On Fundamental Groups of Fibered Complex Manifolds

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Introduction

Let X be a (not necessarily compact) complex manifold. A *fibration* on X is by definition a proper flat holomorphic map $f: X \to Y$ with connected fibers from X onto a complex manifold Y. A fiber $f^{-1}(y)$ is called *singular* if y is a critical value of f and *smooth* otherwise. We denote by Sing f the locus of the points of X where f is not smooth.

Remmert's proper mapping theorem [1] implies that the set $f(\text{Sing } f)$ of critical values of the fibration $f: X \to Y$ is a closed analytic subset of Y.

Let $f: X \to Y$ be a fibration. Then, for each $y \in Y$, we have the following sequence of natural maps:

$$\pi_1(f^{-1}(y)) \to \pi_1(X) \to \pi_1(Y) \to 1. \tag{$*$}$$

If the fibration f has no singular fiber, then it can be viewed as a topological fiber bundle (submersion lemma or Morse theory [4]) and hence the sequence $(*)$ is exact regardless of the choice of $y \in Y$.

The following result due to Nori gives a sufficient condition for the exactness of the sequence $(*)$ for the smooth fibers.

Nori's Lemma [5, Lemma 1.5(C)]. *Let X and Y be smooth connected varieties over $\mathbf{C}$ and let $f: X \to Y$ be an arbitrary morphism. Suppose that the general smooth fiber of f is connected, and let the locus of all $y \in Y$ such that $f^{-1}(y)$ does not have any simple component be of codimension ≥ 2 in Y. Then the sequence $(*)$ is exact for all smooth fibers $f^{-1}(y)$.*

First, we give sufficient conditions for the exactness of the sequence $(*)$ for any choice of fiber.

Theorem 1. *Let $f: X \to Y$ be a fibration. Let the locus of all y such that $f^{-1}(y)$ is not simply connected and does not have any simple component be of codimension ≥ 2 in Y. Then the sequence $(*)$ is exact for all $y \in Y$.*

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THEOREM 2. *Let $f: X \to Y$ be a fibration from a singular variety X onto a Riemann surface Y. Suppose that the singular locus of X is contained in finitely many fibers of f and that, for each critical value c, either $f^{-1}(c)$ is simply connected or there exists a local section of f near c. Then $(*)$ is exact for all $y \in Y$.*

The conditions in Theorems 1 and 2 are somewhat stricter than Nori's, so we do not use Nori's result in our proofs.

Next we consider the following sequence of maps:

$$1 \to \pi_1(f^{-1}(y)) \to \pi_1(X) \to \pi_1(Y) \to 1. \tag{$**$}$$

Shimada has given a sufficient condition for the exactness of the sequence $(**)$ for smooth fibers as follows.

THEOREM A [7; 8]. *Let $f: X \to Y$ be a locally projective fibration. Suppose that:*

(i) *there exists a (topological) continuous section $s: Y \to X$ of f such that $s(Y) \cap \operatorname{Sing} f = \emptyset$; and*
(ii) $\operatorname{codim}_X(\operatorname{Sing} f) \geq 3$.

Then the sequence $(**)$ is exact for all smooth fibers $f^{-1}(y)$.

REMARKS. (1) In fact, Shimada [7] studied the more general case where X is the complement of a divisor of a complex manifold. We simply restate his result in our case to derive the simplified form given here.

(2) If the fibration $f: X \to Y$ is of relative dimension 1, then the theorem of purity—due to Dolgachev, Ramanujam, and Simha [3; 6; 9]—states that

$$\operatorname{codim}_X(\operatorname{Sing} f) \leq 2$$

if $\operatorname{Sing} f \neq \emptyset$. So, in Theorem A, if f is not smooth then we need to assume that $\dim X \geq \dim Y + 2$.

Here, we refine Theorem A as follows.

THEOREM 3. *Under the same condition as in Theorem A, the sequence $(**)$ is exact for all $y \in Y$. In particular, all fibers have the same fundamental group.*

Theorem 3 and the following corollary tell us that, under a "mild" deformation, the fundamental group remains invariant.

COROLLARY. *Let $f: X \to Y$ be a locally projective fibration, and let $f^{-1}(c)$ be an isolated singular fiber. Then we have an isomorphism*

$$\pi_1(f^{-1}(c)) \cong \pi_1(\text{a general smooth fiber})$$

under one of the following three conditions:

(1) $\dim Y = 1$, $\dim X \geq 3$, *and* $\operatorname{codim}_{f^{-1}(c)}(\operatorname{Sing} f \cap f^{-1}(c)) \geq 2$;
(2) $\dim Y = 2$, $\dim X \geq 4$, *and* $\operatorname{codim}_{f^{-1}(c)}(\operatorname{Sing} f \cap f^{-1}(c)) \geq 1$; *or*
(3) $\dim Y \geq 3$, $\dim X \geq \dim Y + 2$, *and at least one component of $f^{-1}(c)$ is simple.*

1. Proofs of Theorems 1 and 2

In the proofs of Theorems 1 and 2 we shall use the following fact, whose proof can be found, for example, in [2].

PROPOSITION A. *Let $f\colon X \to \Delta$ be a proper flat holomorphic map from a complex variety X onto a disk that is smooth over $\Delta \setminus \{0\}$. Then the central fiber is a deformation retract of the total space X.*

The following lemma is also needed.

LEMMA 1. *Let X be a real manifold, and let Z be the support of a cycle in X of real codimension ≥ 3. Then $\pi_1(X - Z) \cong \pi_1(X)$.*

Proof. The inclusion $X - Z \hookrightarrow X$ induces a homomorphism $\pi_1(X - Z) \to \pi_1(X)$. This map is surjective if $\operatorname{codim}_X Z \geq 2$ (any loop in X can be removed from Z having codimension ≥ 2). This map is injective if $\operatorname{codim}_X Z \geq 3$: if $\alpha - \beta$ is the boundary of a disk in X, then it is the boundary of a disc in $X - Z$.

Proof of Theorem 1

We note that if a fiber has a simple component then there always exists a local section near the fiber. This can be seen as follows. Pick a general smooth point of a simple component at which the map f is smooth; then, near this point, f looks like a projection from a product space onto a factor and hence has a section.

We will show first that the sequence $(*)$ is exact for all smooth fibers $f^{-1}(y)$. Let C be the codimension-2 subset of Y outside which all the fibers of f are either simply connected or have a simple component. By Lemma 1,

$$\pi_1(Y) \cong \pi_1(Y \setminus C) \quad \text{and} \quad \pi_1(X) \cong \pi_1(f^{-1}(Y \setminus C)).$$

We may therefore assume that each singular fiber of f is either simply connected or has a simple component. Let $Y' = Y - \{\text{critical values}\}$ and $X' = f^{-1}(Y')$. Then $f\colon X' \to Y'$ becomes a topological fiber bundle and hence, for all $y \in Y'$, the sequence

$$\pi_1(f^{-1}(y)) \to \pi_1(X') \to \pi_1(Y') \to 1$$

is exact.

Consider the following commutative diagram of natural maps:

$$\begin{array}{ccccccc}
\pi_1(f^{-1}(y)) & \xrightarrow{i'_*} & \pi_1(X') & \xrightarrow{f'_*} & \pi_1(Y') & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} & & \\
\pi_1(f^{-1}(y)) & \xrightarrow{i_*} & \pi_1(X) & \xrightarrow{f_*} & \pi_1(Y) & \longrightarrow & 1.
\end{array}$$

Since ν and μ are surjective (see the proof of Lemma 1), and since the top row is exact, f_* in the bottom is surjective. To show the exactness of the bottom row, we need to show that $\ker(f_*) \subset \operatorname{im}(i_*)$.

Suppose that $\gamma \in \ker(f_*)$. Pick up a loop γ' in X' such that $\nu(\gamma') = \gamma$. Then $\mu f'_*(\gamma')$ is the boundary of a disk D in Y. Futhermore, we may assume that the disk D meets the locus of critical values transversally in finitely many points $c_1, \dots, c_r$.

We write

$$f'_*(\gamma') = \prod_i \delta_i,$$

where δ_i is a small loop in Y around the critical point c_i. Lift δ_i to a loop δ'_i which is null homotopic in X. This can be done as follows. If $f^{-1}(c_i)$ has a simple component then a local section lifts the small disc surrounded by δ_i into X, and δ'_i can then be contracted along the lifted small disc. If $f^{-1}(c_i)$ is simply connected, then any lifting δ'_i of δ_i, being homotopic in X to a loop in $f^{-1}(c_i)$, is null homotopic.

Now we see that

$$\gamma'\Big(\prod(\delta'_i)^{-1}\Big) \in \ker(f'_*),$$

and there consequently exists a loop β in the fiber $f^{-1}(y)$ such that

$$i'_*(\beta) = \gamma'\Big(\prod(\delta'_i)^{-1}\Big).$$

Now, since all $\nu(\delta'_i)$ are null homotopic, we see that

$$i_*(\beta) = \nu(i'_*(\beta)) = \nu(\gamma') = \gamma.$$

Hence $\gamma \in \operatorname{im}(i_*)$ and so the bottom row is exact.

Now, let c be a critical value of f, that is, $f^{-1}(c)$ is a singular fiber. Then there exists an analytic disk Δ in Y passing through c such that $f|_{f^{-1}(\Delta)}$ is smooth over $\Delta \setminus \{c\}$. Then, by Proposition A, $f^{-1}(c)$ is a deformation retract of $f^{-1}(\Delta)$. Pick $y \in \Delta \setminus \{c\}$. Then every loop in $f^{-1}(y)$ can be deformed into a loop in $f^{-1}(c)$. Thus we can replace $\pi_1(f^{-1}(y))$ by $\pi_1(f^{-1}(c))$, yielding the desired exact sequence. □

Proof of Theorem 2

Let $c_1, \dots, c_n$ denote the critical values of f or the images of the singular locus of X. Let $Y' = Y \setminus \{c_1, \dots, c_n\}$ and $X' = f^{-1}(Y')$. Consider the commutative diagram in the proof of Theorem 1. The proof in this case is exactly the same as the one given for Theorem 1; we have only to prove the surjectivity of ν. Toward this end, we take a smooth resolution $\sigma: \bar{X} \to X$ and let $\bar{X}' = (f \circ \sigma)^{-1}(Y')$; then we see that $\bar{X}' = X'$ and that the map ν is the composite

$$\pi_1(X') = \pi_1(\bar{X}') \to \pi_1(\bar{X}) \to \pi_1(X)$$

of two surjective maps. □

COROLLARY 1.1. *Under the assumptions of Theorem 1 or Theorem 2, and if, in addition, at least one fiber is simply connected, then* $\pi_1(X) \cong \pi_1(Y)$.

COROLLARY 1.2. *Let* $f: X \to \Delta$ *be a fibration from an irreducible complex analytic variety* X *onto a disc* Δ. *Suppose that the singular locus of* X *is contained in the central fiber* X_0 *and that* f *is smooth away from the central fiber* X_0. *Suppose that there exists a section of* f. *Then the composition of maps*

$$\pi_1(X_y) \to \pi_1(X) \cong \pi_1(X_0)$$

is surjective for all $y \in \Delta$, where the first map is induced by inclusion and the second by deformation retract.

Proof. Apply Theorem 2 to the fibration f and note that $\pi_1(\Delta) = 1$. □

2. Invariance of Fundamental Group under a Deformation

Proof of Theorem 3

Let $c \in Y$ be a critical value of f. By Theorem A it suffices to show that $\pi_1(f^{-1}(c)) \cong \pi_1(f^{-1}(y))$ for a regular value $y \in Y$. To show this, pick a disc Δ in Y passing through the point c such that the restriction $f|_\Delta \colon X|_\Delta := f^{-1}(\Delta) \to \Delta$ is smooth away from the central fiber $f^{-1}(c)$. Then $X|_\Delta$ is nonsingular away from the central fiber.

Let $y \in \Delta - \{c\}$. Because there exists a section $s|_\Delta \colon \Delta \to X|_\Delta$, Corollary 1.2 implies that the map $\pi_1(X_y) \to \pi_1(X_c)$ is surjective. Consider the following commutative diagram of maps:

$$\begin{array}{ccccccccc} & & & & \pi_1(X|_\Delta) & & & & \\ & & & \nearrow & \downarrow & & & & \\ 1 & \longrightarrow & \pi_1(X_y) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(Y) & \longrightarrow & 1. \end{array}$$

Here the bottom row is exact by Theorem A, so the map $\pi_1(X_y) \to \pi_1(X|_\Delta)$ is injective. Therefore $\pi_1(X_y) \cong \pi_1(X_c)$ (Proposition A), and the theorem is proved. □

Proof of Corollary

Since the fiber $f^{-1}(c)$ is isolated, we may assume, by shrinking Y if necessary, that $f^{-1}(c)$ is the only singular fiber of f. By Theorem 3, it is enough to show that each of the three conditions of the corollary implies the two conditions (i) and (ii) of Theorem 3.

Suppose that one of conditions (1), (2), and (3) holds. Then at least one component of $f^{-1}(c)$ is simple and hence there exists a local section of f near the point c. By shrinking Y if necessary, we obtain condition (i) of Theorem 3; condition (ii) follows obviously. □

EXAMPLE 2.1. Let X_0 be a projective surface with isolated singularities, and consider a 1-parameter deformation $f \colon X \to \Delta$ that is smooth away from the central fiber X_0. Suppose that the central fiber is nonmultiple and that X is smooth. Then, by Theorem 3, $\pi_1(X_0) \cong \pi_1(X_y)$ for all $y \in \Delta$.

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