(1 + k)$$

at $z_0 = f(i/\sqrt{k})$, as desired. □

3. The Infimum

The canonical simply connected domain for which $\inf \lambda \mathrm{d} = 1/2$ is the complement of a ray—for example, if $D = \mathbb{C} \setminus (-\infty, 0] = \Omega_1$, then

$$\lambda_D(z) = 1/(2|z|\cos(\arg(z)/2))$$

and hence $\lambda_D(x)\,\mathrm{dist}(x, \partial D) = 1/2$ for all $x > 0$. Beardon and Pommerenke [BP] exhibit a geometric characterization of hyperbolic domains D with inf λd positive, however, their lower bound is always strictly less than $1/2$; see also [HM; P1]. Employing Carathéodory's kernel convergence theorem, Pommerenke [P2, Lemma 4] established a necessary and sufficient condition for inf $\lambda\mathrm{d} > 1/2$ to hold in a simply connected domain, albeit in a disguised form. In addition to applying solely to simply connected domains, Pommerenke's result fails to provide any quantitative information about inf λd. Here we offer a different geometric characterization for these simply connected domains that, in addition, furnishes estimates for inf λd as well as supplying some information about the multiply connected case.

We begin by mentioning the following due to Hilditch [Hi, Thms. 2.1, 2.2]; see also Minda [M1, Thm. 4], Meija and Minda [MM, Thms. 2, 3] and Harmelin and Minda [HM, Thm. 4].

3.1. FACT. *For any hyperbolic domain D,* inf $\lambda\mathrm{d} \le 1$, *and equality holds if and only if D is convex.*

To verify the inequality, let z approach a closest boundary point. Convex domains possess supporting half-planes, so the equality is a necessary condition for convexity; that it is also sufficient follows from a result of Keogh's [K].

Next we examine domains that enjoy a certain arcwise connectivity property. We declare D to be *c-quasiconvex* provided each pair of points z_1, z_2 in D can be joined by an arc γ in D whose Euclidean arclength satisfies

$$\ell(\gamma) \le c|z_1 - z_2| \tag{3.2}$$

for some constant $c \ge 1$. We record the following observation, in part because of the explicit lower bound it furnishes for inf λd. Blevins establishes a similar result for k-domains [B, Thm. 2.2], and Meija and Minda produce such an estimate for k-convex domains [MM, Thm. 1]. Notice that, when $c = 1$ we recover the Hilditch–Minda result for convex domains.

3.3. PROPOSITION. *Suppose D is simply connected and c-quasiconvex. Then*

$$\lambda_D(z)\,\mathrm{dist}(z, \partial D) \ge 1/2\beta > 1/2 \quad \textit{for all } z \in D,$$

where $\beta = 1 - \alpha$ and $\alpha\pi = \arcsin(1/c)$. Moreover, equality holds at a single point $z \in D$ if and only if there is a similarity transformation φ mapping D onto the infinite wedge Ω_β with $\varphi(z) > 0$.

Proof. First we note that the hypotheses on D ensure that either D is a bounded Jordan domain or that ∂D consists of a finite number of distinct infinite Jordan curves (cf. [P3, 5.6]). Fix $z_0 \in D$. Assume that $0 \in \partial D$ and that $z_0 = 1 = \mathrm{dist}(z_0, \partial D)$. Let C be the component of ∂D containing 0 and let G be the component of the complement of C that lies in the complement of D. Next let Γ be an arc joining 0 to ∞ in G.

Fix $r > 0$. Let w be the "first" point where Γ meets the circle $|z| = r$, let A be the subarc of $\{|z| = r\} \cap G$ containing w, and let w_1, w_2 be the endpoints of A. We claim that the angular measure of A is at least $2\alpha\pi$. Choose points $z_i \in D$ on $|z| = r$ close enough to w_i so that the smaller subarcs κ_i between z_i and w_i lie in D. Let γ be an arc in D joining z_1, z_2 and satisfying (3.2). Since γ cannot meet Γ, the origin must lie inside the curve $A \cup \kappa_1 \cup \gamma \cup \kappa_2$, so $\ell(\gamma) \geq 2r$. Thus

$$2r/c \leq |z_1 - z_2| = 2r\sin(\theta/2) \quad \text{or} \quad \theta \geq 2\alpha\pi,$$

where θ is the angle between z_1 and z_2. Letting z_i approach w_i yields our assertion.

Now let D^* be the circular symmetrization of D with respect to the positive real axis (cf. [W] or [Ha, p. 69]). Then D^* is a domain, $z_0 = 1 \in D^*$, $0 \in \partial D^*$, and (since each circle $|z| = r$ contains a subarc $A \subset \mathbb{C}\setminus D$ of angular measure $2\alpha\pi$) we see that $D^* \subset \Omega_\beta$. Thus, by [W] and (2.3),

$$\lambda_D(z_0)\,\mathrm{dist}(z_0, \partial D) = \lambda_D(z_0) \geq \lambda_{D^*}(z_0) \geq \lambda_{\Omega_\beta}(z_0) = \frac{1}{2\beta},$$

as desired. According to Fact 2.2, equality forces $D = D^* = \Omega_\beta$. □

3.4. Remarks. (a) The punctured unit disk $\mathbb{B}^* = \mathbb{B}\setminus\{0\}$ is c-quasiconvex for all $c > 1$, yet $\inf_{\mathbb{B}^*} \lambda\mathrm{d} = 0$. Thus, the simple connectivity hypothesis is essential. (b) The domain $D = \{1 < |z| < 2, |\arg(z)| < \pi\}$ has $\inf \lambda\mathrm{d} > 1/2$ (by 3.11), but is not quasiconvex. (c) An analog of Proposition 3.3 holds with the arclength of γ replaced by its diameter, although in this situation we only obtain the nonsharp lower bound $\inf \lambda\mathrm{d} \geq \pi/[2(\pi - \arcsin(1/2c))]$. However, there are domains that satisfy such a diameter condition but not the corresponding length condition. Another alternative arises if instead of joining interior points we just require that boundary points be joinable; but then we must insist that D be Jordan, or we must consider prime ends. (d) Finally, we mention that one can obtain this result by way of Hölder continuity of conformal mappings; see [NP1, Thm. 2] and [NP2, p. 439].

We now turn to the problem of characterizing the condition $\inf \lambda\mathrm{d} > 1/2$. Roughly speaking, we show that this holds if and only if the boundary of D near each "closest boundary point" oscillates with a minimum amplitude (the constant θ) and a minimum frequency (the constant ε). To be more precise, let $\Theta(w; w_1, w_2) \in [0, \pi]$ denote the angle between the segments $[w, w_1]$ and $[w, w_2]$ (where w, w_1, w_2 are distinct points). Given constants $0 < \varepsilon < 1$ and $0 < \theta < \pi$, we say that D satisfies an (ε, θ)-*annular wedge condition* if, whenever $z \in D$ and $w \in \partial D$ are such that $d = |z - w| = \mathrm{dist}(z, \partial D)$, there then exist points $w_1, w_2 \in \mathbb{C}\setminus D$ with $\varepsilon d \leq |w_i - w| \leq d$ and $\Theta(w; w_1, w_2) \geq \theta$. Thus, if D satisfies some annular wedge condition then D has no boundary points that are endpoints of internal cusps, and in fact there is a quantitative estimate describing how far away from being such a point each exposed boundary point is. Note that our annular wedge condition is equivalent to Pommerenke's half-strip condition [P2, Lemma 4].

We establish that every hyperbolic domain with $\inf \lambda d > 1/2$ must satisfy some annular wedge condition by employing a conformal mapping and domain monotonicity of the Poincaré metric. We verify the converse for simply connected domains by using a result of Kuz′mina [K1]. First we derive an upper bound for $\inf \lambda d$ in domains that fail to satisfy a specific annular wedge condition.

3.5. THEOREM. *Suppose a hyperbolic domain D fails to satisfy the annular wedge condition for a particular pair of constants* $(\varepsilon, \theta) \in (0,1) \times (0,\pi)$. *Then there exists a point $z \in D$ such that*

$$\lambda_D(z)\,\mathrm{dist}(z, \partial D) \le 2K \frac{1+k}{2\pi - \theta}, \tag{3.6}$$

where $0 < k = k(\varepsilon, \theta) < 1$ *is chosen so that* $(2\pi - \theta)/\log(1/\varepsilon) = 2K/K'$; $K = K(k)$; *and* $K' = K(\sqrt{1-k^2})$.

Proof. By definition of the annular wedge condition and similarity invariance of $\lambda_D(z)\,\mathrm{dist}(z, \partial D)$, we may assume that there exist points $z_0 \in D$ and $w_0 = 0 \in \partial D$ with $|z_0| = 1 = \mathrm{dist}(z_0, \partial D)$ and such that either $(\mathbb{C} \setminus D) \cap \{\varepsilon \le |w| \le 1\}$ contains at most one point or $\Theta(0; w_1, w_2) < \theta$ for any two points $w_1, w_2 \in (\mathbb{C} \setminus D) \cap \{\varepsilon \le |w| \le 1\}$. Thus, using a rotation if necessary, we may assume that the annular wedge

$$B = \{\, re^{it} : \varepsilon < r < 1,\ |t| < \pi - \theta/2 \,\}$$

is contained in D. Domain monotonicity of the Poincaré metric in conjunction with Lemma 2.5 now yields

$$|z|\lambda_D(z) \le |z|\lambda_B(z) = 2K \frac{1+k}{2\pi - \theta}$$

at the center point $z = \sqrt{\varepsilon}$ of B, which establishes (3.6) since $0 \in \partial D$. □

It is now easy to demonstrate that a hyperbolic plane domain D that does not satisfy some annular wedge condition must have $\inf \lambda d \le 1/2$. Thus, a necessary condition for $\inf \lambda d > 1/2$ to hold is that D satisfy an annular wedge condition for *some* constants. More precisely, we obtain the following.

3.7. COROLLARY. *Suppose* $\tau = \inf \lambda d > 1/2$. *Then, for each* $0 < \theta < (2 - 1/\tau)\pi$, *there is an* $\varepsilon = \varepsilon(\theta)$, $0 < \varepsilon < 1$, *such that D satisfies an* (ε, θ)*-annular wedge condition*.

Proof. First note that, for fixed θ, the right-hand side of (3.6) tends to $\pi/(2\pi - \theta)$ as $\varepsilon \to 0$ because $k(\varepsilon, \theta) \to 0$ and $K \to \pi/2$. Now when $0 < \theta < (2 - 1/\tau)\pi$ we see that $\pi/(2\pi - \theta) < \tau$, so Theorem 3.5 guarantees that D must satisfy an (ε, θ)-annular wedge condition from some $0 < \varepsilon < 1$. □

We now prove that the annular wedge condition is sufficient for $\inf \lambda d > 1/2$ to hold, provided D is simply connected. It is *not* sufficient even in the doubly connected case. For example, the domain $D = \{\, z : |\arg(z)| < 3\pi/4 \,\} \cup \{\, z : e^{-2\pi} <$

$|z| < 1\}$ satisfies an $(\varepsilon, 3\pi/8)$-annular wedge condition when $\varepsilon > 0$ is sufficiently small, but at the point $z = e^{-\pi}$ we find that $\lambda_D(z)\,\mathrm{dist}(z, \partial D) < 1/2$.

We require the following result of Kuz′mina [K1, Thm. 1′].

3.8. FACT. *Let f be univalent in $\mathbb{B}$ and normalized by $f(0) = 0$. Suppose f does not assume the value 1 nor the value $w = a^{-2}e^{2i\alpha}$, where $0 < a \le 1$ and $|\alpha| \le \pi/2$. Then $|f'(0)| \le 1/h(a, \alpha)$, and this bound is sharp.*

The function $h(a, \alpha)$ is defined via theta functions, other elliptic functions, and various parameters determined by a nonlinear system of equations involving a and α. Since these equations provide no direct information regarding $h(a, \alpha)$, we refer the interested reader to [K1, p. 55] (and to [K2, pp. 77–80], where a detailed analysis is presented); here one also finds information regarding the extremal cases. However, we mention that $h(a, \alpha)$ can be realized as the logarithmic capacity (or transfinite diameter) of the extremal continuum that contains the points $0, 1, a^{-2}e^{2i\alpha}$ and has minimal capacity; see [K2, Chap. 1].

We make a few remarks concerning certain properties of this function. First, define $h(0, \alpha) = 1/4$ for $|\alpha| \le \pi/2$; then h is continuous on $[0, 1] \times [-\pi/2, \pi/2]$ with $1/4 \le h(a, \alpha) = h(a, -\alpha) \le 1/2$. Next, since Fact 3.8 is sharp, $h(a, \alpha) = 1/4$ if and only if either $a = 0$ or $\alpha = 0$. Thus, for $(b, \beta) \in (0, 1) \times (0, \pi/2)$ we have

$$H(b, \beta) = \min\{\, h(a, \alpha) : (a, |\alpha|) \in [b, 1] \times [\beta, \pi/2]\,\} > 1/4. \tag{3.9}$$

We are now in position to announce a converse to Theorem 3.5.

3.10. THEOREM. *Given $(\varepsilon, \theta) \in (0, 1) \times (0, \pi)$, there exist $b = b(\theta) \in (0, 1)$ and $\beta = \beta(\varepsilon, \theta) \in (0, \pi/2)$ such that, if D is a simply connected hyperbolic domain satisfying an (ε, θ)-annular wedge condition, then* $\inf \lambda\mathrm{d} \ge 2H(b, \beta)$, *where $H(b, \beta)$ is defined by* (3.9).

Proof. Fix $z \in D$ and choose $w \in \partial D$ so that $|z - w| = \mathrm{dist}(z, \partial D)$. By similarity invariance we may assume that $z = 0$ and $w = 1$. Then $\lambda_D(z) = 2/|f'(0)|$, where $f : \mathbb{B} \to D$ is conformal with $f(0) = z = 0$.

Suppose D satisfies an (ε, θ)-annular wedge condition. This guarantees the existence of points $w_j = 1 + r_j e^{i\theta_j} \in \mathbb{C}\setminus D \subset \mathbb{C}\setminus\mathbb{B}$ such that $\varepsilon \le r_j \le 1$ and $\Theta(w; w_1, w_2) \ge \theta$. Since $w_j \notin \mathbb{B}$, we may choose θ_j with $|\theta_j| \le 2\pi/3$. As $\Theta(w; w_1, w_2) \ge \theta$, one of the θ_j (say, θ_1) satisfies $|\theta_1| \ge \theta/2$. By symmetry, we may assume $\theta_1 \ge \theta/2$.

Next we exhibit constants $b = b(\theta)$ and $\beta = \beta(\varepsilon, \theta)$ that satisfy

$$0 < b \le a = |w_1|^{-1/2} \le 1 \quad \text{and} \quad 0 < \beta \le \alpha = |\arg \sqrt{w_1}\,| \le \pi/2.$$

The bounds on θ_1 and r_1 yield

$$1 \le |w_1| \le |1 + e^{i\theta/2}| = 2\cos(\theta/4);$$

consequently, the first inequality holds for $b = b(\theta) = [2\cos(\theta/4)]^{-1/2}$. Also, since $\alpha = \arg\sqrt{w_1} \ge \arg\big((1 + \varepsilon e^{i\theta/2})^{1/2}\big)$, we find that the second inequality is valid when $\beta = \beta(\varepsilon, \theta) = \frac{1}{2}\arctan\big(\varepsilon\sin(\theta/2)/[1 + \varepsilon\cos(\theta/2)]\big)$.

Finally, Fact 3.8 and (3.9) now permit us to assert

$$\lambda_D(z)\,\mathrm{dist}(z,\partial D) = 2/|f'(0)| \geq 2h(a,\alpha) \geq 2H(b,\beta),$$

which completes the proof. □

3.11. Corollary. *For a simply connected hyperbolic domain D*, $\inf \lambda \mathrm{d} > 1/2$ *if and only if D satisfies some annular wedge condition*.

Proof. The necessity follows from Corollary 3.7. The sufficiency is a consequence of Theorem 3.10 and (3.9). □

We close this section with an example illustrating the usefulness of Theorem 3.10. Consider $D = \mathbb{C}\setminus F$, where F is a fractal constructed as follows. Start with $[0,+\infty)\bigcup_{n\in\mathbb{Z}}[2^n(1-i), 2^n(1+i)]$. Add appropriate horizontal segments to the "end" of each vertical segment, and so on. In the limit we obtain a "feathery" closed connected set F consisting of $[0,+\infty)$ together with many vertical and horizontal line segments. We see that D satisfies some annular wedge condition, and thus $\inf \lambda \mathrm{d} > 1/2$.

4. The Supremum

First we verify that $\sup \lambda \mathrm{d} \geq k$ for any hyperbolic domain D, where $k > 0$ is an absolute constant. Hilditch [Hi] conjectures that we can take $k = \frac{1}{2}\lambda_{0,1}(\frac{1}{2})$, where $\lambda_{0,1} = \lambda_{\Omega_{0,1}}$ and $\Omega_{0,1} = \mathbb{C}\setminus\{0,1\}$. We exhibit precise values of k that are valid in certain domains.

Recall that D is a *Bloch* domain if $R(D) = \sup_{z\in D} \mathrm{dist}(z,\partial D)$ is finite. Minda [M2, Thm. 2] demonstrates that $2/R(D) \geq \Lambda(D) \geq 1/R(D)$, where $\Lambda(D) = \inf_{z\in D}\lambda_D(z)$; thus D is Bloch if and only if $\Lambda(D)$ is positive .

4.1. Proposition. *There is an absolute constant $k > 0$ such that* $\sup \lambda \mathrm{d} \geq k$ *for any hyperbolic domain D. If D is a Bloch domain, then* $\sup \lambda \mathrm{d} \geq 1$. *If D is a convex Bloch domain, then* $\sup \lambda \mathrm{d} \geq \pi/2$ *and this estimate is best possible. If D is a simply connected hyperbolic domain and $\Lambda(D)$ is attained in D, then* $\sup \lambda \mathrm{d} \geq 1.04176\ldots$

Proof. The asserted estimate for $\sup \lambda \mathrm{d}$ for simply connected domains in which $\Lambda(D)$ is attained is a consequence of a result due to Minda and Overholt [MO, Thm. 3]. Suppose that D is a Bloch domain. Then, according to [M2, Thm. 2], we have

$$\lambda_D(z)\,\mathrm{dist}(z,\partial D) \geq \Lambda(D)\,\mathrm{dist}(z,\partial D) \geq \mathrm{dist}(z,\partial D)/R(D)$$

for any $z\in D$. Letting $\mathrm{dist}(z,\partial D)\to R(D)$ yields $\sup \lambda \mathrm{d} \geq 1$. When D is also convex, [M2, Thm. 3] similarly furnishes $\sup \lambda \mathrm{d} \geq \pi/2$; see also [M1, Thm. 5]. The infinite strip Σ is a convex Bloch domain with $\sup \lambda_\Sigma \mathrm{d} = \pi/2$; see (2.4).

Now we consider a general hyperbolic domain D. Suppose first that D has an isolated boundary point w_0, and let w_1 be any point of $\partial D\setminus\{w_0\}$ closest to w_0. Put

$z_0 = (w_0 + w_1)/2$. Using a similarity transformation, we can assume that $w_0 = 0$ and $w_1 = 1$, so $z_0 = 1/2$. Then we see that $D \subset \Omega_{0,1} = \mathbb{C} \setminus \{0, 1\}$ and thus

$$\sup \lambda d \geq \lambda_D(z_0)\,\mathrm{dist}(z_0, \partial D) \geq (1/2)\lambda_{0,1}(1/2).$$

Next assume that no point of ∂D is isolated. Suppose that $w_0 = 0 \in \partial D$ is the closest point of ∂D to some point $z_0 \in D$. Since w_0 is not isolated, there is a point $w_1 \in \partial D \cap B(w_0; |z_0|)$, $w_1 \neq w_0$. Let $z_1 = |w_1/z_0| z_0$. Using a similarity transformation, we can assume that $z_1 = -1$. Let D^* be the circular symmetrization of D with respect to the positive real axis. Then $D^* \subset \Omega_{0,1}$ and so (by [Ha, Thm. 4.8] or [W]) we obtain

$$\sup \lambda d \geq \lambda_D(z_1)\,\mathrm{dist}(z_1, \partial D) = \lambda_D(z_1) \geq \lambda_{D^*}(z_1) \geq \lambda_{0,1}(-1).$$

Hence, in all cases $\sup \lambda d \geq k = \lambda_{0,1}(-1) = 0.22847\dots$ □

It is true that equality holds in (1.1) at a point z if and only if D is the disk $B(z; \mathrm{dist}(z, \partial D))$. However, as our next result indicates, there are plenty of nondisk domains D with $\sup \lambda d = 2$.

4.2. Theorem. *A hyperbolic domain D satisfies* $\sup \lambda d < 2$ *if and only if there exists a constant $a > 0$ such that, for each point $z \in D$, there is a point $\zeta \in D$ with $|\zeta - z| = \mathrm{dist}(z, \partial D) \leq (1/a)\,\mathrm{dist}(\zeta, \partial D)$.*

Proof. First, we verify the sufficiency; assume such a constant a exists. Observe that $a \leq 2$. If $a = 2$ then, by taking an increasing union of disks, we deduce that for each $z \in D$ there is a half-plane H with $B(z; \mathrm{dist}(z, \partial D)) \subset H \subset D$, and therefore $\sup \lambda d \leq 1$. Suppose $a < 2$. Fix $z_0 \in D$, set $d = \mathrm{dist}(z_0, \partial D)$, and choose $\zeta_0 \in D$ with $|\zeta_0 - z_0| = d \leq (1/a)\,\mathrm{dist}(\zeta_0, \partial D)$. Then $B = B(z_0; d) \cup B(\zeta_0; ad) \subset D$. Similarity invariance of $\lambda_D(z)\,\mathrm{dist}(z, \partial D)$ allows us to assume that $z_0 = 0$, $d = 1$, and $\zeta_0 = 1$. In order to calculate $\lambda_B(0)$, we map B conformally onto the upper half-plane via $z \mapsto w$, where

$$w = \left(\frac{z - e^{i\theta}}{e^{i\theta} z - 1}\right)^p, \quad p = \frac{2\pi}{3\pi - \theta}, \quad \text{and} \quad \theta = 2\arcsin\left(\frac{a}{2}\right).$$

Then we evaluate

$$\lambda_B(z) = \frac{1}{\Im(w)}\left|\frac{dw}{dz}\right| \quad \text{at } z = 0,\ w = e^{ip\theta}$$

to obtain $\lambda_B(0) = 2p\sin(\theta)/\sin(p\theta) < 2$. Domain monotonicity of the Poincaré metric yields $\lambda_D(0)d \leq \lambda_B(0)$, which, in conjunction with Fact 2.2, produces

$$\sup \lambda d \leq 2p\sin(\theta)/\sin(p\theta) < 2;$$

notice that this (strictly decreasing) bound on $\sup \lambda d$ depends only on a.

In the opposite direction, suppose there exist points $z_n \in D$ with the property that, for all points $\zeta \in D$ with $|\zeta - z_n| = d_n = \mathrm{dist}(z_n, \partial D)$, we always have $\mathrm{dist}(\zeta, \partial D) \leq d_n/n$. We claim that $\lambda_D(z_n)d_n \to 2$ as $n \to \infty$. Since $D \subset G_n = \mathbb{C} \setminus (\partial D \cap \bar{B}(z_n; (1+1/n)d_n))$, it suffices to show that $\lambda_{G_n}(z_n)d_n \to 2$ as $n \to \infty$.

Consider the image H_n of G_n under the similarity transformation $w = (z - z_n)/d_n$. Observe that the kernel of $\{H_n\}$ with respect to the origin is the unit disk $\mathbb{B}$ and, moreover, that $H_n \to \mathbb{B}$. Appealing to Fact 2.1, we obtain

$$\lambda_{G_n}(z_n)d_n = \lambda_{H_n}(0) \to \lambda_{\mathbb{B}}(0) = 2,$$

as desired. □

It would be useful to have a quantitative estimate for the constant a in terms of $\sup \lambda \mathrm{d}$. The difficulty in obtaining such information stems from allowing D to be a completely arbitrary hyperbolic domain.

With a view toward later applications, we now examine a condition that in some sense is dual to the quasiconvexity condition (3.2). We consider when it is possible to join points z_1, z_2 by a rectifiable arc γ in D satisfying

$$\min\{\ell(\gamma(\zeta, z_1)), \ell(\gamma(\zeta, z_2))\} \le c\, \mathrm{dist}(\zeta, \partial D) \quad \text{for all } \zeta \in \gamma \tag{4.3}$$

for some constant $c \ge 1$. Here $\gamma(\zeta, z)$ denotes the subarc of γ between ζ, z. We can view (4.3) as describing a *curvilinear double wedge* joining the points z_1, z_2 in D.

We also utilize the geometric quantity

$$b(D) = \inf_{z \in D} \sup_{\zeta \in D} \frac{|z - \zeta|}{\mathrm{dist}(z, \partial D)},$$

which enjoys the following properties.

4.4. LEMMA. *The quantity $b = b(D)$ satisfies $1 \le b \le \infty$, where $b = \infty$ if and only if D is unbounded and $b = 1$ if and only if D is a disk.*

Proof. We verify the last assertion. Assume D is bounded, but suppose that for each positive integer n there exists a point $z_n \in D$ with $D \subset B(z_n; (1 + 1/n)d_n)$, where $d_n = \mathrm{dist}(z_n, \partial D)$. Note that $\mathrm{diam}(D) \le 2(1+1/n)d_n$ for all n. Passing to subsequences, we can assume that $z_n \to z_0$ and $d_n \to d_0$ as $n \to \infty$. Appealing to the continuity of $\mathrm{dist}(z, \partial D)$, we find that $\mathrm{dist}(z_0, \partial D) = d_0 \ge \mathrm{diam}(D)/2 > 0$, so in particular $z_0 \in D$. We assert that $D = B(z_0; d_0)$. For if $z \in D$ and $\varepsilon > 0$, then for n sufficiently large we obtain

$$|z - z_0| \le (1 + 1/n)d_n + |z_n - z_0| \le (1 + \varepsilon)(d_0 + \varepsilon) + \varepsilon;$$

letting $\varepsilon \to 0$ yields $|z - z_0| \le d_0$, whence $D \subset \bar{B}(z_0; d_0)$. □

Here is an analog of Proposition 3.3.

4.5. PROPOSITION. *Suppose there exists a constant $c \ge 1$ such that each pair of points $z_1, z_2 \in D$ can be joined by an arc $\gamma \subset D$ satisfying* (4.3). *Then either D is a disk or* $\sup \lambda \mathrm{d} \le \sigma < 2$, *where σ depends only on c and possibly on $b(D)$.*

Proof. We assume that D is not a disk, so $b = b(D) > 1$. Fix a point $z_1 \in D$ and set $d = \mathrm{dist}(z_1, \partial D)$. Select a point $z_2 \in \bar{D}$ with $|z_2 - z_1| \ge bd$. According to

[NV, 2.19], there is an arc $\gamma \subset D$ joining z_1 and z_2 with (4.3) valid. Pick a point $\zeta \in \gamma \cap \partial B(z_1; d)$. Now

$$\ell_1 = \ell(\gamma(\zeta, z_1)) \geq d \quad \text{and} \quad \ell_2 = \ell(\gamma(\zeta, z_2)) \geq (b-1)d,$$

so employing condition (4.3) we obtain $\operatorname{dist}(\zeta, \partial D) \geq ad$, where $a = \min\{1, b-1\}/c$. Thus Theorem 4.2 yields $\sup \lambda \mathrm{d} \leq \sigma(c, b) < 2$. □

5. Quasidisks

Here we apply Propositions 3.3 and 4.5 to verify that nonround quasidisks satisfy both $\inf \lambda \mathrm{d} > 1/2$ and $\sup \lambda \mathrm{d} < 2$. Then we characterize unbounded convex quasidisks in terms of sharp estimates on $\inf \lambda \mathrm{d}$ and $\sup \lambda \mathrm{d}$.

Väisälä [V, 2.21] demonstrates that condition (4.3) describes the class of *John domains* (see also [NV, 2.14; P3, pp. 96–102]). An especially important proper subclass of the John domains are the *uniform* domains, which also enjoy the quasiconvex property (3.2). Martio and Sarvas coined this terminology to describe concepts introduced by John. In general, a simply connected hyperbolic John domain D need not satisfy $\inf \lambda \mathrm{d} > 1/2$. However, a simply connected hyperbolic domain is uniform if and only if it is a quasidisk; see [NV, 9.2; P3, Chap. 5]. Thus we obtain the following consequences of Propositions 3.3 and 4.5.

5.1. Corollary. *Every John or uniform domain D is either a disk or satisfies* $\sup \lambda \mathrm{d} < 2$. *In addition, every quasidisk D satisfies* $\inf \lambda \mathrm{d} > 1/2$.

In particular, every *unbounded* quasidisk enjoys both $1/\rho \leq \inf \lambda \mathrm{d}$ and $\sup \lambda \mathrm{d} \leq \rho$, where $\rho = \rho(c) < 2$ and c is the constant in (3.2) and (4.3). We conclude this work by further investigating this situation. We shall write Θ for the inverse of the function $F(\theta) = \sin(\theta)/\theta$, $0 \leq \theta \leq \pi/2$. Notice that Θ decreases from $\Theta(2/\pi) = \pi/2$ to $\Theta(1) = 0$. Our description for unbounded convex quasidisks is in terms of the following estimate on the hyperbolic metric:

$$1 \leq \lambda_D(z) \operatorname{dist}(z, \partial D) \leq \sigma \quad \text{for all } z \in D. \tag{5.2}$$

We leave the proof of the following to the interested reader.

5.3. Lemma. *If D is unbounded and convex, then each point of D is the endpoint of some infinite ray in D.*

5.4. Theorem. *Suppose* (5.2) *holds with $\sigma < \pi/2$. Then D is an unbounded convex domain and each point of D is the vertex of an infinite wedge in D with apex angle $\theta \geq \Theta(2\sigma/\pi)$ (so $\sin(\theta)/\theta \leq 2\sigma/\pi$). Conversely, if D is an unbounded convex domain and each point of D is the vertex of an infinite wedge in D with apex angle θ, then* (5.2) *holds with $\sigma = \sigma(\theta) < \sigma(0)$ and $\sin(\theta)/\theta \leq 2\sigma/\pi \leq 1/\theta$.*

5.5. Remarks. (a) For unbounded convex domains, the infinite wedge condition is equivalent to the domain being a quasidisk. (b) The infinite strip example shows that $\pi/2$ is sharp for the wedge condition (of course, 1 is sharp for convexity); see (2.4). (c) According to (2.3), the estimate $\sin(\theta)/\theta \le 2\sigma/\pi$ gives best possible lower bounds both for θ in terms of σ and for σ in terms of θ. (d) The constant $\sigma(\theta)$ is simply $\lambda_{\Delta_\alpha}(0)$, where Δ_α is the convex hull of $\mathbb{B} \cup \Omega_\alpha$ and $\theta = \alpha\pi$. Also, $\Delta_0 = \mathbb{B} \cup \{\, x + iy : x > 0,\ |y| < 1 \}$, and we see that as θ decreases to 0, $\sigma(\theta)$ increases to $\sigma(0)$ and, since $\Delta_0 \subset \Sigma$, Fact 2.2 forces $\sigma(0) > \pi/2 = \lambda_\Sigma(0)$. (e) For the second half of 5.4, equality holds at some point z in (5.2) if and only if either D is a half-plane or D is affine equivalent to Ω_α.

Proof of Sufficiency. We assume that the hyperbolic metric in D satisfies (5.2) with $\sigma < \pi/2$. Then D is convex by Fact 3.1 and hence non-Bloch (so unbounded) according to Proposition 4.1. Next, we verify existence of the infinite wedges. Last, we estimate the apex angles.

Fix an arbitrary point $z_0 \in D$. Assume $z_0 = 0$ and $\operatorname{dist}(z_0, \partial D) = 1$; thus $\mathbb{B} \subset D$. By Lemma 5.3, D contains an infinite ray from z_0, which we assume to be the positive real axis $\mathbf{R}_+$; so D contains the convex hull of $\mathbb{B} \cup \mathbf{R}_+$ (which is Δ_0). Notice that, by convexity, if w_1 and w_2 are points of ∂D with, say, $0 < \Re(w_1) < \Re(w_2)$, then necessarily either $0 < \Im(w_1) \le \Im(w_2)$ or $0 > \Im(w_1) \ge \Im(w_2)$.

We proceed to verify that z_0 is the vertex of an infinite wedge in D. Fix $x \in \mathbf{R}_+$ and consider the vertical line $L = \{\Re(z) = x\}$. If L meets no point of ∂D in the upper half-plane, then D contains the first quadrant, which is an infinite wedge; a similar conclusion holds if L meets no point of ∂D in the lower half-plane. Thus we may assume that there are $y_\pm = y_\pm(x)$ with $x + iy_+$ and $x + iy_-$ points of $L \cap \partial D$ in the upper and lower half-planes, respectively. Since D is non-Bloch, we must have

$$d(x) = \max\{y_+(x), -y_-(x)\} \to \infty \quad \text{as } x \to \infty.$$

Notice that D contains an infinite wedge if and only if $x/d(x)$ is bounded as $x \to \infty$.

Suppose $x/d(x) \to \infty$ as $x \to \infty$. Put

$$G_x = \{\Re(z) < x,\ y_-(x) < \Im(z) < y_+(x)\} \cup \{\Re(z) > x\}$$

and let $z(x) = (x - ah) + iy$, where

$$a = a(x) = \log \frac{x}{d(x)}, \quad h = h(x) = \frac{y_+ - y_-}{2}, \quad y = y(x) = \frac{y_+ + y_-}{2}.$$

Under the change of variables $w = (z - z(x))/h(x)$, we see that G_x is mapped onto a domain H_x, $z(x)$ corresponds to $w = 0$, and $x + iy(x)$ corresponds to $a(x)$. Since $a(x) \to \infty$ as $x \to \infty$, we find that H_x converges to the infinite strip Σ with respect to the origin, in the sense of kernel convergence. Now $G_x \supset D$, so

$$\begin{aligned}\lambda_D(z(x)) \operatorname{dist}(z(x), \partial D) &\ge \lambda_{G_x}(z(x)) \operatorname{dist}(z(x), \partial D) \\ &= \lambda_{H_x}(0) \operatorname{dist}(z(x), \partial D)/h(x).\end{aligned}$$

We claim that $\limsup_{x\to\infty} \operatorname{dist}(z(x), \partial D)/h(x) \ge 1$, and thus—by Fact 2.1 and (2.4)—we deduce that $\sup \lambda\mathrm{d} \ge \pi/2$, which contradicts our hypotheses. (To check our claim: Assume that $d = d(x) = y_+$. Then $\operatorname{dist}(z(x), \partial D) \geq t = (x - ad)h/\sqrt{x^2 + d^2}$, where t is the distance from $z(x)$ to the line through z_0 and $x + iy_+(x)$.)

We have now established that D must contain an infinite wedge with vertex z_0; it remains to estimate the apex angle of the largest such wedge. Using another rotation, if necessary, we may assume that Ω_α is the largest wedge contained in D with vertex z_0, where $\theta = \alpha\pi$. This means that $\{ re^{\pm i(\theta+\varepsilon)} : r \geq 0 \} \cap \partial D \neq \emptyset$ for all small $\varepsilon > 0$. We show that, as $x \to \infty$, $\lambda_D(x) \operatorname{dist}(x, \partial D)$ is asymptotically equal to $\lambda_{\Omega_\alpha}(x) \operatorname{dist}(x, \partial\Omega_\alpha) = (\pi/2)(\sin\theta/\theta)$ (see (2.3)).

Take $\varepsilon = \pi/n$, and let u_n and v_n be the "first" points of $\partial D \cap \{ re^{-i(\theta+\pi/n)} : r \geq 0 \}$ and $\partial D \cap \{ re^{i(\theta+\pi/n)} : r \geq 0 \}$, respectively. Put $x_n = \max\{\Re(u_n), \Re(v_n)\}$ and $y_n = \max\{-\Im(u_n), \Im(v_n)\}$, so $\tan(\theta + \pi/n) = y_n/x_n$ and either $x_n + iy_n = \bar{u}_n$ or $x_n + iy_n = v_n$. We assume that $x_n \to \infty$, for otherwise we easily conclude that $D = \Omega_\alpha$.

Using convexity again we see that $D \subset G_n$, where now

$$G_n = \Omega_{\alpha+1/n} \cup \{ x + iy : x < x_n,\ |y| < y_n \}.$$

Let $z_n = x_n^2$ and $d_n = \operatorname{dist}(z_n, \partial G_n)$; notice that $\operatorname{dist}(z_n, \partial D)/d_n \to 1$ as $n \to \infty$. Consider the change of variable $w = (z - z_n)/d_n$; G_n is mapped onto a domain H_n with z_n and x_n corresponding to 0 and $(x_n - x_n^2)/d_n$, respectively. Now $y_n/d_n \to 0$ and $(x_n - z_n)/d_n \to -\csc\theta$, from which we deduce that $H_n \to \Omega_\alpha - \csc\theta = \{ z - \csc\theta : z \in \Omega_\alpha \}$ with respect to the origin, in the sense of kernel convergence. All of this in conjunction with Fact 2.1, (2.3), and

$$\lambda_D(z_n) \operatorname{dist}(z_n, \partial D) \geq \lambda_{G_n}(z_n) \operatorname{dist}(z_n, \partial D) = \lambda_{H_n}(0) \operatorname{dist}(z_n, \partial D)/d_n$$

yields $\sin\theta/\theta \leq 2\sigma/\pi$, as desired. □

Proof of Necessity. Now we assume that D is an unbounded convex domain that enjoys the infinite wedge condition for some apex angle $\theta = \alpha\pi$. Since D is convex, the lower bound for $\lambda_D(z) \operatorname{dist}(z, \partial D)$ follows from Fact 3.1; we establish an upper bound and provide the indicated estimates.

Fix $z_0 \in D$. Assume $z_0 = 0$, $\operatorname{dist}(z_0, \partial D) = 1$, and Ω_α is the infinite wedge joining z_0 to infinity in D. Then the convex hull $\Delta = \Delta_\alpha$ of $\mathbb{B} \cup \Omega_\alpha$ is contained in D, and

$$\lambda_D(z_0) \operatorname{dist}(z_0, \partial D) = \lambda_D(z_0) \leq \lambda_\Delta(0) = \sigma(\theta).$$

It remains to estimate $\sigma = \sigma(\theta)$; for this we utilize a result [MW, Thm. 2] of Minda and Wright which asserts that $1/\lambda_D$ is concave on lines in D when D is convex. To obtain an upper bound for σ we write $0 = (1-t)y + tx$, where $-1 < y < 0$, $x > 0$, and $0 < t < 1$; solve for t; and then let $y \to -1$ and $x \to \infty$. Since $\lambda_\Delta(x)$ is asymptotic to $[2\alpha(x + c)]^{-1}$ as $x \to \infty$ (use kernel convergence), we obtain

$$\frac{1}{\lambda_\Delta(0)} \geq \frac{1-t}{\lambda_\Delta(y)} + \frac{t}{\lambda_\Delta(x)} \geq 2\alpha t(x + c) \to 2\alpha$$

and hence $2\sigma/\pi \leq 1/\theta$; here $c = \csc\theta$.

For the lower bound we write $y = \log x = (1-t)0 + tx$ (so $0 < t < 1$); let $x \to \infty$; and again use the fact that $\lambda_\Delta(x)$ is asymptotic to $[2\alpha(x+c)]^{-1}$ as $x \to \infty$, where $c = \csc\theta$. We find that

$$\begin{aligned}\frac{1}{\lambda_\Delta(0)} &\le \frac{1}{1-t}\left(\frac{1}{\lambda_\Delta(y)} - \frac{t}{\lambda_\Delta(x)}\right) = \frac{1}{x-y}\left(\frac{x}{\lambda_\Delta(y)} - \frac{y}{\lambda_\Delta(x)}\right)\\ &= \frac{2\alpha}{x-y}\left(\frac{xy+cx}{q} - \frac{xy+cy)}{p}\right)\\ &= 2\alpha\left[\frac{xy}{x-y}\left(\frac{1}{q}-\frac{1}{p}\right) + c\left(\frac{x}{x-y}\frac{1}{q} - \frac{y}{x-y}\frac{1}{p}\right)\right],\end{aligned}$$

where $p = 2\alpha(x+c)\lambda_\Delta(x)$ and $q = 2\alpha(y+c)\lambda_\Delta(y)$. Since p and q both tend to 1 as $x \to \infty$, we see that $1/\lambda_\Delta(0) \le 2\alpha c$ and so $\sin\theta/\theta \le 2\sigma/\pi$ as desired. □

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