Harmonic Cohomology Classes of Almost Cosymplectic Manifolds

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1. Introduction

The concept of Poisson structure plays an important role in mathematics and physics. Apparently, Poisson structures in local coordinates were first considered in 1875 in the work of Lie [20]; from the mathematical viewpoint, such a theory has been developed since the early 1970s by Lichnerowicz [19], Weinstein [26], and others. A Poisson manifold is a smooth manifold M endowed with a Poisson bracket, that is, a Lie bracket $\{\ ,\ \}$ on the algebra of smooth functions on M satisfying Leibniz's rule. The existence of a Poisson bracket on M is equivalent to the existence of a skew-symmetric contravariant 2-tensor G on M satisfying $[G, G] = 0$, where $[\ ,\ ]$ denotes the Schouten–Nijenhuis bracket [1].

For a Poisson manifold M, Koszul [15] introduced a differential operator Δ: $\Lambda^k(M) \to \Lambda^{k-1}(M)$, defined by $\Delta = [i(G), d]$, where $i(G)$ denotes the contraction by G and d is the exterior derivative of M. We call it the Koszul differential and shall write δ instead of Δ. Since $\delta^2 = 0$ [4; 15], it defines the so-called canonical homology of M. Moreover, as in the Riemannian case, a Poisson Laplacian $\Delta = d\delta + \delta d$, which is identically zero, can be defined [15]. A k-form α is called harmonic (with respect to the Poisson structure) if $d\alpha = \delta\alpha = 0$. In [4], Brylinski proposed the following.

Problem. Give conditions on a compact Poisson manifold M ensuring that any de Rham cohomology class has a harmonic (with respect to the Poisson structure) representative α, that is, $d\alpha = \delta\alpha = 0$.

In the particular case of symplectic manifolds, this problem has already been solved [4; 8; 21]. More precisely, Brylinski [4] proved that for compact Kähler manifolds this problem has an affirmative solution. However, we exhibit in [8] an example of a compact symplectic manifold M^4 and a de Rham cohomology class $\mathfrak{a}$ on M^4 such that $\mathfrak{a}$ does not admit harmonic representatives. Independently, Mathieu [21] proved that a compact symplectic manifold has the conditions of Brylinski's problem if and only if it satisfies the hard Lefschetz theorem.

Almost cosymplectic manifolds are another important class of Poisson manifolds. Remember that an almost cosymplectic manifold is a $(2n+1)$-dimensional

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manifold with a closed 2-form Φ and a closed 1-form η such that $\eta \wedge \Phi^n \neq 0$ [2]. Roughly speaking, almost cosymplectic manifolds may be considered as the odd-dimensional counterpart of symplectic manifolds.

The aim of this paper is to study Brylinski's problem for almost cosymplectic manifolds. The main result, proved in Section 5 (see Theorem 5.4), is that an important class of almost cosymplectic manifolds satisfies Brylinski's problem.

MAIN THEOREM. *Let M be a compact cosymplectic manifold. Then any de Rham cohomology class of M has a harmonic* (*with respect to the Poisson structure*) *representative.*

Cosymplectic manifolds appear to be the closest odd-dimensional analog of Kähler manifolds; several known results from Kähler geometry carry over to the cosymplectic manifolds [23; 24], particularly topological properties [3; 6]. However, the proof of our main theorem is not a consequence of the corresponding result for Kähler manifolds. We need to introduce and study, in Section 3, new operators for almost cosymplectic manifolds. Moreover, in Section 4 we prove the cosymplectic version of the Hodge decomposition theorem for compact Kähler manifolds [12; 27].

We show in Section 6 that the conditions of Brylinski's problem are not satisfied for any arbitrary almost cosymplectic manifold. In fact, we construct a 5-dimensional compact manifold M^5 and an almost cosymplectic structure on M^5 such that M^5 has a de Rham cohomology class (of degree 3) not admitting a harmonic representative.

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2. The Koszul Differential of a Poisson Manifold

Let M be a C^∞ manifold. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M, and by $\mathfrak{F}(M)$ the algebra of C^∞ functions on M. A *Poisson bracket* $\{\ ,\ \}$ on M is a bilinear mapping

$$\{\ ,\ \}: \mathfrak{F}(M) \times \mathfrak{F}(M) \to \mathfrak{F}(M)$$

satisfying the following properties:

(i) $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (Leibniz's rule),
(ii) $\{f, g\} = -\{g, f\}$, and
(iii) $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$ (Jacobi's identity)
for $f, g, h \in \mathfrak{F}(M)$.

Properties (ii) and (iii) mean that $\{\ ,\ \}$ endows $\mathfrak{F}(M)$ with a Lie algebra structure. Moreover, for fixed $f \in \mathfrak{F}(M)$, property (i) implies that the mapping $g \mapsto \{f, g\}$ defines a vector field X_f, which is called the *Hamiltonian vector field* corresponding to f. Thus,

$$X_f(g) = \{f, g\} \quad \text{for } g \in \mathfrak{F}(M).$$

A manifold M endowed with a Poisson bracket is called a *Poisson manifold.*

Poisson manifolds were introduced by Lichnerowicz [19], who remarked that a Poisson bracket gives rise to a skew-symmetric tensor field of type (2, 0) on M such that

$$G(df, dg) = \{f, g\} \quad \text{for } f, g \in \mathfrak{F}(M). \tag{1}$$

G is called a *Poisson tensor* and satisfies $[G, G] = 0$, where $[\ ,\ ]$ is the Schouten–Nijenhuis bracket [1]. Conversely, given such a tensor G we can recover the Poisson bracket by means of (1). The rank of G is called the rank of the Poisson structure. (In general, the rank of G is not constant.)

The local structure of a Poisson manifold M was elucidated by Weinstein [26]. Concretely, if the Poisson structure has constant rank, then there exist local coordinates $\{q^1, \ldots, q^r, p_1, \ldots, p_r, z^1, \ldots, z^{m-2r}\}$ around each point of M such that

$$\begin{aligned} G &= \sum_{i=1}^{r} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \\ X_f &= \sum_{i=1}^{r} \left\{ \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} \right\}, \\ \{f, g\} &= \sum_{i=1}^{r} \left\{ \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right\}. \end{aligned} \tag{2}$$

(Such local coordinates are called *Darboux coordinates*.)

Next, we shall denote by $\Lambda^k(M)$ the space of the differential k-forms on M. Koszul [15] introduced the differential operator $\delta\colon \Lambda^k(M) \to \Lambda^{k-1}(M)$ given by the commutator of $i(G)$ and the exterior differential d; that is,

$$\delta = [i(G), d] = i(G) \circ d - d \circ i(G). \tag{3}$$

On the other hand, Brylinski [4], inspired by the differential of the Chevalley–Eilenberg complex $C_*(\mathfrak{F}(M), \mathfrak{F}(M))$, has proved that δ can be alternatively defined by the formula

$$\begin{aligned} \delta(f_0\, df_1 \wedge \cdots \wedge df_k) = {} & \sum_{1 \le i \le k} (-1)^{i+1} \{f_0, f_i\}\, df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k \\ & + \sum_{1 \le i < j \le k} (-1)^{i+j} f_0\, d\{f_i, f_j\} \\ & \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k. \end{aligned} \tag{4}$$

Since $\delta^2 = 0$ [4; 15], the *canonical complex* of M is the complex

$$\cdots \to \Lambda^{k+1}(M) \xrightarrow{\delta} \Lambda^k(M) \xrightarrow{\delta} \Lambda^{k-1}(M) \to \cdots.$$

The homology of this complex is denoted by $H_*^{\mathrm{can}}(M)$ and is called the *canonical homology* of M.

REMARK 2.1. The canonical homology was called Poisson homology by Huebschmann [13] and (Koszul–Brylinski)–Poisson homology by Vaisman [25].

Taking into acount the Hodge theory for Riemannian manifolds, we may consider a Laplacian related to the Poisson structure $\Delta = d\delta + \delta d$, which identically vanishes [15]. We can define the notion of harmonicity for a compact Poisson manifold as follows: A k-form α on a compact Poisson manifold M is *harmonic* (with respect to the Poisson structure) if $d\alpha = \delta\alpha = 0$. Brylinski [4] stated the following problem.

BRYLINSKI'S PROBLEM. Give conditions on a compact Poisson manifold M ensuring that any de Rham cohomology class in $H^k(M)$ has a harmonic (with respect to the Poisson structure) representative α, that is, $d\alpha = \delta\alpha = 0$.

Brylinski's problem has already been studied for compact symplectic manifolds (see [4; 8; 21]).

In [4] Brylinski also defined, for symplectic manifolds, the *symplectic star operator* $\star$, imitating the definition of the Hodge star operator for Riemannian manifolds. He proved that

$$\star^2 = I \quad \text{and} \quad \delta = (-1)^{k+1} \star d \star$$

on $\Lambda^k(M)$. The symplectic star operator $\star$ permits us to relate the canonical homology with the de Rham cohomology of M.

PROPOSITION 2.2 [4]. *The symplectic star operator $\star$ establishes an isomorphism of the canonical homology group $H_k^{\mathrm{can}}(M)$ with the de Rham cohomology group $H^{2n-k}(M)$ for M a symplectic manifold of dimension $2n$.*

REMARK 2.3. The symplectic star operator was first considered in the 1950s by Libermann [17]. Take the isomorphism $\mu\colon \mathfrak{X}(M) \to \Lambda^1(M)$ defined by $\mu(X) = i(X)\omega$ for any $X \in \mathfrak{X}(M)$, and extend it to an algebra isomorphism $\mu\colon \bigoplus_{k\geq 0} \mathfrak{X}^k(M) \to \bigoplus_{k\geq 0} \Lambda^k(M)$. Hence, the symplectic star operator is defined for each k-form α by

$$\star(\alpha) = (-1)^k i(\mu^{-1}(\alpha))v_M, \tag{5}$$

where v_M is the volume form on M given by $v_M = \omega^n/n!$.

3. Almost Cosymplectic Manifolds

A $(2n+1)$-dimensional manifold M is said to have an *almost contact structure* if there exists on M a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1.$$

Moreover, if the almost contact structure (ϕ, ξ, η) on M admits a compatible Riemannian metric g—that is, if

$$g(\phi(X), \phi(Y)) = g(X,Y) - \eta(X)\eta(Y)$$

for $X, Y \in \mathfrak{X}(M)$—then M is said to have an almost contact metric structure and a 2-form Φ on M can be defined by

$$\Phi(X, Y) = g(\phi(X), Y)$$

for $X, Y \in \mathfrak{X}(M)$. This is called the fundamental 2-form of the almost contact metric structure, and satisfies $\eta \wedge \Phi^n \neq 0$. Moreover, $v_M = (\eta \wedge \Phi^n)/n!$ is a volume form on M.

A $(2n+1)$-dimensional manifold M with an almost contact metric structure (ϕ, ξ, η, g) is called:

(i) *almost cosymplectic* iff Φ and η are closed;
(ii) *normal* iff $N_\phi + 2d\eta \otimes \xi = 0$, where N_ϕ is the Nijenhuis torsion of ϕ;
(iii) *cosymplectic* iff it is normal and almost cosymplectic (or equivalently iff $\nabla\phi = 0$).

From now on, we suppose that M is an almost cosymplectic manifold of dimension $(2n+1)$, with almost contact metric structure (ϕ, ξ, η, g) and fundamental 2-form Φ.

As in the symplectic case, almost cosymplectic manifolds have an associated Poisson structure [9, 11]. For each function $f \in \mathfrak{F}(M)$, there exists a unique vector field X_f on M such that

$$i(X_f)\Phi = df - \xi(f)\eta,$$
$$\eta(X_f) = 0,$$

called the Hamiltonian vector field associated to f. Then, the Poisson bracket $\{\ ,\ \}$ on M is defined by

$$\{f, g\} = -\Phi(X_f, X_g)$$

for any $f, g \in \mathfrak{F}(M)$. Furthermore, the Poisson tensor G is given by (1) and its rank is $2n$. If we consider Darboux coordinates $\{q^1, \dots, q^n, p_1, \dots, p_n, z\}$ in a neighborhood of every point of M, then the fundamental 2-form Φ, the vector field ξ, and the 1-form η may be written by means of these coordinates as

$$\Phi = \sum_{i=1}^{n} dp_i \wedge dq^i, \quad \xi = \frac{\partial}{\partial z}, \quad \eta = dz;$$

the other elements are given by (2). We then have the Koszul differential for almost cosymplectic manifolds, and we may ask for the solution of Brylinski's problem for such Poisson manifolds.

First, Libermann's definition (5) of the symplectic star operator permits us to extend this concept to almost cosymplectic manifolds. We consider the isomorphism [9] $\mu\colon \mathfrak{X}(M) \to \Lambda^1(M)$ defined by $\mu(X) = i(X)\Phi + (i(X)\eta)\eta$ for any $X \in \mathfrak{X}(M)$, and we extend it to an algebra isomorphism $\mu\colon \bigoplus_{k\geq 0} \mathfrak{X}^k(M) \to \bigoplus_{k\geq 0} \Lambda^k(M)$. Hence, we define the *almost cosymplectic star operator* as the isomorphism $\star\colon \Lambda^k(M) \to \Lambda^{(2n+1)-k}(M)$ given by

$$\star(\alpha) = (-1)^k i(\mu^{-1}(\alpha))v_M \tag{6}$$

for any $\alpha \in \Lambda^k(M)$.

In order to study the almost cosymplectic star operator and the Koszul differential on almost cosymplectic manifolds, we must recall some known facts. Fernández, Ibáñez, and de León [9; 11] have shown that the space $\Lambda^k(M)$ may be decomposed as

$$\Lambda^k(M) = \Lambda^k_\xi(M) \oplus \Lambda^k_\eta(M), \tag{7}$$

where the spaces $\Lambda^k_\xi(M)$ and $\Lambda^k_\eta(M)$ are defined as follows:

$$\Lambda^k_\xi(M) = \{\alpha \in \Lambda^k(M) \mid i(\xi)\alpha = 0\},$$
$$\Lambda^k_\eta(M) = \{\alpha \in \Lambda^k(M) \mid \eta \wedge \alpha = 0\}.$$

In fact, in [9] it is proved that for any $\alpha \in \Lambda^k(M)$,

$$\alpha = (\alpha - \eta \wedge i(\xi)\alpha) + \eta \wedge i(\xi)\alpha,$$

with $(\alpha - \eta \wedge i(\xi)\alpha) \in \Lambda^k_\xi(M)$ and $(\eta \wedge i(\xi)\alpha) \in \Lambda^k_\eta(M)$. In that paper the following operators on $\Lambda^*_\xi(M)$ were defined: the almost cosymplectic ξ-star operator $\star_\xi$, defined in a similar way to the symplectic star operator; and the differential operator d_ξ of degree $+1$, which it is the projection of the exterior differential d over the space $\Lambda^*_\xi(M)$. It was also shown in [9] that

$$\delta = (-1)^{k+1} \star_\xi d_\xi \star_\xi \quad \text{and} \quad {\star_\xi}^2 = I, \tag{8}$$

on $\Lambda^*_\xi(M)$.

Both operators $\star$ and $\star_\xi$ are related in the following way.

PROPOSITION 3.1. *Let M be a $(2n+1)$-dimensional almost cosymplectic manifold. Then*

(i) $\star(\alpha) = (-1)^k \eta \wedge (\star_\xi \alpha)$ *for* $\alpha \in \Lambda^k_\xi(M)$;
(ii) $\star(\alpha) = -\star_\xi(i(\xi)\alpha)$ *for* $\alpha \in \Lambda^k_\eta(M)$.

Proof. First, we shall explain the Libermann-type definition of the almost cosymplectic ξ-star operator. We consider the restriction of the isomorphism $\mu\colon \mathfrak{X}(M) \to \Lambda^1(M)$ to the subspace $\mathfrak{X}_\eta(M) = \{X \in \mathfrak{X}(M) \mid \eta(X) = 0\}$, that is, the isomorphism $\hat{\mu}\colon \mathfrak{X}_\eta(M) \to \Lambda^1_\xi(M)$, which we extend to an algebra isomorphism. Then

$$\star_\xi(\alpha) = (-1)^k i(\hat{\mu}^{-1}(\alpha)) v^\xi_M \tag{9}$$

for any $\alpha \in \Lambda^k_\xi(M)$ where $v^\xi_M = i(\xi)v_M$.

Now, since $v_M = \eta \wedge v^\xi_M$ and $\hat{\mu}$ is the restriction of μ, from (6) and (9) we have the following.

(i) For $\alpha \in \Lambda^k_\xi(M)$:

$$\begin{aligned}\star\alpha &= (-1)^k i(\mu^{-1}\alpha)v_M = (-1)^k i(\hat{\mu}^{-1}\alpha)(v^\xi_M \wedge \eta)\\ &= (-1)^k (i(\hat{\mu}^{-1}\alpha)v^\xi_M) \wedge \eta = \star_\xi(\alpha) \wedge \eta\\ &= (-1)^k \eta \wedge \star_\xi(\alpha).\end{aligned}$$

(ii) For $\alpha \in \Lambda^k_\eta(M)$: $\alpha = \eta \wedge i(\xi)\alpha$ and

$$\begin{aligned}\star\alpha &= \star(\eta \wedge i(\xi)\alpha) = (-1)^k i(\mu^{-1}(\eta \wedge i(\xi)\alpha))v_M \\ &= (-1)^k i(\xi \wedge \hat{\mu}^{-1}(i(\xi)\alpha))(\eta \wedge v_M^\xi) \\ &= (-1)^k \hat{\mu}^{-1}(i(\xi)\alpha)v_M^\xi = -\star_\xi(i(\xi)\alpha).\end{aligned}$$ □

COROLLARY 3.2.

(i) $\star^2\alpha = (-1)^{k+1}\alpha$ *for* $\alpha \in \Lambda^k_\xi(M)$;
(ii) $\star^2\alpha = (-1)^k\alpha$ *for* $\alpha \in \Lambda^k_\eta(M)$.

Proof. This follows directly from (8) and Proposition 3.1. □

Notice that, because of the canonical decomposition (7), we deduce from Corollary 3.2 that $\star^2 \neq I$, but

$$\star^4 = I. \tag{10}$$

Next, we introduce a differential operator of degree -1 defined by

$$\delta_2 = (-1)^{k+1} \star^3 d\star \tag{11}$$

on $\Lambda^k(M)$; we call it the *second Koszul differential* of an almost cosymplectic manifold. From (10), ${\delta_2}^2 = 0$ and we consider the complex

$$\cdots \to \Lambda^{k+1}(M) \xrightarrow{\delta_2} \Lambda^k(M) \xrightarrow{\delta_2} \Lambda^{k-1}(M) \to \cdots.$$

The homology of this complex will be called *second canonical homology* of M; it is denoted by $H_*^{\mathrm{can2}}(M)$.

PROPOSITION 3.3. *The almost cosymplectic star operator $\star$ establishes an isomorphism of the second canonical homology group $H_k^{\mathrm{can2}}(M)$ with the de Rham cohomology group $H^{(2n+1)-k}(M)$, for M an almost cosymplectic manifold of dimension $(2n+1)$.*

REMARK 3.4. The operators δ and δ_2 are different, as are their homology groups; that is,

$$H_k^{\mathrm{can2}}(M) \ncong H_k^{\mathrm{can}}(M),$$

as we shall show in Section 6. Nevertheless, there exists a relation between both operators, as we shall prove in the following proposition.

PROPOSITION 3.5. *Let M be a $(2n+1)$-dimensional almost cosymplectic manifold. Then:*

(i) $\delta_2 = \delta$ *on* $\Lambda^*_\xi(M)$;
(ii) *for* $\alpha \in \Lambda^k_\eta(M)$,

$$\delta_2(\alpha) = \eta \wedge \delta(i(\xi)\alpha) + \beta,$$

where $\beta \in \Lambda^k_\xi(M)$.

Proof. From (8), (11), Proposition 3.1, and Corollary 3.2, we have the following.

(i) For $\alpha \in \Lambda_\xi^k(M)$:

$$\begin{aligned}\delta_2\alpha &= (-1)^{k+1} \star^3 d \star (\alpha) = (-1)^{k+1} \star^3 d((-1)^k \eta \wedge \star_\xi \alpha) \\ &= \star^3(\eta \wedge d(\star_\xi \alpha)) = (-1)^k \star (\eta \wedge d(\star_\xi \alpha)) \\ &= (-1)^k \star (\eta \wedge d_\xi(\star_\xi \alpha)) = (-1)^{k+1} \star_\xi d_\xi \star_\xi (\alpha) \\ &= \delta\alpha.\end{aligned}$$

(ii) For $\alpha \in \Lambda_\eta^k(M)$:

$$\begin{aligned}\delta_2\alpha &= (-1)^{k+1} \star^3 d \star (\alpha) = (-1)^k \star^3 d \star_\xi (i(\xi)\alpha) \\ &= (-1)^k \star^3 \left[d_\xi \star_\xi (i(\xi)\alpha) + (d - d_\xi) \star_\xi (i(\xi)\alpha)\right] \\ &= \star\left[-d_\xi \star_\xi (i(\xi)\alpha) + (d - d_\xi) \star_\xi (i(\xi)\alpha)\right] \\ &= (-1)^{k+1}\eta \wedge \star_\xi d_\xi \star_\xi (i(\xi)\alpha) - \star_\xi i(\xi)(d - d_\xi) \star_\xi (i(\xi)\alpha) \\ &= \eta \wedge \delta(i(\xi)\alpha) - \star_\xi i(\xi)(d - d_\xi) \star_\xi (i(\xi)\alpha),\end{aligned}$$

where $\star_\xi i(\xi)(d - d_\xi) \star_\xi (i(\xi)\alpha) \in \Lambda_\xi^k(M)$; it may be seen locally that this is equal to $i(\xi)di(\xi)(\alpha)$. □

REMARK 3.6. Since an almost cosymplectic manifold is a Poisson manifold, the "Poisson" Laplacian $\Delta = d\delta + \delta d$ identically vanishes. However, we can define a new Laplacian Δ_2 for δ_2: $\Delta_2 = d\delta_2 + \delta_2 d$. A direct computation in Darboux coordinates $\{q^1, \dots, q^n, p_1, \dots, p_n, z\}$ shows that, for any $f \in \mathfrak{F}(M)$,

$$\Delta_2(f) = -\frac{\partial^2 f}{\partial z^2}.$$

For the Koszul differential δ we have Brylinski's problem asking for conditions to ensure that any de Rham cohomology class in $H^k(M)$ has a representative α such that $d\alpha = \delta\alpha = 0$. Therefore, we can formulate a similar problem for the second Koszul differential δ_2 on almost cosymplectic manifolds.

NEW PROBLEM. Give conditions on a compact almost cosymplectic manifold M ensuring that any de Rham cohomology class in $H^k(M)$ has a representative α such that $d\alpha = \delta_2\alpha = 0$.

4. A Decomposition Hodge's Theorem for Cosymplectic Manifolds

This section is devoted to giving a similar result for compact cosymplectic manifolds to the Hodge decomposition theorem for compact Kähler manifolds [12; 27]. To prove this result, we follow Chinea, de León, and Marrero [6], where the authors developed a detailed study of topological properties of cosymplectic manifolds and introduced the trigraduation of complex forms.

First of all, we recall the following result about the harmonicity with respect to the Riemannian metric.

PROPOSITION 4.1 [6, Prop. 1]. *Let M be a $(2n+1)$-dimensional compact cosymplectic manifold, with almost contact metric structure (ϕ, ξ, η, g) and fundamental 2-form Φ. Then:*

(i) *if α is a harmonic k-form then $\eta \wedge \alpha$ is a harmonic $(k+1)$-form;*
(ii) *α is a harmonic k-form iff $\alpha - \eta \wedge i(\xi)\alpha$ is a harmonic k-form and $i(\xi)\alpha$ is a harmonic $(k-1)$-form;*
(iii) *$\Lambda^k_H(M) = \Lambda^k_{H,\xi}(M) \oplus \Lambda^k_{H,\eta}(M)$, where $\Lambda^k_H(M)$ is the space of harmonic k-forms and $\Lambda^k_{H,\xi}(M)$ (resp. $\Lambda^k_{H,\eta}(M)$) is the subspace of $\Lambda^k_\xi(M)$ (resp. $\Lambda^k_\eta(M)$) of harmonic k-forms.*

COROLLARY 4.2. *The induced mapping*

$$\widehat{i(\xi)}\colon \Lambda^k_{H,\eta}(M) \to \Lambda^{k-1}_{H,\xi}(M) \tag{12}$$

is a linear isomorphism.

Proof. Recall [9] that the mapping $\widehat{i(\xi)}\colon \Lambda^k_\eta(M) \to \Lambda^{k-1}_\xi(M)$ is defined by $\widehat{i(\xi)}(\alpha) = i(\xi)\alpha$ for $\alpha \in \Lambda^k_\eta(M)$. The result then follows from Proposition 4.1. □

Next, we shall present the trigraduation of complex forms. Let M be a $(2n+1)$-dimensional manifold with an almost contact metric structure (ϕ, ξ, η, g) and fundamental 2-form Φ. We extend ϕ and η to a complex tensor field of type $(1,1)$ and a complex 1-form, respectively, and denote them by the same symbols. Hence, the eigenvalues of ϕ are $\sqrt{-1}$, $-\sqrt{-1}$, and 0. As in the almost Hermitian manifolds, we obtain a trigraduation of complex forms on M:

$$\Lambda^k_{\mathbb{C}}(M) = \bigoplus_{\substack{p,q=0,\dots,n;\, r=0,1\\ p+q+r=k}} \Lambda^{p,q,r}(M).$$

In fact, for a cosymplectic manifold there exist local coordinates $\{z_1, \dots, z_n, z\}$ around each point of M (with z_j complex coordinates and z a real coordinate) such that $\{dz_1, \dots, dz_n\}$ is a local basis of complex forms of tridegree $(1,0,0)$, $\{d\bar{z}_1, \dots, d\bar{z}_n\}$ of tridegree $(0,1,0)$, and $\{dz\}$ of tridegree $(0,0,1)$. It follows that the set of complex forms $\{dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}\}$ and $\{dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q} \wedge dz\}$ with $1 \le j_1 \le \cdots \le j_p \le n$ and $1 \le k_1 \le \cdots \le k_q \le n$ are local bases for $\Lambda^{p,q,0}(M)$ and $\Lambda^{p,q,1}(M)$, respectively. Moreover,

$$\Phi = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j, \qquad \eta = dz. \tag{13}$$

(See [6] for a complete description of the trigraduation.)

Denote by $\Lambda^k_{\mathbb{C},\xi}(M)$ (resp. $\Lambda^k_{\mathbb{C}H,\xi}(M)$) the space of the (resp. harmonic) complex k-forms α on M such that $i(\xi)\alpha = 0$, and denote by $\Lambda^k_{\mathbb{C},\eta}(M)$ (resp. $\Lambda^k_{\mathbb{C}H,\eta}(M)$)

the space of the (resp. harmonic) complex k-forms α on M such that $\alpha \wedge \eta = 0$. Then, Proposition 4.1 and Corollary 4.2 still hold for complex forms. Furthermore, the induced mapping (12) preserves the trigraduation in the following sense:

$$\widehat{i(\xi)}\colon \Lambda_H^{p,q,1}(M) \to \Lambda_H^{p,q,0}(M)$$

is a linear isomorphism.

The following result was proved in [6].

PROPOSITION 4.3 [6, Prop. 11]. *Let M be a $(2n+1)$-dimensional compact cosymplectic manifold with almost contact metric structure (ϕ, ξ, η, g) and fundamental 2-form Φ. Then*

$$\Lambda^k_{\mathbb{C},\xi}(M) = \bigoplus_{p+q=k} \Lambda^{p,q,0}(M),$$
$$\Lambda^k_{\mathbb{C}H,\xi}(M) = \bigoplus_{p+q=k} \Lambda_H^{p,q,0}(M),$$

where $\Lambda_H^{p,q,0}(M)$ denotes the space of harmonic complex forms of tridegree $(p, q, 0)$.

Thus, we have the following corollary.

COROLLARY 4.4 (cosymplectic Hodge decomposition theorem). *For a compact cosymplectic manifold M, every harmonic k-form α can be uniquely decomposed as a sum of harmonic forms of pure type. That is,*

$$\alpha = \sum_{p+q+r=k} \alpha_{p,q,r},$$

where $\alpha_{p,q,r} \in \Lambda_H^{p,q,r}(M)$.

Proof.

$$\begin{aligned}
\Lambda^k_{\mathbb{C}H}(M) &= \Lambda^k_{\mathbb{C}H,\xi}(M) \oplus \Lambda^k_{\mathbb{C}H,\eta}(M) \\
&= \Lambda^k_{\mathbb{C}H,\xi}(M) \oplus \widehat{i(\xi)}^{-1}(\Lambda^{k-1}_{\mathbb{C}H,\xi}(M)) \\
&= \bigoplus_{p+q=k} \Lambda_H^{p,q,0}(M) \oplus \widehat{i(\xi)}^{-1}\left(\bigoplus_{p+q=k-1} \Lambda_H^{p,q,0}(M)\right) \\
&= \bigoplus_{p+q=k} \Lambda_H^{p,q,0}(M) \oplus \left(\bigoplus_{p+q=k-1} \Lambda_H^{p,q,1}(M)\right) \\
&= \bigoplus_{p+q+r=p} \Lambda_H^{p,q,r}(M).
\end{aligned}$$

□

5. Harmonic Cohomology Classes for Cosymplectic Manifolds

The aim of this section is to prove that any compact cosymplectic manifold satisfies Brylinski's problem. We shall presume throughout that M is a $(2n+1)$-dimensional compact cosymplectic manifold with almost contact metric structure (ϕ, ξ, η, g) and fundamental 2-form Φ.

THEOREM 5.1. *For $\alpha \in \Lambda^{p,q,r}(M)$, we have*

$$\star(\alpha) = (\sqrt{-1})^{q-p+2r} \star_g (\alpha).$$

Proof. We take local coordinates $\{z_1, \dots, z_n, z\}$ such that Φ and η are given by (13). Putting $z_j = x_j + \sqrt{-1}y_j$, we have $\Phi = \sum_{j=1}^n dx_j \wedge dy_j$ and hence $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial z}\}$ form an orthonormal basis with respect to g. Furthermore, g^{-1} is given by

$$g^{-1} = \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^2 + \sum_{j=1}^n \left(\frac{\partial}{\partial y_j}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2;$$

then $g^{-1}(dz_j, d\bar{z}_j) = 2$. If we remember that the Hodge star operator is given by

$$\beta \wedge (\star_g \alpha) = \Lambda^k g^{-1}(\beta, \alpha) \cdot v_M,$$

then for $\alpha, \beta \in \Lambda^k(M)$ we have

$$\begin{aligned}\star_g &\left[dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_j} \wedge dz_{p_1} \wedge d\bar{z}_{p_1} \wedge \cdots \wedge dz_{p_s} \wedge d\bar{z}_{p_s}\right] \\ &= 2^{i+j+2s}(-1)^a dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_j} \wedge dz_{q_1} \wedge d\bar{z}_{q_1} \\ &\qquad \wedge \cdots \wedge dz_{q_t} \wedge d\bar{z}_{q_t} \wedge dz,\\ \star_g &\left[dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_j} \wedge dz_{p_1} \wedge d\bar{z}_{p_1} \wedge \cdots \wedge dz_{p_s} \wedge d\bar{z}_{p_s} \wedge dz\right] \\ &= 2^{i+j+2s}(-1)^{a+i+j} dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_j} \wedge dz_{q_1} \wedge d\bar{z}_{q_1} \\ &\qquad \wedge \cdots \wedge dz_{q_t} \wedge d\bar{z}_{q_t},\end{aligned}$$

where $a = \frac{1}{2}[i(i+1) + j(j-1)] + ij + s$ and $\{k_1, \dots, k_i, l_1, \dots, l_j, p_1, \dots, p_s, q_1, \dots, q_t\} = \{1, \dots, n\}$.

To compute the cosymplectic star operator, we observe that

$$\mu^{-1}(dz_j) = 2\sqrt{-1}\frac{\partial}{\partial \bar{z}_j}, \quad \mu^{-1}(d\bar{z}_j) = -2\sqrt{-1}\frac{\partial}{\partial z_j}, \quad \mu^{-1}(dz) = \frac{\partial}{\partial z}.$$

Then, from (6),

$$\begin{aligned}\star &\left[dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_j} \wedge dz_{p_1} \wedge d\bar{z}_{p_1} \wedge \cdots \wedge dz_{p_s} \wedge d\bar{z}_{p_s}\right] \\ &= 2^{i+j+2s}(\sqrt{-1})^{i+j+2s}(-1)^b dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_j} \wedge dz_{q_1} \\ &\qquad \wedge d\bar{z}_{q_1} \wedge \cdots \wedge dz_{q_t} \wedge d\bar{z}_{q_t} \wedge dz,\\ \star &\left[dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_j} \wedge dz_{p_1} \wedge d\bar{z}_{p_1} \wedge \cdots \wedge dz_{p_s} \wedge d\bar{z}_{p_s} \wedge dz\right] \\ &= 2^{i+j+2s}(\sqrt{-1})^{i+j+2s}(-1)^{b+i+j} dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge d\bar{z}_{l_1} \wedge \cdots \wedge d\bar{z}_{l_j} \\ &\qquad \wedge dz_{q_1} \wedge d\bar{z}_{q_1} \wedge \cdots \wedge dz_{q_t} \wedge d\bar{z}_{q_t},\end{aligned}$$

where $b = \frac{1}{2}[i(i+1) + j(j-1)] + j(i+1) + (i+j+r)$.

Thus, for a complex form α of pure type (p, q, r),

$$\star(\alpha) = (\sqrt{-1})^{q-p+2r} \star_g (\alpha). \qquad \square$$

COROLLARY 5.2. *Let α be a harmonic form of pure type (p, q, r) on a cosymplectic manifold. Then $\delta_2\alpha = \delta\alpha = 0$.*

Proof. If α is harmonic, then $\star_g d \star_g (\alpha) = 0$. It follows from Theorem 5.1 and the fact that α is of pure type that $\delta_2(\alpha) = \pm \star^3 d \star (\alpha)$ is also 0.

Now, if α is harmonic and of pure type $(p, q, 0)$, then $\delta\alpha = 0$ from Proposition 3.5(i). If α is harmonic and of pure type $(p, q, 1)$, then $i(\xi)\alpha$ is harmonic (from Proposition 4.1) and of pure type $(p, q, 0)$, so $\delta(i(\xi)\alpha) = 0$ and, realizing that δ preserves the decomposition (7) (see [9]), we have

$$\delta(\alpha) = \delta(\eta \wedge (i(\xi)\alpha)) = \eta \wedge \delta(i(\xi)\alpha) = 0. \qquad \square$$

REMARK 5.3. Theorem 5.1 and Corollary 5.2 hold also for almost cosymplectic manifolds.

THEOREM 5.4 (Main Theorem). *If M is a compact cosymplectic manifold, then any de Rham cohomology class of M has a representative α such that $\delta_2\alpha = \delta\alpha = 0$.*

Proof. Taking into account that any de Rham cohomology class is generated by a harmonic form, Corollary 4.4 and Corollary 5.2 yield the result. $\square$

Note that we have actually proved that compact cosymplectic manifolds satisfy Brylinski's problem and the new problem.

6. The Almost Cosymplectic Nilmanifold M^5

The purpose of this section is to show that Brylinski's problem is not satisfied for any compact almost cosymplectic manifold.

Consider the 5-dimensional compact nilmanifold $M^5 = \Gamma\backslash K$ described in [9; 11], where K is a simply connected nilpotent Lie group of dimension 5 defined by the left invariant 1-forms $\{\alpha_i \mid 1 \leq i \leq 5\}$ such that

$$\begin{aligned} d\alpha_1 &= d\alpha_2 = d\alpha_5 = 0, \\ d\alpha_3 &= \alpha_2 \wedge \alpha_5, \\ d\alpha_4 &= \alpha_1 \wedge \alpha_2, \end{aligned}$$

and where Γ is a discrete and uniform subgroup of K. On account of Kobayashi's theorem [14], the manifold M^5 can be alternatively described as the total space at the top of a tower $M^5 \to M^4 \to \mathbb{T}^3$ of principal S^1-bundles.

An easy computation using Nomizu's theorem [22] shows that the de Rham cohomology of M^5 is

$$
\begin{aligned}
H^0(M^5) &= \{1\},\\
H^1(M^5) &= \{[\alpha_1], [\alpha_2], [\alpha_5]\},\\
H^2(M^5) &= \{[\alpha_1 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5], [\alpha_1 \wedge \alpha_4], [\alpha_1 \wedge \alpha_5],\\
&\qquad [\alpha_2 \wedge \alpha_3], [\alpha_2 \wedge \alpha_4], [\alpha_3 \wedge \alpha_5]\},\\
H^3(M^5) &= \{[\alpha_1 \wedge \alpha_2 \wedge \alpha_3], [\alpha_1 \wedge \alpha_2 \wedge \alpha_4], [\alpha_1 \wedge \alpha_3 \wedge \alpha_5],\\
&\qquad [\alpha_1 \wedge \alpha_4 \wedge \alpha_5], [\alpha_2 \wedge \alpha_3 \wedge \alpha_4], [\alpha_2 \wedge \alpha_3 \wedge \alpha_5]\},\\
H^4(M^5) &= \{[\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4], [\alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5],\\
&\qquad [\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5]\},\\
H^5(M^5) &= \{[\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5]\}.
\end{aligned}
\tag{14}
$$

The compact manifold M^5 does not admit cosymplectic structures because its minimal model is not formal (see [6]); in particular, the triple Massey product $\langle[\alpha_2], [\alpha_2], [\alpha_5]\rangle$ is nonzero. However, M^5 does admit almost cosymplectic structures.

Let $\{X_i\}$ be the basis dual to $\{\alpha_i\}$. Then we define the tensor field of type (1, 1) over M by

$$
\begin{aligned}
\phi(X_1) = X_4, \quad \phi(X_2) = X_3, \quad \phi(X_5) = 0,\\
\phi(X_4) = -X_1, \quad \phi(X_3) = -X_2.
\end{aligned}
$$

Let the vector field $\xi = X_5$, the 1-form $\eta = \alpha_5$, and the compatible Riemannian metric $g = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M^5 (arising from a left invariant almost contact metric structure on K) whose fundamental 2-form Φ is

$$\Phi = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3.$$

It is now a simple matter to check that $d\Phi = d\eta = 0$, from which it follows that M^5 is a compact almost cosymplectic manifold.

Moreover, it can be shown that the Poisson tensor is $G = X_4 \wedge X_1 + X_3 \wedge X_2$. It is also easy to verify that, for the Koszul differential δ of M^5 with Poisson tensor G,

$$
\begin{aligned}
\delta(\alpha_3 \wedge \alpha_4) = \alpha_1, \quad \delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4) = -\alpha_1 \wedge \alpha_2,\\
\delta(\alpha_3 \wedge \alpha_4 \wedge \eta) = \alpha_1 \wedge \eta, \quad \delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta) = -\alpha_1 \wedge \alpha_2 \wedge \eta,
\end{aligned}
$$

and $\delta(\beta) = 0$ for the other left invariant forms β.

THEOREM 6.1. *The de Rham cohomology class* $[\alpha_2 \wedge \alpha_3 \wedge \alpha_4]$ *on* $H^3(M^5)$ *does not admit a representative* ν *harmonic* (*with respect to the Poisson structure*); *that is,* $d\nu = \delta\nu = 0$.

Proof. Consider the de Rham cohomology class $[\alpha_2 \wedge \alpha_3 \wedge \alpha_4] \in H^3(M^5)$. As we know, $\delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4) = -\alpha_1 \wedge \alpha_2 \neq 0$.

Suppose that there exists a representative of $[\alpha_2 \wedge \alpha_3 \wedge \alpha_4]$ that is harmonic (with respect to the Poisson structure). Hence, there exists a form $\theta \in \Lambda^2(M^5)$ such that

$$\delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4 + d\theta) = 0.$$

Since $d\delta + \delta d = 0$, we deduce that

$$d\delta\theta = \delta(\alpha_2 \wedge \alpha_3 \wedge \alpha_4) = -\alpha_1 \wedge \alpha_2 = d(-\alpha_4). \tag{15}$$

Multiplying (15) by η, we obtain that $d(\alpha_4 \wedge \eta + \delta\theta \wedge \eta) = 0$. Then (see [9]), since δ preserves the decomposition (7)—that is, $\delta(\theta) \wedge \eta = \delta(\theta \wedge \eta)$—we obtain that

$$[\alpha_4 \wedge \eta + \delta(\theta \wedge \eta)] \in H^2(M^5).$$

We consider two possibilities.

Case 1: $[\alpha_4 \wedge \eta + \delta(\theta \wedge \eta)] = 0$ in $H^2(M^5)$. Here, there exists a $\beta \in \Lambda^1(M^5)$ such that $\alpha_4 \wedge \eta = \delta(-\theta \wedge \eta) + d\beta$. We now apply the operator L (defined by $L(\alpha) = \alpha \wedge \Phi$) to both sides, and from (3) and that $Ld = dL$ we obtain

$$\begin{aligned}\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta &= L\delta(-\theta \wedge \eta) + L(d\beta)\\ &= L(di(G) - i(G)d)(\theta \wedge \eta) + d(L\beta)\\ &= d(L\beta + Li(G)(\theta \wedge \eta)) - Li(G)d(\theta \wedge \eta)\\ &= d(L\beta + Li(G)(\theta \wedge \eta) + \theta \wedge \eta)\end{aligned}$$

via direct computation, if we realize that $d(\theta \wedge \eta)$ is a combination of $\{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \eta,\ \alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \eta,\ \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta,\ \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta\}$. Hence we have obtained a contradiction with (14), because the form $\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta$ defines a nonzero de Rham cohomology class on M^5.

Case 2: $[\alpha_4 \wedge \eta + \delta(\theta \wedge \eta)] \neq 0$ in $H^2(M^5)$. In this case, there exist real numbers $\lambda_i \in \mathbb{R}$ $(i = 1, \dots, 6)$ and a form $\beta \in \Lambda^1(M^5)$ such that

$$\begin{aligned}\alpha_4 \wedge \eta = \delta(-\theta \wedge \eta) + d\beta + \lambda_1(\alpha_1 \wedge \alpha_3 + \alpha_4 \wedge \eta) + \lambda_2\alpha_1 \wedge \alpha_4\\ + \lambda_3\alpha_1 \wedge \eta + \lambda_4\alpha_2 \wedge \alpha_3 + \lambda_5\alpha_2 \wedge \alpha_4 + \lambda_6\alpha_3 \wedge \eta.\end{aligned} \tag{16}$$

Next, we apply L to both sides of (16) and obtain

$$(1 - \lambda_1)\alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta - (\lambda_2 + \lambda_4)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 + \lambda_6\alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \eta = d\gamma$$

for some 3-form γ. We then obtain a contradiction with (14), except when $\lambda_1 = 1$, $\lambda_2 + \lambda_4 = 0$, and $\lambda_6 = 0$. In that case, we multiply (16) by η and obtain

$$0 = \alpha_1 \wedge \alpha_3 \wedge \eta + \lambda_2\alpha_1 \wedge \alpha_4 \wedge \eta - \lambda_2\alpha_2 \wedge \alpha_3 \wedge \eta + \lambda_5\alpha_2 \wedge \alpha_4 \wedge \eta + d(\beta \wedge \eta).$$

Again we have a contradiction with (14) (notice that $\alpha_2 \wedge \alpha_4 \wedge \eta$ belongs to $[\alpha_1 \wedge \alpha_2 \wedge \alpha_3]$). □

Therefore, Brylinski's problem is not satisfied for any compact almost cosymplectic manifold.

REMARK 6.2. Notice that the differentials δ and δ_2 have different behaviors with regard to Brylinski's problem and the new problem, respectively. In fact, for M^5 the de Rham cohomology class $[\alpha_1 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_4] \in H^3(M^5)$ satisfies the new problem because $\delta_2(\alpha_1 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_4) = 0$, but does not satisfy Brylinski's problem because $\delta(\alpha_1 \wedge \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_3 \wedge \alpha_4) = -\alpha_1 \wedge \alpha_2 \neq 0$, as follows from the method used in Theorem 6.1.

We also have the following related questions.

(i) Is the result of the new problem true for any compact noncosymplectic almost cosymplectic manifolds?

(ii) Taking into account Mathieu's result [21], is it true for compact almost cosymplectic manifolds that Brylinski's problem is satisfied iff the cosymplectic hard Lefschetz theorem is satisfied (see [6])? Does the same result hold for the new problem?

Finally we shall show that there does not exist an isomorphism between the canonical homology and the second canonical homology for compact almost cosymplectic manifolds.

COROLLARY 6.3. *Let $M = \Gamma\backslash K$ be a compact almost cosymplectic nilmanifold whose almost contact metric structure arises from a left invariant almost contact metric structure on K. Then there exists an isomorphism*

$$\nu\colon H_k^{\mathrm{can2}}(\mathfrak{K}^*) \to H_k^{\mathrm{can2}}(\Gamma\backslash K), \quad k \geq 0, \tag{17}$$

where $\mathfrak{K}$ is the Lie algebra of K.

Proof. This follows easily from Proposition 3.3 and Nomizu's theorem [22] for the de Rham cohomology of a compact nilmanifold. □

In [9; 11] the authors have also proved that under the conditions of Corollary 6.3, there exists an injective homomorphism

$$\nu\colon H_k^{\mathrm{can}}(\mathfrak{K}^*) \to H_k^{\mathrm{can}}(\Gamma\backslash K), \quad k \geq 0. \tag{18}$$

Thus, from Corollary 6.3 applied to M^5, we calculate

$$H_1^{\mathrm{can2}}(M^5) = \{\{\alpha_3\}, \{\alpha_4\}, \{\alpha_5\}\}.$$

From (18),

$$H_1^{\mathrm{can}}(\mathfrak{K}^*) = \{\{\alpha_2\}, \{\alpha_3\}, \{\alpha_4\}, \{\alpha_5\}\} \subseteq H_1^{\mathrm{can}}(M^5),$$

so

$$\dim H_1^{\mathrm{can2}}(M^5) = 3 \neq 4 = \dim H_1^{\mathrm{can}}(\mathfrak{K}^*) \leq \dim H_1^{\mathrm{can}}(M^5).$$

Consequently,

$$H_1^{\mathrm{can2}}(M^5) \ncong H_1^{\mathrm{can}}(M^5).$$

REMARK 6.4. Notice that $H_1^{\mathrm{can}}(M^5) \ncong H^4(M^5)$.

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