Equivariant Chern Character for the Invariant Dirac Operator

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1. Introduction

The attributes of elliptic pseudodifferential systems that have relevance within Atiyah–Singer index theory have been abstracted to the concept of Fredholm modules, or K-cycles in the language of K-homology. Pairing of a K-cycle with a K-cocycle gives the index. Connes invented cyclic cohomology while looking for a homology–cohomology formula of this pairing. The Chern–Connes character relates finitely summable Fredholm modules to cyclic cocycles and, more generally, relates Θ-summable Fredholm modules to entire cyclic cocycles. We refer to [5], and to [4] for the background and definitions.

The prototype for Θ-summable Fredholm modules is the Dirac K-cycle $(\mathcal{A}, \mathcal{H}, \mathrm{D})$, where $\mathcal{A} = C^\infty(M)$ for an even-dimensional Spin-manifold M, $\mathcal{H}$ is the graded Hilbert space of L^2 sections of the spinor bundle, and D is the Dirac operator, unbounded on $\mathcal{H}$, which is Θ-summable by Weyl's asymptotics for heat kernels. The Chern character is represented by the JLO-cocycle in the entire cyclic cohomology of $\mathcal{A}$. The Chern character has been computed in [2] by using symbol calculus [8] since $e^{-t^2\mathrm{D}^2}$ is a Getzler's asymptotic operator for which the leading symbol is known [2; 8; 16; 15]. In this paper we discuss the equivariant case; if G is a connected compact Lie group acting on M by isometries, then G-action commutes with Clifford multiplication and D is G-invariant, so we have an equivariant Θ-summable Fredholm module over the G-algebra $\mathcal{A}$. The corresponding equivariant Chern character in the equivariant entire cyclic cohomology (JLO-version) was defined in [10] and [13]. Apparently Getzler's symbol calculus does not apply directly here since $e^{-t^2\mathrm{D}^2} \cdot T$ is not an asymptotic operator in the sense of [2] for an orientation-preserving isometry T on M.

On the other hand, for classical geometric operators, direct heat-kernel asymptotic techniques were applied to the Hodge–de Rham and Riemann–Roch operators by Patodi and later improved by Yu [17] into a more essential form, called Clifford asymptotics for heat kernels. Using this technique, we are able to derive a cyclic cohomological formula for the equivariant Chern character for the equivariant entire cyclic cohomology.

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In fact, the Dirac K-cycle is finitely summable, so one would naturally consider the Chern character in the equivariant cyclic cohomology. The reason we consider the entire cyclic cohomology first is to take advantage of the JLO-cocycle. In [6], Connes and Moscovici introduced a universal local index formula, based on (generalized) Wodzicki residues, for the Chern character in the cyclic cohomology for finitely summable Fredholm modules. This formula has implications in many areas, such as quantization and index theorem on infinite-dimensional manifolds. By the results in [6] and using the Mellin transform, one can reduce the computation of the Chern character in the equivariant cyclic cohomology to its entire equivalent. Thus we also give a formula for the equivariant Chern character in the equivariant cyclic cohomology.

In Section 2 we recall the definition (JLO-version) of Chern character in the entire cyclic cohomology by [13; 12]. We then derive, from the JLO-version, an asymptotic expansion formula that combines many heat operators into one so we can handle it with classical heat-kernel asymptotic techniques. We are indebted to H. Moscovici for this formula in its original form for the nonequivariant case, though our proof uses a different approach. In Section 3 we apply the Clifford asymptotics method to the Chern character formula. Finally, in Section 4 we give the cohomological formula for the equivariant Chern character, both in the equivariant entire cyclic cohomology and in the equivariant cyclic cohomology.

2. The Equivariant Chern Character

The definitions of equivariant (finitely or Θ-summable) Fredholm modules and of equivariant cyclic cohomology are discussed in [13; 12; 10]. For the convenience of the reader we will reproduce the definitions here.

DEFINITION 2.1. Let G be a compact Lie group. A G-equivariant Θ-summable Fredholm module is a triple $(\mathcal{A}, \mathcal{H}, \mathrm{D})$, where the following conditions hold.

(i) $\mathcal{A} = (\mathcal{A}, \tau, G)$ is a unital G-Banach algebra; that is, G acts on $\mathcal{A}$ by continuous automorphisms $\tau: G \to \mathrm{Aut}(\mathcal{A})$ such that $\tau(g)$ is unitary for all $g \in G$.

(ii) $\mathcal{H} = (\mathcal{H}, \gamma, \rho, G)$ is a graded Hilbert space with γ denoting the grading operator, and $\rho: G \to \mathcal{L}(H)$ is an even-graded unitary representation of G; the induced G-action on $\mathcal{L}(H)$ is $\rho_*(g)P = \rho(g)P\rho(g)^{-1}$.

(iii) Moreover, $\mathcal{H}$ is an even-graded G-equivariant $\mathcal{A}$ module; there is a continuous action $\mu: \mathcal{A} \to \mathcal{L}(H)$ such that $\mu(\tau(g)a) = \rho_*(g)(a)$.

(iv) D is an unbounded odd-graded self-adjoint operator on $\mathcal{H}$ which is G-invariant, that is, G commutes with $\rho(g)$ for any $g \in G$.

(v) For any $a \in \mathcal{A}$, $[\mathrm{D}, \mu(a)]$ is densely defined, extending to a bounded operator on $\mathcal{H}$, and there is a constant $N(\mathrm{D})$ such that

$$\|\mu(a)\| + \|[\mathrm{D}, \mu(a)]\| \leq N(\mathrm{D})\|a\|_{\mathcal{A}}.$$

(vi) $\mathrm{tr}(e^{-\mathrm{D}^2}) < \infty$.

DEFINITION 2.2. A triple $(\mathcal{A}, \mathcal{H}, \mathrm{D})$ is a G-equivariant p-summable Fredholm module if we replace (vi) in Definition 2.1 by

(vi′) $(\mathrm{D}+i)^{-1}$ is in $\mathcal{L}^p(\mathcal{H})$, the Schatten–von Neumann ideal of compact operators.

DEFINITION 2.3. For a unital G-Banach algebra $\mathcal{A}$, we define:

(i) $C^n_G(\mathcal{A}) = \mathrm{Hom}_G(\mathcal{A}^{\tilde{\otimes}(n+1)}, C(G))$, where $\tilde{\otimes}$ is the projective tensor product. We set $C^n_G = 0$ for $n < 0$, with the induced action τ_* of G,

$$\tau_*(g)\phi(a^0, \dots, a^n)(h) = \phi(\tau(g)^{-1}a^0, \dots, \tau(g)^{-1}a^n)(\tau(g)^{-1}h\tau(g));$$

(ii) $b\colon C^n_G(\mathcal{A}) \to C^{n+1}_G(\mathcal{A})$, where, for any $\phi \in C^n_G(\mathcal{A})$,

$$b\phi(a^0, \dots, a^{n+1}) = \sum_{j=0}^{n} (-1)^n (a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1}(a^{n+1}a^0, \dots, a^n);$$

(iii) $B\colon C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$, $B\phi = AB_0\phi$, where, for any $\phi \in C^n_G(\mathcal{A})$ and $\psi \in C^{n-1}_G(\mathcal{A})$,

$$B_0\phi(a^0, \dots, a^{n-1}) = \phi(1, a^0, \dots, a^{n-1}) - (-1)^n(a^0, \dots, a^{n-1}, 1),$$

$$A\psi(a^0, \dots, a^{n-1}) = \sum_{j=0}^{n-1} \psi(a^j, a^{j+1}, \dots, a^{j-1});$$

(iv) $HC^n_G(\mathcal{A}) = H^n(C^{*,*}, b, B)(\mathcal{A})$, where $C^{n,m} = C^{n-m}$ is a (b, B)-bicomplex.

Before we define the G-equivariant entire cyclic cohomology, note that the norm on C^n_G is the restriction of the norm on $\mathrm{Hom}(\mathcal{A}^{\tilde{\otimes}(n+1)}, C(G))$:

$$\|\phi\| = \sup_{g \in G;\, a^i \in \mathcal{A}} |\phi(a^0, \dots, a^n)(g)|;$$

moreover, as in the nonequivariant case, on C^n_G we have $\|b\| \le n+2$ and $\|B\| \le 2n$.

DEFINITION 2.4. For a unital G-Banach algebra $\mathcal{A}$,

(i) an even G-equivariant cochain (ϕ_{2n}) is

$$(\phi_{2n}) \in C^{\mathrm{even}}_G(\mathcal{A}) = \{ (\phi_{2n})_{n \in N} \mid \phi_{2n} \in C^{2n}_G(\mathcal{A}) \}.$$

It is entire if the radius of convergence of $\sum \|\phi_{2n}\| z^n/n!$ is infinity;

(ii) an odd G-equivariant cochain (ϕ_{2n+1}) is

$$(\phi_{2n+1}) \in C^{\mathrm{odd}}_G(\mathcal{A}) = \{ (\phi_{2n+1})_{n \in N} \mid \phi_{2n+1} \in C^{2n+1}_G(\mathcal{A}) \}.$$

It is entire if the radius of convergence of $\sum \|\phi_{2n+1}\| z^n/n!$ is infinity.

(iii) The G-equivariant entire cyclic cohomology of $\mathcal{A}$ is the cohomology of the short complex

$$C_G^{\mathrm{even}}(\mathcal{A}) \leftrightarrow C_G^{\mathrm{odd}}(\mathcal{A}),$$

where the boundary operator $\partial = d_1 + d_2$ with $d_1 = (n+1)b$ and $d_2 = B/n$.

NOTATION. For the rest of this paper, let G be a connected compact Lie group acting on a closed even dimensional Spin-manifold M by isometries, $\mathcal{A} = C^\infty(M)$, and let D be the G-invariant Dirac operator. We omit the standard representations.

When the K-cycle $(\mathcal{A}, \mathcal{H}, \mathrm{D})$ is fixed, for a bounded operator B on $\mathcal{H}$ let $\nabla^0 B = B$ and $\nabla^l B = [\mathrm{D}^2, \nabla^{l-1} B]$ inductively. We also use the notation of [11] as follows:

$$B(s) = e^{-s\mathrm{D}^2} B e^{s\mathrm{D}^2}. \tag{1}$$

DEFINITION 2.5. The equivariant Chern character $\mathrm{ch}_*^G(\mathcal{A}, \mathcal{H}, \mathrm{D}) = \{\mathrm{ch}_k^G(\mathrm{D})\}_{k\geq 0}$ in the equivariant entire cyclic cohomology is defined by

$$\begin{aligned}
&\mathrm{ch}_k^G(\mathcal{A}, \mathcal{H}, \mathrm{D})(a^0, a^1, \dots, a^k)(g) \\
&\quad = \int_{\Delta_k} \mathrm{str}\big(a^0[\mathrm{D}, a^1](s_1)[\mathrm{D}, a^2](s_2 - s_1) \\
&\qquad\qquad \cdots [\mathrm{D}, a^k](s_k - s_{k-1}) g\big) e^{-\mathrm{D}^2}\, ds_1\, ds_2 \cdots ds_k \\
&\quad = \int_{\Delta_k} \mathrm{str}\big(a^0 e^{-s_1\mathrm{D}^2}[\mathrm{D}, a^1] e^{-(s_2-s_1)\mathrm{D}^2}[\mathrm{D}, a^2] \\
&\qquad\qquad \cdots e^{-(s_k-s_{k-1})\mathrm{D}^2}[\mathrm{D}, a^k] e^{-(1-s_k)\mathrm{D}^2} g\big)\, ds_1\, ds_2 \cdots ds_k,
\end{aligned} \tag{2}$$

where $a^i \in \mathcal{A}$, $g \in G$, and $\Delta_k = \{\, (s_1, s_2, \dots, s_k) \mid 0 \leq s_1 \leq s_2 \leq \cdots \leq s_k \leq 1 \,\}$.

REMARK 2.6. (i) $\mathrm{ch}_*^G(\mathrm{D}) = \{\mathrm{ch}_k^G(\mathrm{D})\}_{k=\mathrm{even}}$ defines an equivariant entire cyclic cocycle in [13] and [10], which extends the *JLO* cocycle [11] to the equivariant situation.

(ii) If k is odd, $\mathrm{ch}_k^G(\mathrm{D})(a^0, a^1, \dots, a^k)(g) = 0$ since inside the supertrace is an odd operator.

(iii) If we introduce a parameter t to replace D^2 by $t\mathrm{D}^2$, then we obtain an equivariant entire cyclic cocycle $\mathrm{ch}_*^G(\sqrt{t}\mathrm{D}) = \{\mathrm{ch}_k^G(\sqrt{t}\mathrm{D})\}_{k\geq 0}$ defined by

$$\begin{aligned}
&\mathrm{ch}_k^G(\sqrt{t}\mathrm{D})(a^0, a^1, \dots, a^k)(g) \\
&\quad = t^{k/2} \int_{\Delta_k} \mathrm{str}\big(a^0[\mathrm{D}, a^1](s_1 t)[\mathrm{D}, a^2]((s_2 - s_1)t) \\
&\qquad\qquad \cdots [\mathrm{D}, a^k]((1 - s_k)t) e^{-t\mathrm{D}^2} g\big)\, ds_1\, ds_2 \cdots ds_k;
\end{aligned} \tag{3}$$

this cocycle is cohomologous to $\mathrm{ch}_*^G(\mathrm{D})$ in the equivariant entire cyclic cohomology [5; 10].

(iv) We would like to remind the reader that the notation for ch_* may appear as ch^* in other works. We use Connes' notation because, in our case, $\mathcal{A}$ is conceived as a space in noncommutative geometry instead of an abstract algebra, the cyclic cocycles are viewed as closed de Rham currents, and the *JLO*-cocycle is viewed as the local index density for D. In this sense, we are calculating G-equivariant local index for the Dirac operator.

LEMMA 2.7.

$$B(s) = \sum_{l=0}^{N-1} \frac{(-1)^l s^l}{l!} \nabla^l B + s^N \nabla_R^N B(s),$$

where $\nabla_R^N B(s)$ *is given by*

$$\nabla_R^N B(s) = (-1)^N \int_{\Delta_N} \nabla^N B(t_1 s)\, dt_1\, dt_2 \cdots dt_N. \tag{4}$$

Proof. From

$$\frac{d}{du} B(s) = \frac{d}{du}(e^{-us\mathrm{D}^2} B e^{us\mathrm{D}^2}) = -se^{-us\mathrm{D}^2} \nabla(B) e^{us\mathrm{D}^2} = -s\nabla B(us)$$

it follows that

$$B(s) = B - s \int_0^1 \nabla B(us)\, du,$$

which verifies the lemma for $N = 1$. Repeating the above formula, we prove this lemma by induction:

$$\begin{aligned}
& s^N \nabla_R^N B(s) \\
&= (-1)^N \int_{\Delta_N} s^N \nabla^N B(t_1 s)\, dt_1\, dt_2 \cdots dt_N \\
&= (-1)^N \int_{\Delta_N} s^N \Big\{ \nabla^N(B) - \int_0^1 s \nabla^{N+1} B(u t_1 s)\, du \Big\}\, dt_1\, dt_2 \cdots dt_N \\
&= (-1)^N \int_{\Delta_N} s^N \nabla^N B\, dt_1\, dt_2 \cdots dt_N \\
&\quad + (-1)^{N+1} \int_{0 \le t_1 \le t_2 \le \cdots \le t_N \le 1} \int_0^{t_1} s^{N+1} \nabla^{N+1} B(t_0 s)\, dt_0\, dt_1\, dt_2 \cdots dt_N \\
&= \frac{(-1)^N}{N!} s^N \nabla^N B + s^{N+1} \nabla_R^{N+1} B(s).
\end{aligned}$$

□

Now, if we write $[\mathrm{D}, a]$ as da and apply the preceding "Taylor series", we have

$$\begin{aligned}
& \operatorname{str}\big(a^0 [\mathrm{D}, a^1](s_1 t)[\mathrm{D}, a^2]((s_2 - s_1)t) \cdots [\mathrm{D}, a^k]((1 - s_k)t) e^{-t\mathrm{D}^2} g\big) \\
&= \sum_{|\lambda| < N} \frac{(-1)^{|\lambda|} t^{|\lambda|} s^\lambda}{\lambda!} \operatorname{str}\big(a^0 \nabla^{\lambda_1}(da^1) \nabla^{\lambda_2}(da^2) \cdots \nabla^{\lambda_k}(da^k) e^{-t\mathrm{D}^2} g\big) \\
&\quad + \sum_{\substack{1 \le \lambda_1, \ldots, \lambda_k \le N \\ |\lambda| \ge N}} \frac{(-1)^{|\lambda|} t^{|\lambda|} s^\lambda}{\lambda!} \operatorname{str}\big(a^0 \nabla_!^{\lambda_1}(da^1) \nabla_!^{\lambda_2}(da^2) \cdots \nabla_!^{\lambda_k}(da^k) e^{-t\mathrm{D}^2} g\big),
\end{aligned} \tag{5}$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$, $s = (s_1, \ldots, s_k)$, and

$$\nabla_!^{\lambda_j} = \begin{cases} \nabla^{\lambda_j} & \text{if } \lambda_j < N, \\ \nabla_R^{\lambda_j} & \text{if } \lambda_j = N. \end{cases} \tag{6}$$

This observation leads us to the asymptotic formula that is the starting point of Clifford asymptotics.

THEOREM 1.

$$\begin{aligned} &\mathrm{ch}_k^G(\sqrt{t}\mathrm{D})(f_0, f_1, \dots, f_k)(g) \\ &= \sum_{|\lambda| \le 2p-k} \frac{(-1)^{|\lambda|}}{\lambda!\,\tilde{\lambda}!} t^{|\lambda|+k/2} \operatorname{str}\big(f_0 \nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2) \\ &\qquad \cdots \nabla^{\lambda_k}(df_k) e^{-t\mathrm{D}^2} g\big) + O(t), \end{aligned}$$

where k is even, $g \in G$, $f_i \in \mathcal{A}$, $p > \dim M$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, $\tilde{\lambda}! = (\lambda_1 + 1)(\lambda_1 + \lambda_2 + 2) \cdots (\lambda_1 + \lambda_2 + \cdots + \lambda_k + k)$, and $\lambda! = \lambda_1!\,\lambda_2! \cdots \lambda_k!$.

Proof. We need to prove that every term in (5) is integrable and that the second part of it is $O(t)$. The proof is reduced to two lemmas that follow. □

LEMMA 2.8. *For any $0 \le u_1 \le u_2$, let $\|\cdot\|_q$ be the Schatten norm on $\mathcal{L}^q$ (and $\|\cdot\|_\infty = \|\cdot\|$). We have*

$$\|e^{-(u_2-u_1)t\mathrm{D}^2} B(-u_2 t)\|_{(u_2-u_1)^{-1}} \le C_l (u_2 - u_1)^{l/2} t^{l/2} (\operatorname{tr}(e^{-t/\mathrm{D}^2/2}))^{u_2-u_1}$$

for

(i) *B a differential operator of order l (in particular, $B = \nabla^l(da)$ for $a \in \mathcal{A}$) and*

(ii) *$B = \nabla_R^l(da)$ for $a \in \mathcal{A}$.*

Proof.

$$\begin{aligned} &\|e^{-(u_2-u_1)t\mathrm{D}^2} B(-u_2 t)\|_{(u_2-u_1)^{-1}} \\ &= \|e^{u_1 t\mathrm{D}^2} B e^{-u_2 t\mathrm{D}^2}\|_{(u_2-u_1)^{-1}} \\ &\le \|e^{(u_2-u_1)t\mathrm{D}^2/2}\|_{-(u_2-u_1)^{-1}} \cdot \|e^{(u_2+u_1)t\mathrm{D}^2/2} B e^{-u_2 t\mathrm{D}^2}\| \\ &\le C_l^{(1)} (\operatorname{tr}(e^{-t\mathrm{D}^2/2}))^{u_2-u_1} \cdot \|e^{(u_2+u_1)t\mathrm{D}^2/2}(1+\mathrm{D}^2)^{l/2} e^{-u_2 t\mathrm{D}^2}\| \\ &\le C_l^{(2)} \|(1+\mathrm{D}^2)^{l/2} e^{-(u_2-u_1)t\mathrm{D}^2/2}\| \cdot (\operatorname{tr}(e^{-t\mathrm{D}^2/2}))^{u_2-u_1} \\ &\le C_l (u_2 - u_1)^{l/2} t^{l/2} (\operatorname{tr}(e^{-t\mathrm{D}^2/2}))^{u_2-u_1}. \end{aligned}$$

The last step follows from

$$\sup\{(1+x)^{l/2} e^{-utx/2}\} = (ut)^{-l/2} e^{-(l-ut)/2}.$$

This proves (i).

By Hölder's inequality and (i), for $0 \le t_1 \le 1$ we have

$$\begin{aligned} &\|e^{-(u_2-u_1)t\mathrm{D}^2}(\nabla^l(da))(-t_1u_2t)\|_{(u_2-u_1)^{-1}} \\ &\quad\le \|e^{-(1-t_1)(u_2-u_1)t\mathrm{D}^2}\|_{((u_2-u_1)(1-t_1))^{-1}} \\ &\qquad\cdot \|e^{-t_1(u_2-u_1)t\mathrm{D}^2}(\nabla^l(da))(-t_1u_2t)\|_{((u_2-u_1)t_1)^{-1}} \\ &\quad\le (\mathrm{tr}(e^{-t\mathrm{D}^2/2}))^{(1-t_1)(u_2-u_1)}C_l(u_2-u_1)^{l/2}(t_1t)^{l/2}(\mathrm{tr}(e^{-t\mathrm{D}^2/2}))^{t_1(u_2-u_1)} \\ &\quad= C_l(u_2-u_1)^{l/2}(t_1t)^{l/2}(\mathrm{tr}(e^{-t\mathrm{D}^2/2}))^{u_2-u_1}. \end{aligned}$$

Then the proof of (ii) is straightforward by integrating on Δ_N. □

LEMMA 2.9. *For $s \in \Delta_k$ and small $t > 0$, with $\nabla_!^l$ as in (6) for a fixed N,*

$$\left|t^{|\lambda|}s^\lambda\,\mathrm{str}\big(a^0\nabla_!^{\lambda_1}(da^1)\nabla_!^{\lambda_2}(da^2)\cdots\nabla_!^{\lambda_k}(da^k)e^{-t\mathrm{D}^2}g\big)\right| \le C\cdot s^{\lambda/2}t^{|\lambda|/2-p}; \tag{7}$$

here p is a fixed number such that $p > \dim M$. Hence all the terms in (5) are integrable and we have

$$t^{|\lambda|}\left|\int_{\Delta_k} s_1^l\,\mathrm{str}\big(a^0\nabla_!^{\lambda_1}(da^1)\nabla_!^{\lambda_2}(da^2)\cdots\nabla_!^{\lambda_k}(da^k)e^{-t\mathrm{D}^2}g\big)\right| \le C_\lambda\cdot t^{|\lambda|/2-p}.$$

Proof. Let γ be the grading operator. Then

$$\begin{aligned} &\left|t^{|\lambda|}s^\lambda\,\mathrm{str}\big(a^0\nabla_!^{\lambda_1}(da^1)\nabla_!^{\lambda_2}(da^2)\cdots\nabla_!^{\lambda_k}(da^k)e^{-t\mathrm{D}^2}g\big)\right| \\ &\quad\le \Big|t^{|\lambda|}s^\lambda\,\mathrm{tr}\big(a^0e^{-s_1t\mathrm{D}^2}(\nabla_!^{\alpha_1}da^1)(-s_1t)e^{-(s_2-s_1)t\mathrm{D}^2}(\nabla_!^{\alpha_2}da^2)(-s_2t)) \\ &\qquad\qquad\cdots e^{-(s_k-s_{k-1})t\mathrm{D}^2}(\nabla_!^{\alpha_k}da^k)(-s_kt)e^{-(1-s_k)t\mathrm{D}^2}g\gamma\big)\Big| \\ &\quad\le s^\lambda t^{|\lambda|}\|a^0\|\cdot\|e^{-s_1t\mathrm{D}^2}(\nabla_!^{\alpha_1}da^1)(-s_1t)\|_{s_1^{-1}} \\ &\qquad\cdot\|e^{-(s_2-s_1)t\mathrm{D}^2}(\nabla_!^{\alpha_2}da^2)(-s_2t)\|_{(s_2-s_1)^{-1}} \\ &\qquad\cdots\|e^{-(s_k-s_{k-1})t\mathrm{D}^2}(\nabla_!^{\alpha_k}da^k)(-s_kt)\|_{(s_k-s_{k-1})^{-1}}\cdot\|e^{-(1-s_k)t\mathrm{D}^2}\|_{(1-s_k)^{-1}}\cdot\|\gamma\|. \end{aligned}$$

By the Weyl asymptotics on the heat kernel we have

$$\mathrm{tr}(e^{-t\mathrm{D}^2}) \sim (4\pi t)^{-(\dim M)/2}2^{(\dim M)/2}\,\mathrm{vol}(M),$$

so as $t \to 0$ we may relax the estimate to $p > \dim M$ and obtain

$$\mathrm{tr}(e^{-t\mathrm{D}^2/2}) \le Ct^{-p/2}.$$

Then, using Hölder's inequality and Lemma 2.8 for $(u_1, u_2) = (0, s_1), (s_1, s_2), \ldots,$ we find the desired upper bound for (2.9). □

3. Clifford Asymptotics for Heat Kernels

In this section we apply the Clifford asymptotics developed in [14] and [17]. First we prove that

$$\lim_{t\to 0} t^{|\lambda|+k/2}\,\mathrm{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)e^{-t\mathrm{D}^2}g\big) = 0, \tag{8}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \neq 0$ and k is even. This is a consequence of

$$\bar{\chi}\Big(t^{|\lambda|+k/2}\,\mathrm{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)e^{-t\mathrm{D}^2}g\big)\Big) < 0, \tag{9}$$

where $\bar{\chi}$ is a degree function (defined in what follows) under a particular moving frame.

NOTATION. *Since we use the following expression frequently, we make a shortcut for the operator*

$$\mathcal{D}_t^{\lambda} = t^{|\lambda|+k/2} f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k). \tag{10}$$

REMARK 3.1. Condition (9) corresponds to

$$\lim_{t\to 0}\sigma_{t^{-1}}(\mathcal{D}_t^{\lambda}) = 0 \tag{11}$$

in the sense of [8] and [2].

Subsequently we compute

$$\lim_{t\to 0} t^{k/2}\,\mathrm{str}\big(f_0c(df_1)c(df_2)\cdots c(df_k)e^{-t\mathrm{D}^2}g\big)$$

by Clifford asymptotics under certain normal coordinate frames in [14] and [17]. The frames E and E^{Tx} (introduced in Lemma 3.2) are key to the computation.

Let T be an orientation-preserving isometry acting on a Spin$(2n)$-manifold M^{2n}, and let $M^T = \{x \in M \mid Tx = x\}$ be the fixed point set of T. Then M^T is a disjoint union of a finite number of even-dimensional totally geodesic submanifolds $M_1^T, M_2^T, \dots, M_r^T$. For our purpose (see Proposition 3.7 below, whose proof, we emphasize, is independent of the choice of normal coordinate system), we may assume $r \neq 0$. Without loss of generality, we also assume $r = 1$ and $M^T = M_1^T$. Let ν be the normal bundle of M^T, $\dim M^T = 2n_0$, and $\dim \nu_{x_0} = 2n_1$ for $x_0 \in M^T$ (hence $2n = 2n_0 + 2n_1$).

LEMMA 3.2. (i) [14] *Let $x_0 \in M^T$. Then there exists an oriented orthonormal frame field $E = (E_1, E_2, \dots, E_{2n})$ in a neighborhood of x_0 such that*:

(a) *for $x_0' \in M^T$, $(E_1(x_0'), E_2(x_0'), \dots, E_{2n_0}(x_0'))$ are tangent to M^T and $(E_{2n_0+1}(x_0'), E_{2n_0+2}(x_0'), \dots, E_{2n}(x_0'))$ are normal to M^T*;

(b) *E is parallel along geodesics normal to M^T*;

(c) *$dT\colon \mathrm{SO}(M)_x \to \mathrm{SO}(M)_{Tx}$ is expressed as a matrix-valued function $\mathcal{T}$,*

$$dTE(x) = E(Tx)\mathcal{T}(x),$$

where x is in a neighborhood of x_0 in M;

(d) $$\mathcal{T}(x_0) = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \cos\theta_1 & \sin\theta_1 & & & \\ & & & -\sin\theta_1 & \cos\theta_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \cos\theta_{n_1} & \sin\theta_{n_1} \\ & & & & & & -\sin\theta_{n_1} & \cos\theta_{n_1} \end{bmatrix},$$

where $0 < \theta_i < 2\pi$ *for* $i = 1, 2, \ldots, n_1$; *and*

(e) *the orientation of* E *is the same as the orientation of* M.

(ii) [14] *There is a neighborhood* V *of* x_0 *in* M^T *such that* E *is defined on* $U = \exp(\nu|_V \cap \nu(\varepsilon))$ *for sufficiently small* ε, *where* $\nu(\varepsilon) = \{ v \in \nu \mid \|v\| < \varepsilon \}$. *If* $B_0(\varepsilon)$ *is the ball of radius* ε *in* $\mathbb{R}^{2n_1}$, *define the homeomorphism* $\Phi\colon V \times B_0(\varepsilon) \to U$ *by setting*

$$\Phi(x_0'; c_1, c_2, \ldots, c_{2n_1}) = x = \exp_{x_0'}\left(\sum_{r=1}^{2n_1} c_r E_{2n_0+r}(x_0')\right).$$

We then see $(x_0'; c)$ *as the coordinates of* x *with respect to* $E = (E_1, \ldots, E_{2n})$ *at* x_0'. *Moreover, we have that*

(a) $\mathcal{T}(x_0'; c) = \mathcal{T}(x_0')$ *and*

(b) *the isometry* T *has the form* $T(x_0'; c) = (x_0'; ce^{-\Theta(x_0')})$, *where* $\Theta(x_0') \in \mathfrak{so}(2n_1)$ *defined by*

$$\mathcal{T}(x_0') = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & e^{\Theta(x_0')} \end{bmatrix}.$$

(c) *Let* E^{Tx} *be a oriented orthonormal frame field defined over the patch* U *by requiring that* $E^{Tx}(Tx) = E(Tx)$ *and that* E^{Tx} *be parallel along geodesics through* Tx. *Define the coordinates* $\{y_i\}$ *of* x *as*

$$(x_0'; c) = x = \exp\left(\sum_{i=1}^{2n} y_i E_i^{Tx}(Tx)\right).$$

Then

$$\begin{cases} y_i = o(|c|), & 1 \le i \le 2n_0, \\ y_{2n_0+r} = c_r - \bar{c}_r + o(|c|), & 1 \le r \le 2n_1, \end{cases}$$

where $\bar{c} = ce^{-\Theta(x_0')}$.

DEFINITION 3.3. (i) [17] In the normal coordinates $y_1, y_2, \ldots, y_{2n}$ at Tx with respect to the frame field $E^{Tx} = (E_1^{Tx}, E_2^{Tx}, \ldots, E_{2n}^{Tx})$, the operator χ is defined on monomials on the Clifford algebra bundle

$$\chi(y^\alpha D_y^\beta e^\gamma) = |\beta| - |\alpha| + |\gamma| \tag{12}$$

for multi-indices α, β, and γ, with $y^\alpha = y_1^{\alpha_1} \cdots y_{2n}^{\alpha_{2n}}$,

$$D_y^\beta = \left(\frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial y_{2n}}\right)^{\beta_{2n}},$$

and $e^\gamma = e_1^{\gamma_1} \cdots e_{2n}^{\gamma_{2n}}$ for $\gamma_i \in \{0, 1\}$.

We denote by $(\chi < m)$ the linear space spanned by the foregoing monomials ω_0 with $\chi(\omega_0) < m$, and by $\omega_1 = \omega_2 + (\chi < m)$ we mean that there exists a $\omega_3 \in (\chi < m)$ such that $\omega_1 = \omega_2 + \omega_3$.

(ii) [14] In the coordinates $(x_0'; c)$ at $x_0' = x_0 \in M^T$ with respect to the frame field $E = (E_1, E_2, \dots, E_{2n})$, set $c = \sqrt{t}b$ and define operator $\bar{\chi}$ on the monomials $\phi(t)e_{i_1}e_{i_2}\cdots e_{i_s}$ by

$$\bar{\chi}(\phi(t)e_{i_1}e_{i_2}\cdots e_{i_s}) = s - \sup\left\{ l \in \mathbb{Z} \;\middle|\; \lim_{t\to 0} \frac{|\phi(t)|}{t^{l/2}} < \infty \right\}, \tag{13}$$

where $\phi(t) \in \mathbb{R}$. In particular, we have $\bar{\chi}(t) = -2$ and $\bar{\chi}(e_i) = 1$. We denote by $P = Q + (\bar{\chi} < m)$ the congruence of P and Q modulo the space generated by monomials with $\bar{\chi} < m$.

REMARK 3.4. (i) In (i) of Definition 3.3, if we let $\chi(t) = -2$ then

$$\chi(t^l y^\alpha D_y^\beta e^\gamma) = |\beta| - |\alpha| + |\gamma| - 2l.$$

Up to a sign, this is the same as the "deg" function defined in [9] and [1].

(ii) By Lemma 3.2(ii)(c) we have

$$\begin{cases} y_i = o(|c|) = 0 + \sqrt{t}o(|b|), & 1 \le i \le 2n_0, \\ y_{2n_0+r} = c_r - \bar{c}_r + o(|c|) = \sqrt{t}(b_r - \bar{b}_r + o(|b|)), & 1 \le r \le 2n_1. \end{cases}$$

Thus

$$\begin{cases} \bar{\chi}(y_i) = \chi(y_i) = -1, & 1 \le i \le 2n, \\ \bar{\chi}(\frac{\partial}{\partial y_i}) = \chi(\frac{\partial}{\partial y_i}) = 1, & 1 \le i \le 2n, \end{cases}$$

which means that operator χ and $\bar{\chi}$ are compatible.

LEMMA 3.5. *For $f \in \mathcal{A} = C^\infty(M)$ we have* $[\mathrm{D}^2, [\mathrm{D}, f]] = 0 + (\chi < 3)$.

Proof. By Proposition 13 in [17],

$$\begin{aligned} \mathrm{D}^2 = &-\sum_i \frac{\partial^2}{\partial y_i^2} + \frac{1}{4}\sum_{i,j,\alpha_1,\alpha_2} R_{ij\alpha_1,\alpha_2}(Tx)y_i\frac{\partial}{\partial y_j}e_{\alpha_1}e_{\alpha_2} \\ &+ \frac{1}{64}\sum_{i,j,k,\alpha_1,\alpha_2,\alpha_3,\alpha_4} y_iy_jR_{ik\alpha_1\alpha_2}(Tx)R_{kj\alpha_3\alpha_4}(Tx)e_{\alpha_1}e_{\alpha_2}e_{\alpha_3}e_{\alpha_4} \\ &+ (\chi < 2). \end{aligned} \tag{14}$$

In general we have

$$[ab, df] = a[b, df] + [a, df]b, \tag{15}$$

$$\bar{\chi}(\omega_1\omega_2) \le \bar{\chi}(\omega_1) + \bar{\chi}(\omega_2). \tag{16}$$

Since $df = \sum_{i=1}^{2n} e_i(f)e_i$, it follows that $\chi(df) \le 1$. Note that:

(i) $[e_{\alpha_1}\cdots e_{\alpha_k}, df] = 0 + (\chi < k)$ (which follows from $e_ie_j = -e_je_i - 2\delta_{ij}$);
(ii) $\left[\frac{\partial}{\partial y_s}, df\right] = \sum_{s,i}\frac{\partial(e_i(f))}{\partial y_s}e_i = 0 + (\chi < 2)$;
(iii) $[y_s, df] = 0$.

In other words, for each factor in every monomial in (14), the operation under $[\cdot, df]$ lowers or fixes its degree. Together with (15), this is also true for every monomial in (14). □

LEMMA 3.6. *For $f_0, f_1, \ldots, f_k \in C^\infty(M)$ (k even) we have*

$$t^{|\lambda|+k/2} f_0 \nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k) = 0 + (\bar{\chi} < 0),$$

where $\lambda \neq 0$.

Proof. Note that

$$\nabla^{\lambda_i}(df_i) = \underbrace{[\mathrm{D}^2, [\mathrm{D}^2, \ldots, [\mathrm{D}^2, [\mathrm{D}, f]] \ldots]]}_{\lambda_i \text{ times}}$$
$$= 0 + (\chi < 2\lambda_i + 1).$$

Since $\bar{\chi}$ and χ are compatible, we have

$$t^{|\lambda|+k/2} f_0 \nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k) < -2|\lambda| - k + 2|\lambda| + k = 0$$

for $\lambda \neq 0$. □

Now, choose a spin frame field $\sigma: U \to \mathrm{Spin}(M)$ such that

$$\pi_\rho \sigma = (E_1^{Tx}, E_2^{Tx}, \ldots, E_{2n}^{Tx}),$$

where $\pi_\rho: \mathrm{Spin}(M) \to \mathrm{SO}(M)$ is the covering map of bundles. Let $K_t(x, y)$ be the kernel of the heat operator $e^{-t\mathrm{D}^2}$. For $x \in U$, let $\bar{K}_t(x)$ and $\bar{T}^*(x) \in \mathrm{Hom}(S^\pm, S^\pm)$ be defined by

$$K_t(x, Tx)[(\sigma(Tx), v)] = [(\sigma(x), \bar{K}_t(x)v)] \tag{17}$$

and

$$T^*[(\sigma(x), u)] = [(\sigma(Tx), \bar{T}^*(x)u)], \tag{18}$$

where $[(\sigma(\cdot), v)]$ (resp. $[(\sigma(\cdot), u)]$) is a section of the bundle $\mathrm{Spin}(M) \times_{\mathrm{Spin}(2n)} S^\pm$, $v \in S^\pm$ (resp. $u \in S^\pm$) and $T^*: \mathrm{Spin}(M) \to \mathrm{Spin}(M)$ is induced by the map $T: M \to M$. We also define $\bar{K}_t^\lambda(x) \in \mathrm{Hom}(S^\pm, S^\pm)$ by setting

$$\mathcal{D}_t^\lambda K_t(x', Tx)|_{x'=x}[(\sigma(Tx), v)] = [(\sigma(x), \bar{K}_t^\lambda(x)v)], \tag{19}$$

where the differential operator $\mathcal{D}_t^\lambda$ acts on the first variable in $K_t(x', Tx)$, that is, on x'. Then we have the following proposition.

PROPOSITION 3.7.

$$\lim_{t\to 0} t^{|\lambda|+k/2} \mathrm{str}\big(f_0 \nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k) T e^{-t\mathrm{D}^2}\big)$$
$$= \int_{M^T} \lim_{t\to 0} \int_{\nu_{x_0}(\varepsilon)} \mathrm{str}(\mathcal{D}_t^\lambda e^{-t\mathrm{D}^2} T)(x, x)\, dx\, dx_0, \tag{20}$$

where

$$\lim_{t\to 0} \int_{\nu_{x_0}(\varepsilon)} \mathrm{str}(\mathcal{D}_t^\lambda e^{-t\mathrm{D}^2} T)(x, x)\, dx = \lim_{t\to 0} \int_{\nu_{x_0}(\varepsilon)} \mathrm{str}\big(\bar{K}_t^\lambda(x)\bar{T}^*(x)\big)\, dx, \tag{21}$$

ε is sufficiently small, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\lambda_i \geq 0$.

Proof. This is an analog of Theorem 2.2 in [14]. We need a stronger proof, since in this case we need estimates on the derivatives of the heat kernel. However, the proof is made easy by the following estimates of the derivatives for the heat kernel due to [3], the statement of which is simplified for a compact Riemannian manifold:

$$|\nabla_y^l K_t(y,x)| \le C(n,T)\tilde{\delta}^{\delta(\dim M, l)} t^{-(\dim M+1)/2} e^{-\alpha d^2(y,x)/t}, \tag{22}$$

where $d(y,x)$ is the distance between y and x, α is a positive constant depending only on $\dim M$, and $t \in (0,T)$ for any fixed $T > 0$. On the compact set $\overline{M\backslash\nu(\varepsilon)}$ let

$$\varepsilon = \min\{\, d(x,Tx) \mid x \in \overline{M\backslash\nu(\varepsilon)}\,\}.$$

We see that the factor $e^{-\alpha\varepsilon^2/t}$ for the upper bound of $\mathcal{D}_t^\lambda K_t(x', Tx)$ rapidly decays; hence the limit of its integral is 0 when $t \to 0$, regardless of the coefficients of any power of t. □

REMARK 3.8. Note that (22) is a global estimation; in the neighborhood of the fixed-point set we have the Minakshisundaram–Pleijel asymptotic expansion (see [7], [17], and Lemma 3.11 below), which is the basis for Clifford asymptotics for the heat kernel.

From Proposition 3.7, in order to compute $\lim_{t\to 0} \operatorname{str}(\mathcal{D}_t^\lambda T e^{-t\mathrm{D}^2})$ we should find

$$\lim_{t\to 0} \int_{\nu_{x_0}(\varepsilon)} \operatorname{str}\big(\bar{K}_t^\lambda(x)\bar{T}^*(x)\big)\,dx. \tag{23}$$

Next, we find the Clifford asymptotics of $\bar{T}^*(x)$ and $\bar{K}_t^\lambda(x)$.

LEMMA 3.9 [14]. *The operator $\bar{T}^*(x)$ is given by*

$$\begin{aligned}\bar{T}^*(x) &= (-1)^{n_1}\prod_{\alpha=1}^{n_1}\sin\frac{\theta_\alpha}{2}\\ &\quad\cdot \exp\Big\{-\frac{t}{4}\sum_{\alpha,\beta=1}^{2n_1} b_\alpha b_\beta (e^{-\Theta(x_0)}A^\perp)_{\alpha\beta}\Big\}(e_{2n_0+1}e_{2n_0+2}\cdots e_{2n})\\ &\quad + (\bar{\chi} < 2n_1),\end{aligned} \tag{24}$$

where

(i) $\Theta(x_0) \in \mathfrak{so}(2n_1)$ *is in the form*

$$\Theta(x_0) = \begin{bmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_{n_1} \\ & & & -\theta_{n_1} & 0 \end{bmatrix}_{2n_1\times 2n_1}$$

and

$$\mathcal{T}(x_0) = \begin{bmatrix} I & 0 \\ 0 & e^{\Theta(x_0)} \end{bmatrix};$$

(ii) $x = (x_0'; c) = (x_0'; \sqrt{t}b)$, *and* $A^\perp$ *is a* $2n_1 \times 2n_1$ *matrix whose* (α, β) *element is given by*

$$(A^\perp)_{\alpha\beta} = -\frac{1}{2}\sum_{i,j=1}^{2n_0} R_{(2n_0+\alpha)(2n_0+\beta)ij}(x_0)e_ie_j.$$

To apply the Clifford asymptotics of $\bar{K}_t^\lambda(x)$, we shall introduce some notation. Let x' be a point near $x = (x_0'; c)$, and let $y = (y_1', y_2', \ldots, y_{2n}')$ be the normal coordinates of x' at the point Tx with respect to the orthonormal frame field $E^{Tx} = (E_1^{Tx}, E_2^{Tx}, \ldots, E_{2n}^{Tx})$ defined in Lemma 3.2(ii)(c); that is,

$$x' = \exp_{Tx}\left(\sum_{i=1}^{2n} y_i' E_i^{Tx}(Tx)\right) = \exp_{Tx}\left(\sum_{i=1}^{2n} y_i' E_i(Tx)\right).$$

DEFINITION 3.10 [14]. Let $\tilde{A}$ be the $2n \times 2n$ matrix defined by

$$\tilde{A}_{ij} = -\frac{1}{2}\sum_{k,l=1}^{2n} R_{ijkl}^{Tx}(Tx)e_ke_l, \tag{25}$$

where $R_{ijkl}^{Tx}(Tx)$ are the coefficients of the Riemannian curvature tensor under the frame field $E^{Tx} = (E_1^{Tx}, E_2^{Tx}, \ldots, E_{2n}^{Tx})$ at point Tx. We define $\tilde{A}^l(y')$ as

$$\tilde{A}^l(y') = \sum_{i,j=1}^{2n} y_i' y_j' (\tilde{A}^l)_{ij} \tag{26}$$

for $l = 1, 2, \ldots$.

LEMMA 3.11 [17]. *There exists a universal function* $F(t; z_1, z_2, \ldots; w_1, w_2, \ldots)$ *that is a power series in t with coefficient polynomials in* z_i *and* w_j *such that the following conditions hold.*

(i) *If* $\bar{K}_t(x')$ *is determined by the relation*

$$K_t(x', Tx)[(\sigma(Tx), v)] = [(\sigma(x'), \bar{K}_t(x')v)], \tag{27}$$

then $\bar{K}_t(x')$ *has the following Clifford asymptotics at the point* Tx:

$$\bar{K}_t(x') = \left(\frac{1}{4\pi t}\right)^n \exp\left\{-\frac{d^2(x', Tx)}{4t}\right\} \cdot \left\{F\big(t; \operatorname{tr}(\tilde{A}^2), \ldots, \operatorname{tr}(\tilde{A}^{2n}); \tilde{A}^2(y'), \ldots, \tilde{A}^{2n}(y')\big) + \sum_{m\geq 0} t^m(\chi < 2m)\right\}.$$

(ii) *The function F is determined by*

$$F\Bigg(t; -2(u_1^2 + \cdots + u_n^2), \ldots, (-1)^l 2(u_1^{2l} + \cdots + u_n^{2l}), \ldots; (-1)\sum_{\alpha=1}^{n}(v_{2\alpha-1}^2 + v_{2\alpha}^2)u_\alpha^2, \ldots, (-1)^l\sum_{\alpha=1}^{n}(v_{2\alpha-1}^2 + v_{2\alpha}^2)u_\alpha^{2l}, \ldots\Bigg)$$
$$= (4\pi t)^n e^{\|v\|^2/4t}\prod_{\alpha=1}^{n}\frac{iu_\alpha}{8\pi\sinh iu_\alpha t/2}\exp\left\{-\frac{iu_\alpha}{8}(v_{2\alpha-1}^2 + v_{2\alpha}^2)\coth\frac{iu_\alpha t}{2}\right\},$$

where $v_1, \dots, v_{2n}$ are variables and $u_1, \dots, u_n$ are the generators of the polynomial ring $\mathbb{R}[u_1, \dots, u_n]$.

We refer to [17] for a proof of the lemma.

THEOREM 2. *If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \neq 0$, then we have the following equalities:*

$$\bar{K}_t^\lambda(x)\bar{T}^*(x) = e^{-\|b-\bar{b}\|^2/4} e^{\sqrt{t}f} (\bar{\chi} < 2n + 2n_1)_b, \tag{28}$$

where f is a bounded continuous function and $(\bar{\chi} < 2n+2n_1)_b$ denotes the space spanned by monomials that are polynomials in b satisfying $\bar{\chi} < 2n + 2n_1$;

$$\lim_{t\to 0} \int_{\nu_{x_0}(\varepsilon)} \operatorname{str}\big(\bar{K}_t^\lambda(x)\bar{T}^*(x)\big)\, dx = 0; \tag{29}$$

and

$$\lim_{t\to 0} t^{|\lambda|+k/2} \operatorname{str}\big(f_0 \nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)(Te^{-t\mathrm{D}^2})\big) = 0. \tag{30}$$

Proof. First of all, by part (ii) of Lemma 3.2, we have

$$\begin{aligned} e^{-\|y\|^2/4t} &= \exp\left\{ -\frac{\sum_{\alpha=2n_0+1}^{2n} \|y_\alpha\|^2}{4t} - \frac{\sum_{i=1}^{2n_0} \|y_i\|^2}{4t} \right\} \\ &= e^{-\|b-\bar{b}\|^2/4} e^{\sqrt{t}f}, \end{aligned}$$

where $\bar{b} = ce^{-\Theta(x_0)}$, y is the normal coordinate of $x = (x_0'; c)$ under the frame field E^{Tx} at Tx, and f is a bounded continuous function.

Then, by Lemma 3.6, Lemma 3.9, Lemma 3.11, the fact that $\mathcal{D}_t^\lambda$ is a differential operator, and

$$\bar{\chi}\big\{F\big(t; \operatorname{tr}(\tilde{A}^2), \dots, \operatorname{tr}(\tilde{A}^{2n}); \tilde{A}^2(y'), \dots, \tilde{A}^{2n}(y')\big)\big\} = 0,$$

we conclude that

$$\begin{aligned} \bar{K}_t^\lambda(x)\bar{T}^*(x) &= \mathcal{D}_t^\lambda K_t(x', Tx)|_{x'=x}\bar{T}^*(x) \\ &= e^{-\|b-\bar{b}\|^2/4} e^{\sqrt{t}f} (\bar{\chi} < 2n + 2n_1)_b, \end{aligned}$$

where $(\bar{\chi} < 2n + 2n_1)_b$ denotes the space spanned by monomials that are polynomials in b satisfying $\bar{\chi} < 2n + 2n_1$. Thus,

$$\lim_{t\to 0} \int_{\nu_{x_0}(\varepsilon)} \operatorname{str}\big(\bar{K}_t^\lambda(x)\bar{T}^*(x)\big)\, dx = \lim_{t\to 0} \int_{\mathbb{R}^{2n_1}} t^{n_1} e^{-\|b-\bar{b}\|^2/4} \operatorname{str}(\omega)\, db,$$

where $\omega \in (\bar{\chi} < 2n + 2n_1)_b$. The proof amounts to showing that

$$\lim_{t\to 0} \int_{\mathbb{R}^{2n_1}} t^{n_1} e^{-\|b-\bar{b}\|^2/4} \operatorname{str}(\omega)\, db = 0. \tag{31}$$

Without loss of generality we may assume $\omega = b_{i_1} b_{i_2} \dots b_{i_r} \omega_0$, where ω_0 is independent of $b_{i_1}, b_{i_2}, \dots, b_{i_r}$ and $\bar{\chi}(\omega_0) < 2n + 2n_1$. Then $\bar{\chi}(t^{n_1}\omega_0) < 2n$. If we write $t^{n_1}\omega_0$ as

$$t^{n_1}\omega_0 = t^{n_1} \sum_{I \subset \{1,2,\dots,2n\}} \omega_0^I(t) e_I,$$

where $e_I = e_{i_1} e_{i_2} \dots e_{i_r}$ if $I = \{i_1, i_2, \dots, i_r\}$, then

$$\mathrm{str}(t^{n_1}\omega_0) = \mathrm{str}\big(t^{n_1}\omega_0^{\{1,2,\dots,2n\}}(t) e_1 e_2 \cdots e_{2n}\big) \to 0 \quad \text{as } t \to 0,$$

since $\bar{\chi}\{t^{n_1}\omega_0^{\{1,2,\dots,2n\}}(t)\} < 0$. Hence (31) follows from

$$\int_{\mathbb{R}^{2n_1}} e^{-\|b-\bar{b}\|^2/4} |b_{i_1} b_{i_2} \cdots b_{i_r}|\, db < \infty. \qquad \square$$

Now, in order to find the Chern character we need only compute

$$\lim_{t\to 0} t^{k/2}\, \mathrm{str}\big(f_0 c(df_1) c(df_2) \cdots c(df_k) T e^{-t\mathrm{D}^2}\big). \tag{32}$$

By Proposition 3.7, we need to determine the Clifford asymptotics of the operator

$$t^{k/2} f_0 c(df_1) c(df_2) \cdots c(df_k) \bar{K}_t(x) \bar{T}^*(x), \tag{33}$$

and for this we use Lemma 3.11.

LEMMA 3.12. *For any $f \in \mathcal{A} = C^\infty(M)$ and $x = (x_0; c) = (x_0; \sqrt{t}b)$, we have*
(i) $f(x) = f(x_0) + (\bar{\chi} < 0)$ *and*
(ii) $(df)(x) = (df)(x_0) + (\bar{\chi} < 1)$.

Proof. (i) follows from

$$f(x) = f(x_0) + \sqrt{t} O(|b|) = f(x_0) + (\bar{\chi} < 0);$$

(ii) follows from $df = \sum_i e_i(f)$ and $e_i(f)(x) = e_i(f)(x_0) + (\bar{\chi} < 0)$. $\square$

PROPOSITION 3.13. *We have the following Clifford asymptotics:*

$$\begin{aligned} &t^{k/2} f_0 c(df_1) c(df_2) \cdots c(df_k) \bar{K}_t(x) \bar{T}^*(x) \\ &\quad = t^{k/2} f_0(x_0) c(df_1(x_0)) c(df_2(x_0)) \cdots c(df_k(x_0)) e^{-\|b-\bar{b}\|^2/4} \\ &\quad \cdot \Bigg\{ \frac{(-1)^{n_1}}{(4\pi t)^n} \prod_{\alpha=1}^{n_1} \sin\frac{\theta_\alpha}{2} \exp\Bigg\{ -\frac{t}{4} \sum_{\alpha,\beta=1}^{2n_1} b_\alpha b_\beta (e^{-\Theta(x_0)} A^\perp)_{\alpha\beta} \Bigg\} \\ &\qquad \cdot F\big(t; \mathrm{tr}(A^\top)^2 + \mathrm{tr}(A^\perp)^2, \dots, \mathrm{tr}(A^\top)^{2l} + \mathrm{tr}(A^\perp)^{2l}, \dots; \\ &\qquad\qquad t(A^\top)^2(b-\bar{b}), \dots, t(A^\top)^{2l}(b-\bar{b}), \dots\big) \\ &\qquad \cdot e_{2n_0+1} e_{2n_0+2} \cdots e_{2n} + e^{\sqrt{t}f} (\bar{\chi} < 2n + 2n_1)_b \Bigg\}, \end{aligned} \tag{34}$$

where $A^\top$ is a $2n_0 \times 2n_0$ matrix defined by

$$\begin{aligned} (A^\top)_{ij} &= -\frac{1}{2} \sum_{k,l=1}^{2n_0} R_{ijkl}(x_0) e_k e_l \\ &= -\frac{1}{2} \sum_{k,l=1}^{2n_0} R_{ijkl}^{Tx}(Tx) e_k e_l + (\bar{\chi} < 2) \end{aligned} \tag{35}$$

for $1 \leq i, j \leq 2n_0$, *and* $(A^\top)^{2l}(b-\bar{b})$ *is defined by*

$$(A^\top)^{2l}(b-\bar{b}) = \sum_{\alpha,\beta=1}^{2n_1} (b_\alpha - \bar{b}_\alpha)(b_\beta - \bar{b}_\beta)(A^\top)^{2l}_{\alpha\beta}. \tag{36}$$

Proof. This proposition follows from Lemma 3.2(ii)(c), Lemma 3.9, Lemma 3.11 (for $y' = y$), Lemma 3.12, and the equalities

$$\operatorname{tr}(\tilde{A}^{2l}) = \operatorname{tr}(A^\top)^{2l} + \operatorname{tr}(A^\perp)^{2l} + \sum_{\alpha=1}^{2n_1} e_{2n_0+\alpha}(\bar{\chi} < 4l) \tag{37}$$

and

$$(\tilde{A}^{2l})(y) = t(A^\perp)^{2l}(b-\bar{b}) + t\sum_{\alpha=1}^{2n_1} e_{2n_0+\alpha}(\bar{\chi} < 4l), \tag{38}$$

which are in Lemma 4.3 of [14]. □

LEMMA 3.14. *If* $\phi \in (2n+2n_1)_b$ *then*

$$\lim_{t\to 0}\int_{\nu_{x_0}(\varepsilon)} t^{k/2}\operatorname{str}\big(f_0(x_0)c(df_1(x_0))c(df_2(x_0)) \cdots c(df_k(x_0))e^{-\|b-\bar{b}\|^2/4}e^{\sqrt{t}f}\phi\big) = 0.$$

Proof. Using

$$\bar{\chi}\{t^{k/2}f_0(x_0)c(df_1(x_0))c(df_2(x_0))\cdots c(df_k(x_0))\phi\} = -k+k+\bar{\chi}(\phi) < 2n+2n_1,$$

the lemma can be established in the manner of the proof of Theorem 2. □

COROLLARY 3.15.

$$\begin{aligned}
&\lim_{t\to 0} t^{k/2}\operatorname{str}\big(f_0 c(df_1)c(df_2)\cdots c(df_k)(Te^{-t\mathrm{D}^2})\big)\\
&= \int_{M^T}\lim_{t\to 0}\int_{\mathbb{R}^{2n_1}} t^{n_1}\operatorname{str}\Bigg(\omega^t(x_0)e^{-\|b-\bar{b}\|^2/4}\frac{(-1)^{n_1}}{(4\pi t)^n}\\
&\qquad\cdot\prod_{\alpha=1}^{n_1}\sin\frac{\theta_\alpha}{2}\exp\Big\{-\frac{t}{4}\sum_{\alpha,\beta=1}^{2n_1} b_\alpha b_\beta(e^{-\Theta(x_0)}A^\perp)_{\alpha\beta}\Big\}\\
&\qquad\cdot F\big(t; \operatorname{tr}(A^\top)^2+\operatorname{tr}(A^\perp)^2, \ldots, \operatorname{tr}(A^\top)^{2l}+\operatorname{tr}(A^\perp)^{2l}, \ldots;\\
&\qquad\cdot t(A^\top)^2(b-\bar{b}), \ldots, t(A^\top)^{2l}(b-\bar{b}), \ldots\big)e_{2n_0+1}e_{2n_0+2}\cdots e_{2n}\Bigg)\,db\,dx_0,
\end{aligned}$$

where $\omega^t(x_0) = t^{k/2}f_0(x_0)c(df_1(x_0))c(df_2(x_0))\cdots c(df_k(x_0))$.

THEOREM 3. *The degree-$2n_0$ component of the differential form (with respect to x_0)*

$$\lim_{t\to 0}\int_{\mathbb{R}^{2n_1}} t^{n_1}\,\mathrm{str}\Bigg(\omega^t(x_0)e^{-\|b-\bar b\|^2/4}\frac{(-1)^{n_1}}{(4\pi t)^n}\cdot\prod_{\alpha=1}^{n_1}\sin\frac{\theta_\alpha}{2}\exp\bigg\{-\frac{t}{4}\sum_{\alpha,\beta=1}^{2n_1}b_\alpha b_\beta(e^{-\Theta(x_0)}A^\perp)_{\alpha\beta}\bigg\}$$
$$\cdot F\big(t;\mathrm{tr}(A^\top)^2+\mathrm{tr}(A^\perp)^2,\ldots,\mathrm{tr}(A^\top)^{2l}+\mathrm{tr}(A^\perp)^{2l},\ldots;$$
$$t(A^\top)^2(b-\bar b),\ldots,t(A^\top)^{2l}(b-\bar b),\ldots\big)e_{2n_0+1}e_{2n_0+2}\cdots e_{2n}\Bigg)\,db\,dx_0$$

is the same as the degree-$2n_0$ component of the differential form

$$\frac{1}{(2\pi\sqrt{-1})^{k/2}}f_0(x_0)(df_1)(x_0)\wedge\cdots\wedge df_k)(x_0)\wedge\det{}^{1/2}\frac{\Omega^\top/4\pi}{\sinh(\Omega^\top/4\pi)}\wedge\mathbf{Pf}\bigg\{2\sinh\bigg(\frac{\Omega^\perp}{4\pi}+\frac{\sqrt{-1}\Theta}{2}\bigg)\bigg\}^{-1}$$
$$=\frac{1}{(2\pi\sqrt{-1})^{k/2}}f_0(x_0)(df_1)(x_0)\wedge\cdots\wedge(df_k)(x_0)\wedge\hat{\mathbf{A}}(TM^T)\wedge\mathbf{Pf}\bigg\{2\sinh\bigg(\frac{\Omega}{4\pi}+\frac{\sqrt{-1}\Theta}{2}\bigg)(\nu(M^T))\bigg\}^{-1},$$

where $\Omega^\top$ is a $2n_0\times 2n_0$ matrix defined by

$$(\Omega^\top)_{ij}=-\frac{1}{2}\sum_{k,l=1}^{2n_0}R_{ijkl}(x_0)\omega_k\wedge\omega_l,\quad 1\le i,j\le 2n_0,\tag{39}$$

and $\Omega^\perp$ is a $2n_1\times 2n_1$ matrix defined by

$$(\Omega^\perp)_{ij}=-\frac{1}{2}\sum_{k,l=1}^{2n_0}R_{ijkl}(x_0)\omega_k\wedge\omega_l,\quad 2n_0+1\le i,j\le 2n.\tag{40}$$

Here $\omega=(\omega_1,\omega_2,\ldots,\omega_{2n})$ is the frame dual to $E=(E_1,E_2,\ldots,E_{2n})$ and Ω is given by

$$\Omega=\begin{bmatrix}\Omega^\top & 0\\ 0 & \Omega^\perp\end{bmatrix}.$$

Proof. Since $e_ie_j=-e_je_i+(\bar\chi<1)$, we can replace $A^\top$ and $A^\perp$ by $\Omega^\top$ and $\Omega^\perp$ (respectively) in Corollary 3.15.

If we write $\Omega^\top$ and $\Omega^\perp$ (formally) as

$$\Omega^{\top} = \begin{bmatrix} 0 & u_1 & & & & & \\ -u_1 & 0 & & & & & \\ & & 0 & u_2 & & & \\ & & -u_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & u_{n_0} \\ & & & & & -u_{n_0} & 0 \end{bmatrix},$$

$$\Omega^{\perp} = \begin{bmatrix} 0 & v_1 & & & & & \\ -v_1 & 0 & & & & & \\ & & 0 & v_2 & & & \\ & & -v_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & v_{n_1} \\ & & & & & -v_{n_1} & 0 \end{bmatrix},$$

where u_i and v_j are indeterminants, then we have

$$\sum_{\alpha,\beta=1}^{2n_1} b_\alpha b_\beta (e^{-\Theta(x_0)}\Omega^{\perp})_{\alpha\beta} = \sum_{\alpha=1}^{n_1} \sin\theta_\alpha \cdot \mathbf{v}_\alpha (b_{2\alpha-1}^2 + b_{2\alpha}^2),$$

$$(\Omega^{\perp})^{2l}(y) = (-1)^l \sum_{\alpha=1}^{n_1} 4t \sin^2\frac{\theta_\alpha}{2}\mathbf{v}_\alpha^{2l}(b_{2\alpha-1}^2 + b_{2\alpha}^2),$$

and

$$\operatorname{tr}\Omega^{2l} = 2(-1)^l\left[\sum_{\alpha=1}^{n_0}\mathbf{u}_\alpha^{2l} + \sum_{\beta=1}^{n_1}\mathbf{v}_\beta^{2l}\right].$$

By Lemma 3.11(ii) and Corollary 3.15, it is not difficult to see that the degree-$2n_0$ component of the differential form (with respect to x_0)

$$\begin{aligned} \lim_{t\to 0}\int_{\mathbb{R}^{2n_1}} t^{n_1}\operatorname{str}\Big(\omega^t(x_0)e^{-\|b-\bar b\|^2/4}\frac{(-1)^{n_1}}{(4\pi t)^n} \\ \cdot\prod_{\alpha=1}^{n_1}\sin\frac{\theta_\alpha}{2}\exp\Big\{-\frac{t}{4}\sum_{\alpha,\beta=1}^{2n_1} b_\alpha b_\beta(e^{-\Theta(x_0)}A^{\perp})_{\alpha\beta}\Big\} \\ \cdot F\big(t;\operatorname{tr}(A^{\top})^2+\operatorname{tr}(A^{\perp})^2,\ldots,\operatorname{tr}(A^{\top})^{2l}+\operatorname{tr}(A^{\perp})^{2l},\ldots; \\ t(A^{\top})^2(b-\bar b),\ldots,t(A^{\top})^{2l}(b-\bar b),\ldots\big)e_{2n_0+1}e_{2n_0+2}\cdots e_{2n}\Big)\,db\,dx_0 \end{aligned}$$

is the same as the degree-$2n_0$ component of the differential form

$$\begin{aligned} (-1)^{n_1}\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_{\mathbb{R}^{2n_1}}\omega(x_0)\prod_{\alpha=1}^{n_1}\frac{\sin\theta_\alpha}{2}\prod_{\gamma=1}^{n_0}\frac{\sqrt{-1}\mathbf{u}_\gamma/2}{\sinh(\mathbf{u}_\gamma/2)}\prod_{\beta=1}^{n_1}\frac{\sqrt{-1}\mathbf{v}_\beta/2}{\sinh(\mathbf{v}_\beta/2)} \\ \cdot\exp\left\{\sum_{\gamma=1}^{n_1}\left(-\frac{1}{4}\mathbf{v}_\gamma\sin\theta_\gamma-\frac{\sqrt{-1}\mathbf{v}_\gamma}{2}\coth\frac{\sqrt{-1}\mathbf{v}_\gamma}{2}\sin^2\frac{\theta_\gamma}{2}\right)(b_{2\gamma-1}^2+b_{2\gamma}^2)\right\}db \end{aligned}$$

$$= \frac{1}{(2\pi\sqrt{-1})^{k/2}} f_0(x_0)(df_1)(x_0) \wedge \cdots \wedge (df_k)(x_0)$$

$$\wedge \det{}^{1/2} \frac{\Omega^{\mathrm{T}}/4\pi}{\sinh(\Omega^{\mathrm{T}}/4\pi)} \wedge \mathbf{Pf}\left\{2\sinh\left(\frac{\Omega^{\perp}}{4\pi} + \frac{\sqrt{-1}\Theta}{2}\right)\right\}^{-1}$$

$$= \frac{1}{(2\pi\sqrt{-1})^{k/2}} f_0\, df_1 \wedge df_2 \wedge \cdots \wedge df_k$$

$$\wedge \hat{\mathbf{A}}(TM^T) \wedge \mathbf{Pf}\left\{2\sinh\left(\frac{\Omega}{4\pi} + \frac{\sqrt{-1}\Theta}{2}\right)(\nu(M^T))\right\}^{-1}.$$

For more details about the last few steps, see [14]. □

COROLLARY 3.16.

$$\lim_{t\to 0} t^{k/2} \operatorname{str}\left(f_0 c(df_1)c(df_2)\cdots c(df_k)(Te^{-t\mathrm{D}^2})\right)$$

$$= \int_{M^T} \frac{1}{(2\pi\sqrt{-1})^{k/2}} f_0\, df_1 \wedge df_2 \wedge \cdots \wedge df_k$$

$$\wedge \hat{\mathbf{A}}(TM^T) \wedge \mathbf{Pf}\left\{2\sinh\left(\frac{\Omega}{4\pi} + \frac{\sqrt{-1}\Theta}{2}\right)(\nu(M^T))\right\}^{-1}.$$

4. The Main Results

Combining Theorem 1, Theorem 2, and Theorem 3, we have reached the first goal of this paper as follows.

THEOREM 4.

$$\lim_{t\to 0^+} \mathrm{ch}_k^G(\sqrt{t}\mathrm{D})(f_0, f_1, \ldots, f_k)(T)$$

$$= \frac{1}{k!\,(2\pi\sqrt{-1})^{k/2}} \int_{M^T} f_0\, df_1 \wedge df_2 \wedge \cdots \wedge df_k$$

$$\wedge \hat{\mathbf{A}}(TM^T) \wedge \mathbf{Pf}\left\{2T\sinh\left(\frac{\Omega}{4\pi} + \frac{\sqrt{-1}\Theta}{2}\right)(\nu(M^T))\right\}^{-1},$$

where $ds = ds_1 ds_2 \cdots ds_k$, $k =$ even, $T \in G$, $f_i \in C^\infty(M)$, *and* M^T *is the fixed-point set that is the disjoint union of a finite number of even-dimensional totally geodesic submanifolds* $M_1^T, M_2^T, \ldots, M_r^T$. *For the other notation see Theorem 3.*

Now we derive the equivariant Chern character in the equivariant cyclic cohomology. First recall a theorem by Connes and Moscovici.

THEOREM [6]. *The equivariant Chern character*

$$\Phi_G(\mathrm{D}) = \{\Phi_G^k(\mathrm{D})\}_{k=\text{even}}$$

for the invariant Dirac operator in the equivariant cyclic cohomology is given as follows.

(i) *If $k > 0$, then*

$$\Phi_G^k(\mathrm{D})(f_0, f_1, \dots, f_k)(T) = \sum_\lambda \frac{(-1)^{|\lambda|}}{\lambda!\tilde{\lambda}!}\Gamma\left(|\lambda| + \frac{k}{2}\right) \operatorname*{res}_{s=0} \operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k) \cdot |\mathrm{D}|^{-2s-k-2|\lambda|}T\big),$$

where $T \in G$, $f_i \in C^\infty(M)$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$,

$$\tilde{\lambda}! = (\lambda_1 + 1)(\lambda_1 + \lambda_2 + 2)\cdots(\lambda_1 + \lambda_2 + \cdots + \lambda_k + k),$$

and $\lambda! = \lambda_1!\,\lambda_2!\cdots\lambda_k!$.

(ii) *If $k = 0$, then*

$$\Phi_G^0(\mathrm{D})(f_0)(T) = \operatorname*{res}_{s=0}\{s^{-1}\operatorname{str}(f_0|\mathrm{D}|^{-2s}T)\},$$

where $f_0 \in C^\infty(M)$.

In order to compute this Chern character in the cyclic cohomology, we need the following lemma.

LEMMA 4.1. *If 0 is not an eigenvalue of D^2, and we define an operator $\mathcal{D}^\lambda$ by setting*

$$\mathcal{D}^\lambda = f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k),$$

then $\operatorname{str}(\mathcal{D}^\lambda e^{-t\mathrm{D}^2}T)$ rapidly decays for large time t.

Proof. Since $\mathcal{D}^\lambda$ is a differential operator of order $|\lambda|$, we can replace $\mathcal{D}^\lambda$ by $(1 + \mathrm{D}^2)^{|\lambda|/2}$ for the proof of this lemma. We have

$$\begin{aligned}|\operatorname{str}((1 + \mathrm{D}^2)^{|\lambda|/2}e^{-t\mathrm{D}^2}T)| &= |\operatorname{tr}\{\gamma(1 + \mathrm{D}^2)^{|\lambda|/2}e^{-t\mathrm{D}^2}T\}| \\ &\le \|\gamma(1 + \mathrm{D}^2)^{|\lambda|/2}e^{-t\varepsilon\mathrm{D}^2}\| \cdot \operatorname{tr}(e^{-t(1-\varepsilon)\mathrm{D}^2}).\end{aligned}$$

The maximal value of the function $f(x) = (1 + x)^{l/2}e^{-t\varepsilon x}$ is $(l/t\varepsilon)^{l/2}e^{t\varepsilon-l/2}$, where ε is a positive number such that $\varepsilon < \mu_1/(\mu_1 + 2)$ and μ_1 is the first nonzero eigenvalue of D^2. It follows that

$$\|\gamma(1 + \mathrm{D}^2)^{|\lambda|/2}e^{-t\varepsilon\mathrm{D}^2}\| \le C \cdot t^{-|\lambda|/2} \cdot e^{t\varepsilon-\lambda/2}.$$

Thus,

$$|\operatorname{str}((1 + \mathrm{D}^2)^{|\lambda|/2}e^{-t\mathrm{D}^2}T)| \le C \cdot t^{-|\lambda|/2} \cdot e^{t\varepsilon-\lambda/2-t(1-\varepsilon)\mu_1/2},$$

which is rapidly decaying for large time t. Here, we have applied

$$Tr(e^{-t(1-\varepsilon)\mathrm{D}^2}) \le e^{-t(1-\varepsilon)\mu_1/2}$$

for large time t, which can be found, for example, in [1]. □

THEOREM 5. (i) *If* $k > 0$, *then*

$$\operatorname*{res}_{s=0} \operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)\cdot|\mathrm{D}|^{-2s-k-2|\lambda|}T\big)$$
$$= \frac{1}{\Gamma(|\lambda|+k/2)}\lim_{t\to 0} t^{|\lambda|+k/2}\operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)\cdot e^{-t\mathrm{D}^2}T\big).$$

(ii) *If* $k = 0$, *then*

$$\operatorname*{res}_{s=0}\{s^{-1}\operatorname{str}(f_0|\mathrm{D}|^{-2s}T)\} = \lim_{t\to 0}\operatorname{str}(f_0 e^{-t\mathrm{D}^2}T).$$

(iii) *The equivariant Chern character* $\Phi_G(\mathrm{D}) = \{\Phi_G^k(\mathrm{D})\}_{k=\text{even}}$ *for the invariant Dirac operator in the equivariant cyclic cohomology is given by*

$$\Phi_G^k(\mathrm{D})(f_0, f_1, \ldots, f_k)(T)$$
$$= \frac{1}{k!\,(2\pi\sqrt{-1})^{k/2}}\int_{M^T} f_0\, df_1\wedge df_2\wedge\cdots\wedge df_k$$
$$\wedge\,\hat{\mathbf{A}}(TM^T)\wedge\mathbf{Pf}\left\{2\sinh\left(\frac{\Omega}{4\pi}+\frac{\sqrt{-1}\Theta}{2}\right)(\nu(M^T))\right\}^{-1},$$

where $k =$ even, $T\in G$, $f_i\in C^\infty(M)$, *and* M^T *is the fixed-point set that is the disjoint union of a finite number of even-dimensional totally geodesic submanifolds* $M_1^T, M_2^T, \ldots, M_r^T$.

Proof. By Theorem 2, Theorem 3, and the Connes–Moscovici theorem quoted prior to Lemma 4.1, we need only prove (i) and (ii).

In the case of $k > 0$, applying the Mellin transform yields

$$\operatorname*{res}_{s=0}\operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)\cdot|\mathrm{D}|^{-2s-k-2|\lambda|}T\big)$$
$$= \lim_{s\to 0}\frac{1}{\Gamma(|\lambda|+k/2+s)}\int_0^\infty st^{s-1}$$
$$\cdot\, t^{|\lambda|+k/2}\operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)e^{-t\mathrm{D}^2}T\big)\,dt$$
$$= \lim_{s\to 0}\frac{1}{\Gamma(|\lambda|+k/2+s)}\left(\int_0^1+\int_1^\infty\right)st^{s-1}$$
$$\cdot\, t^{|\lambda|+k/2}\operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)e^{-t\mathrm{D}^2}T\big)\,dt$$
$$= \lim_{s\to 0}\frac{1}{\Gamma(|\lambda|+k/2+s)}\int_0^1 st^{s-1}$$
$$\cdot\, t^{|\lambda|+k/2}\operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)e^{-t\mathrm{D}^2}T\big)\,dt.$$

Here we have used Lemma 4.1.

Because of Theorem 2 and Theorem 3, we have

$$\lim_{t\to 0} t^{|\lambda|+k/2}\operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)\cdot e^{-t\mathrm{D}^2}T\big) = L_\lambda + O(t).$$

Thus,

$$\begin{aligned}&\operatorname*{res}_{s=0} \operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)\cdot|\mathrm{D}|^{-2s-k-2|\lambda|}T\big)\\&\quad=\frac{1}{\Gamma(|\lambda|+k/2)}\lim_{t\to 0} t^{|\lambda|+k/2}\operatorname{str}\big(f_0\nabla^{\lambda_1}(df_1)\nabla^{\lambda_2}(df_2)\cdots\nabla^{\lambda_k}(df_k)\cdot e^{-t\mathrm{D}^2}T\big).\end{aligned}$$

Finally, in the case of $k=0$ (which is similar to the preceding situation), we have

$$\begin{aligned}\operatorname*{res}_{s=0}\{s^{-1}\operatorname{str}(f_0|\mathrm{D}|^{-2s}T)\} &= \lim_{s\to 0}\frac{1}{s\Gamma(s)}\int_0^\infty st^{s-1}\cdot\operatorname{str}(f_0e^{-t\mathrm{D}^2}T)\,dt\\&=\lim_{s\to 0}\frac{1}{\Gamma(s+1)}\int_0^\infty st^{s-1}\cdot\operatorname{str}(f_0e^{-t\mathrm{D}^2}T)\,dt\\&=\lim_{t\to 0}\operatorname{str}(f_0e^{-t\mathrm{D}^2}T).\end{aligned}$$ □

REMARK 4.2. If T is the identity operator, then the formula in part (iii) of Theorem 6 reduces to the corresponding formula in [6], which was computed using Getzler's symbolic calculus.

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