Genus Actions of Finite Groups on 3-Manifolds

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1. Introduction

In [11] (see also [23]) a theory of finite group actions on 3-dimensional handlebodies was developed. In the present paper we extend certain aspects of this theory to closed orientable 3-manifolds using Heegaard decompositions, studying in particular actions of cyclic and dihedral groups of large order relative to the Heegaard genus. Recall that a Heegaard decomposition of genus g of a closed 3-manifold M is a decomposition of M into two handlebodies of genus g intersecting in their common boundary.

In the following, G always denotes a finite group; all manifolds will be orientable and all group actions will be orientation-preserving.

The (handlebody-) *genus* (resp. *strong genus*) of a finite group G is defined as the minimal genus g (resp. $g > 1$) of a handlebody V on which G acts; the corresponding G-action is called a *genus* (resp. *strong genus*) *action*. Note that for most groups the genus and the strong genus coincide, as only very few types of groups act on the handlebodies of genus 0 and 1 (i.e., the 3-ball and the solid torus), among them the cyclic and dihedral groups. Of particular interest is the class of genus actions consisting of those of maximal possible order $12(g-1)$: recall that the order of a finite group acting on a handlebody of genus $g > 1$ is bounded above by $12(g-1)$; the groups of maximal order are called *maximal handlebody groups*. It is shown in [22, Kor. 4.2] that for infinitely many values of g this upper bound is attained (resp. not attained).

In the present paper we will be interested in strong genus actions, in particular also of cyclic and dihedral groups. Central to the paper is the following definition.

Definition.

(i) A closed 3-manifold M is called a *G-manifold of genus g* if it admits an action of the finite group G and g is the minimal genus of a Heegaard splitting of M for which both handlebodies are invariant under the G-action (*equivariant Heegaard genus* of the G-action).

(ii) A G-manifold of genus $g > 1$ is called *minimal* if the induced G-action on each of the two handlebodies of an invariant Heegaard splitting of genus g is a strong genus action (these are the *genus actions* of the title of the paper).

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If moreover G has maximal possible order $12(g-1)$, then the G-manifold M and the G-action are called *maximally symmetric* (m.s.).

Maximally symmetric G-manifolds have been studied in [24] and [27]. Note that, working in the PL category and using G-invariant triangulations, any G-action on a closed 3-manifold has an invariant Heegaard splitting, so we shall restrict ourselves here to the extremal case of minimal G-manifolds. Our main result is a characterization of the minimal G-manifolds for cyclic and dihedral groups G; as an application, there are exactly four types of G-manifolds of genus 2. This is the content of Section 3. In Section 2 we discuss handlebody orbifolds and finite group actions on handlebodies; this leads in a natural way to the notion of Heegaard splitting and Heegaard number (genus) for closed orientable 3-orbifolds. In the last section we discuss maximally symmetric G-manifolds of small genus, in particular the case of hyperbolic 3-manifolds.

2. Handlebody Orbifolds and Finite Group Actions on Handlebodies

Let G be a finite group acting orientation-preservingly on a handlebody V. Let D be a 2-dimensional properly embedded disk in V such that $\partial D = D \cap \partial V$ is a nontrivial closed curve on ∂V. By the equivariant loop theorem/Dehn lemma (see [13]), we can assume that $x(D) = D$ or $x(D) \cap D = \emptyset$ for all $x \in G$. When cutting V along the system of disjoint disks $G(D)$, that is, removing the interior of a G-invariant regular neighborhood of $G(D)$ (which is a collection of 1-handles: products of a 2-disk with an interval), we get again a collection of handlebodies of lower genus on which G acts. Applying inductively the above procedure of cutting along disks, we finally end up with a collection of disjoint 3-balls on which G acts. Thus the quotient orbifold $\mathcal{H} := V/G$ is built up from orbifolds that are quotients of 3-balls by finite groups of homeomorphisms (their stabilizers in G), connected by finite cyclic quotients of 1-handles (*1-handle orbifolds*) which are the projections of the removed regular neighborhoods of the disks (first type of orbifold in Figure 1). The finite orientation-preserving groups that can act on the 3-ball or the 2-sphere are the finite subgroups of the orthogonal group SO(3): cyclic $\mathbf{Z}_n$, dihedral $\mathbf{D}_n$, tetrahedral $\mathbf{A}_4$, octahedral $\mathbf{S}_4$ and dodecahedral $\mathbf{A}_5$, which we will call the *spherical groups*. It is well known that, on the boundary of the 3-ball, the actions are standard—that is, conjugate to orthogonal actions. By Thurston's orbifold geometrization theorem [20], the same is true for the whole 3-ball; in the case of cyclic group actions, instead of the orbifold geometrization theorem one may use the positive solution of the Smith conjecture [16]. The figures of the possible quotient orbifolds are listed in Figure 1; the underlying topological space is the 3-ball in each case.

These quotient orbifolds are connected by the 1-handle orbifolds; the result is called a *handlebody orbifold*. By definition, a handlebody orbifold consists of finitely many orbifolds as in Figure 1 (i.e., quotients of finite orthogonal group actions on the 3-ball) connected by 1-handle orbifolds respecting the singular axes

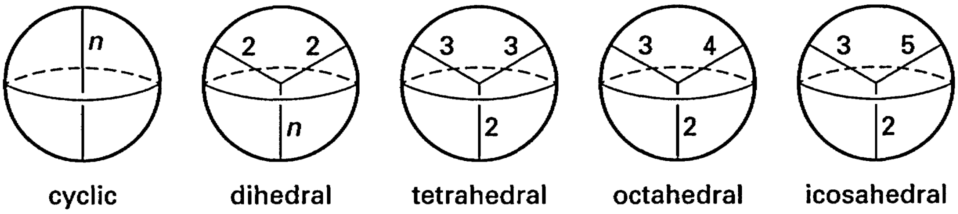

Figure 1

and their orders, and such that topologically the result is an *orientable* handlebody. Therefore, as a consequence of the equivariant Dehn lemma and the orbifold geometrization theorem, we have the following.

PROPOSITION 1. *The quotients of handlebodies by finite group actions are exactly the handlebody orbifolds.*

To each handlebody orbifold $\mathcal{H}$ is associated a graph of groups $(\Gamma, \mathcal{G})$ in a natural way: the vertices (resp. edges) of the graph Γ correspond to the quotients of the 3-balls (resp. of the 1-handles), and to each vertex (resp. edge) is associated the corresponding finite stabilizer. We call such graphs of groups associated to handlebody orbifolds *admissible*. In particular, all vertex groups of an admissible graph of groups $(\Gamma, \mathcal{G})$ are spherical groups, and the edge groups are cyclic groups that are maximally cyclic in the adjacent vertex groups. We can also assume that an admissible graph of groups $(\Gamma, \mathcal{G})$ has no *trivial edges*, that is, edges with two different vertices such that the inclusion from the edge group into one of the two adjacent vertex groups is surjective (i.e., one of the two vertex groups coincides with the edge group).

By the orbifold version of Van Kampen's theorem (see [4]), the orbifold fundamental group $\pi_1\mathcal{H}$ of $\mathcal{H}$ is isomorphic to the fundamental group $\pi_1(\Gamma, \mathcal{G})$ of the corresponding graph of groups (which is the iterated free product with amalgamation and HNN-extension of the vertex groups over the edge groups, starting with a maximal tree in Γ; see [18] or [23] for definitions about graphs of groups). Deleting from Γ the edges whose associated groups are trivial, we get exactly the *singular set* of the handlebody orbifold $\mathcal{H}$ which is a homeomorphism invariant of the handlebody orbifold and therefore also of the equivalence (conjugacy) class of the G-action to which it is associated.

For a finite group G, we say that a graph of groups $(\Gamma, \mathcal{G})$ is *G-admissible* if it is admissible—that is, associated to a handlebody orbifold $\mathcal{H} = \mathcal{H}(\Gamma, \mathcal{G})$ as above—and if moreover there exists an epimorphism from $\pi_1(\Gamma, \mathcal{G}) \cong \pi_1\mathcal{H}$ onto G with torsion-free kernel (or, equivalently, the epimorphism is injective on the vertex groups); we also call such an epimorphism *admissible.*

PROPOSITION 2. *A finite group G acts on a handlebody V_g of genus g if and only if there exists a G-admissible graph of groups $(\Gamma, \mathcal{G})$ such that*

$$g - 1 = -\chi(\Gamma, \mathcal{G})|G|.$$

Here $\chi(\Gamma, \mathcal{G})$ denotes the *Euler characteristic* of $(\Gamma, \mathcal{G})$, which is defined as

$$\chi(\Gamma, \mathcal{G}) := \sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|},$$

where the sum is extended over all vertex groups G_v (resp. edge groups G_e) of $(\Gamma, \mathcal{G})$. For example, the graph of groups

$$\Gamma(B_1, A, B_2)$$

with one edge with edge group A and two vertices with vertex groups B_1 and B_2 has $\chi(\Gamma(B_1, A, B_2)) = 1/|B_1| - 1/|B_2| - 1/|A|$ and $\pi_1(\Gamma(B_1, A, B_2)) = B_1 *_A B_2$. Note that group actions on handlebodies of genus $g > 1$ correspond to admissible graphs of groups with negative Euler characteristic; this is the case that will be of interest to us in the following.

We define the *Euler characteristic* $\chi(\mathcal{H}(\Gamma, \mathcal{G}))$ of a handlebody orbifold $\mathcal{H}(\Gamma, \mathcal{G})$ as the Euler characteristic $\chi(\Gamma, \mathcal{G})$ of the associated graph of groups. This coincides with the definition of orbifold Euler characteristic given in [19].

Proof of Proposition 2. Given an admissible epimorphism $\phi: \pi_1(\Gamma, \mathcal{G}) \to G$, the corresponding G-action on a handlebody is obtained by taking the regular orbifold covering of $\mathcal{H}(\Gamma, \mathcal{G})$ associated to the kernel of ϕ. Conversely, given a G-action on a handlebody, by the equivariant Dehn lemma/loop theorem and the orbifold Van Kampen theorem, the fundamental group of the quotient orbifold is that of an admissible graph of groups, and the quotient map is an orbifold covering corresponding to an admissible epimorphism. (Note that the orbifold geometrization theorem is not needed here; also, the equivariant Dehn lemma/loop theorem can be avoided and replaced by the Stallings structure theorem for groups with infinitely many ends—see [11].)

Finally, the formula in Proposition 2 is just the multiplicativity of the orbifold Euler characteristic under finite orbifold coverings. □

DEFINITION. A *Heegaard decomposition* of a closed orientable 3-orbifold $\mathcal{O}$ is a decomposition of the orbifold into two handlebody orbifolds $\mathcal{H}_1$ and $\mathcal{H}_2$ intersecting in their common boundary (a 2-orbifold). The *Heegaard number* of the orbifold is the largest value of $\chi(\mathcal{H}_1) = \chi(\mathcal{H}_2)$ taken over all Heegaard decompositions of $\mathcal{O}$ (which is also twice the orbifold Euler characteristic of the 2-orbifold that is the common boundary of $\mathcal{H}_1$ and $\mathcal{H}_2$).

Note that with this definition the Heegaard number of a closed 3-manifold is $1-g$, where g is the usual Heegaard genus of the 3-manifold. Also, the G-manifolds of genus g are exactly the G-admissible coverings of 3-orbifolds of Heegaard number $\chi = (1-g)/|G|$. An application of Proposition 2 is the following.

PROPOSITION 3. *Let* $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ *be the prime decomposition of the natural number* $n > 2$, *with* $p_1 < p_2 < \cdots < p_k$. *Then, for the following finite groups* G, *the list gives the* G*-admissible graphs of largest negative Euler characteristic and the strong genus of the group.*

(a) *Cyclic groups* $G = \mathbf{Z}_n$.

(i) *n prime*: $\Gamma(\mathbf{Z}_n, \mathbf{1}, \mathbf{Z}_n), g = n - 1$;

(ii) $m_1 > 1$: $\Gamma(\mathbf{Z}_{p_1}, \mathbf{1}, \mathbf{Z}_n), g = n(p_1 - 1)/p_1$;

(iii) $m_1 = 1$, *n not prime*: $\Gamma(\mathbf{Z}_{p_1}, \mathbf{1}, \mathbf{Z}_{n/p_1}), g = (n - p_1)(p_1 - 1)/p_1$.

(b) *Dihedral groups* $G = \mathbf{D}_n$ *of order* $2n$.

(i) *n prime*: $\Gamma(\mathbf{D}_n, \mathbf{Z}_2, \mathbf{D}_n)$ *or* $\Gamma(\mathbf{Z}_2, \mathbf{1}, \mathbf{Z}_n)$, $g = n - 1$;

(ii) $m_1 > 1$: $\Gamma(\mathbf{D}_{p_1}, \mathbf{Z}_2, \mathbf{D}_n)$, $g = n(p_1 - 1)/p_1$;

(iii) $m_1 = 1$, *n not prime*: $\Gamma(\mathbf{D}_{p_1}, \mathbf{Z}_2, \mathbf{D}_{n/p_1})$, $g = (n - p_1)(p_1 - 1)/p_1$.

(c) $G = \mathbf{D}_n \times \mathbf{Z}_2$: $\Gamma(\mathbf{D}_2, \mathbf{Z}_2, \mathbf{D}_n)$, $g = n - 1$.

Proof. It is easy to see that, in each of the given cases, the given graphs of groups are the G-admissible graphs of groups of largest negative Euler characteristic; see the proof of [10, Thm. 3.2] for basically the same computation. The genus of G is then obtained from these graphs of group $(\Gamma, \mathcal{G})$ by the formula in Proposition 2. □

The maximal negative Euler characteristic of a G-admissible graph of groups, for arbitrary finite groups G, is $-1/12$; there are exactly the following four graphs of this Euler characteristic:

$$\Gamma(\mathbf{D}_2, \mathbf{Z}_2, \mathbf{D}_3), \quad \Gamma(\mathbf{D}_3, \mathbf{Z}_3, \mathbf{A}_4), \quad \Gamma(\mathbf{D}_4, \mathbf{Z}_4, \mathbf{S}_4), \quad \text{and} \quad \Gamma(\mathbf{D}_5, \mathbf{Z}_5, \mathbf{A}_5).$$

As a consequence, the maximal possible order of a finite group G acting on a handlebody of genus $g > 1$ is $12(g - 1)$. Thus the maximal handlebody groups are exactly the finite quotients, by torsion-free subgroups, of one of the following four free products with amalgamation:

$$G_2 := \mathbf{D}_2 *_{\mathbf{Z}_2} \mathbf{D}_3, \quad G_3 := \mathbf{D}_3 *_{\mathbf{Z}_3} \mathbf{A}_4, \quad G_4 := \mathbf{D}_4 *_{\mathbf{Z}_4} \mathbf{S}_4, \quad G_5 := \mathbf{D}_5 *_{\mathbf{Z}_5} \mathbf{A}_5.$$

Such a finite quotient G is called a G_i*-group*, for $i = 2$, 3, 4 or 5; the quotient V/G is the handlebody orbifold $\mathcal{O}_i$ shown in Figure 2a. Now the following is clear.

Proposition 4. *The closed* 3*-orbifolds of largest possible negative Heegaard number* $-1/12$ *are exactly the generalized tetrahedral orbifolds* $\mathcal{O}(\sigma, i, j)$ *obtained by identifying two orbifolds as in Figure 2a along their boundaries. Their underlying topological space is the* 3*-sphere, and the singular set is given in Figure 2b, where* σ *denotes a* 3*-braid and* $2 \le i, j \le 5$. *The maximally symmetric* 3*-manifolds are exactly the regular coverings of these orbifolds.*

We call the orbifolds $\mathcal{O}(\sigma, i, j)$ of *minimal type*, or simply *minimal*. The orbifold fundamental group of $\mathcal{O}(\sigma, i, j)$ can be obtained from Figure 2b exactly in the same way as the Wirtinger presentation is obtained from a projection of a knot or link. We have indicated the corresponding generators in Figure 2b, where we consider the braid σ also as an automorphism of the free group in the three generators a, b, x in the usual way (so $\sigma(abx) = abx$; note also that $t = xy = (ab)^{-1}$ and $s = \sigma(x)y = (\sigma(a)\sigma(b))^{-1}$). Alternatively, one may apply the orbifold version of Van Kampen's theorem. The result is as follows.

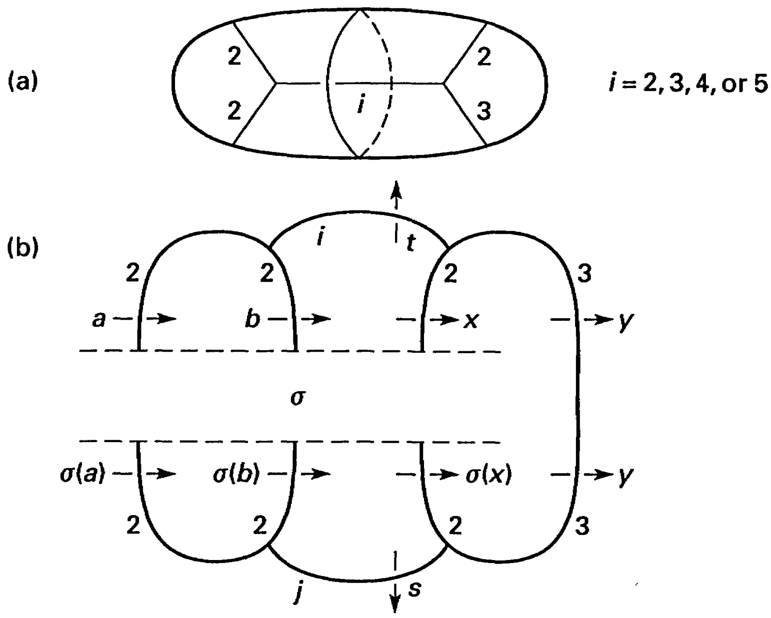

Figure 2

PROPOSITION 5. *The orbifold fundamental group of $\mathcal{O}(\sigma, i, j)$ has a presentation of the form*

$$\pi_1\mathcal{O}(\sigma, i, j) = \langle a, b, x, y \mid abxy = 1,\ a^2 = b^2 = (ab)^i = 1,\ x^2 = y^3 = (xy)^i = 1,\ (\sigma(x)y)^j = 1\rangle \cong G_i/\langle\langle(\sigma(x)y)^j\rangle\rangle, \quad 2 \le i, j \le 5.$$

Each maximally symmetric G-manifold is defined by an *admissible epimorphism* from some $\pi_1\mathcal{O}(\sigma, i, j)$ onto G; here "admissible" means that the epimorphism is injective on all finite groups associated to the singular set of the orbifold $\mathcal{O}(\sigma, i, j)$. If there exists such an epimorphism then we say that the orbifold $\mathcal{O}(\sigma, i, j)$ is *G-admissible.*

3. Minimal G-Manifolds

In this section we characterize the minimal G-manifolds for cyclic and dihedral groups G; we also characterize the possible types of G-manifolds of genus 2.

Before stating our main result we recall some facts about cyclic branched coverings of links. Let $L = K_1 \cup \cdots \cup K_\nu$ be a link in the 3-sphere with ν components. Denote by $m_1, \ldots, m_\nu$ meridians of the components $K_1 \cup \cdots \cup K_\nu$ of the link, oriented in an arbitrary way. The homology $H_1(S^3 - L)$ of the complement of the link, isomorphic to the abelianized fundamental group $\pi_1(S^3 - L)_{ab}$, is isomorphic to $\mathbf{Z}^\nu$ and generated by the homology classes of the meridians. Each surjection

$$\psi: \pi_1(S^3 - L) \to H_1(S^3 - L) \to \mathbf{Z}_n$$

onto the cyclic groups of order n defines an *n-fold cyclic branched covering* of the 3-sphere branched over the link L (or, as we shall say, a cyclic branched covering of the link L). This is a closed 3-manifold $M = M(\psi)$, which can be constructed by compactifying the unbranched regular covering of $S^3 - L$ corresponding to the kernel of ψ by a finite collection of circles (the preimage of the link L); note that the finite cyclic group of covering transformations of the unbranched covering extends to the branched covering. Every n-fold cyclic branched covering of L can be constructed in this way.

Of course a knot has, up to homeomorphism, a unique n-fold cyclic branched covering. In order to fix a preferred n-fold cyclic branched covering of a link, we start with an *oriented* link L. Then, together with an orientation of S^3, the orientations of the components of L define preferred orientations of the meridians of L, and we define the *uniform* n-fold cyclic branched covering of the oriented link L by the condition that the corresponding surjection ψ maps all meridians of L to the same generator of $\mathbf{Z}_n$. Then the preimage of L in M has exactly ν components, and these constitute the fixed point set of the cyclic group of covering transformations.

We have the following characterization of the minimal $\mathbf{Z}_n$ and $\mathbf{D}_n$-manifolds.

PROPOSITION 6. (a) *For $n > 2$, every minimal $\mathbf{Z}_n$-manifold is a minimal $\mathbf{D}_n$-manifold. If $n > 2$ is prime, then every minimal $\mathbf{Z}_n$-manifold is a minimal $(\mathbf{D}_n \times \mathbf{Z}_2)$-manifold.*

(b) *The minimal $(\mathbf{D}_n \times \mathbf{Z}_2)$-manifolds, of genus $g = n - 1$, are exactly the uniform n-fold cyclic branched coverings of the oriented 2-bridge links (one or two components). The minimal $\mathbf{Z}_n$ or $\mathbf{D}_n$-manifolds that are not minimal $(\mathbf{D}_n \times \mathbf{Z}_2)$-manifolds are cyclic branched coverings of 2-bridge links with two components of different branching orders. For infinitely many values of the genus $g = n - 1$, the order $4n$ of $\mathbf{D}_n \times \mathbf{Z}_2$ is maximal for finite group actions on handlebodies of genus g.*

Proof. Let M be a minimal $\mathbf{Z}_n$-manifold. There exists a Heegaard decomposition of M into two $\mathbf{Z}_n$-invariant handlebodies V_i of genus g, $i = 1$ and 2, such that the induced $\mathbf{Z}_n$-actions on these handlebodies are strong genus actions. By Proposition 1, the quotients $\mathcal{H}_i := V_i/\mathbf{Z}_n$ are handlebody orbifolds, and Proposition 3 gives all possibilities for the associated graphs of groups. This determines the handlebody orbifolds $\mathcal{H}_i$, which are of the type shown in Figure 3a: the underlying topological space is the 3-ball, and the singular set consists of two arcs of branching orders a and b, where $(a, b) = (n, n)$, (p_1, n), or $(p_1, n/p_1)$ according to the cases in Proposition 3a.

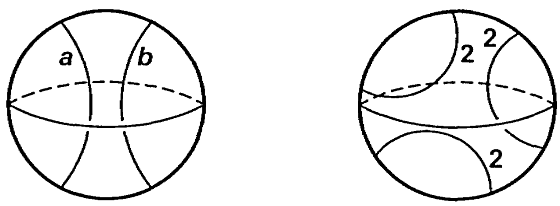

Figure 3

The quotient $M/\mathbf{Z}_n$ is an orbifold obtained by identifying the handlebody orbifolds $\mathcal{H}_1$ and $\mathcal{H}_2$ along their boundaries; the result is the 3-sphere with a 2-bridge link as singular set. Thus M is a cyclic branched covering of a 2-bridge knot or link.

Let L be an arbitrary oriented 2-bridge link. It is well known that such a link can be represented by a closed rational tangle with a $\mathbf{D}_2$-symmetry (see e.g. the pictures in [2] or [21]). The link L has one or two components. If it has one component then the $\mathbf{D}_2$-symmetry of the link lifts to a $\mathbf{D}_2$-action on its cyclic branched covering M. Together with the covering group $\mathbf{Z}_n$ this gives an action of the group $\mathbf{D}_n \times \mathbf{Z}_2$ on M.

If L has two components then any element of the $\mathbf{D}_2$-symmetry preserves or reverses simultaneously the orientations of these components. In case n is prime, the branching orders of the two components are the same and thus the $\mathbf{D}_2$-symmetry lifts again to a $\mathbf{D}_2$-action giving a $(\mathbf{D}_n \times \mathbf{Z}_2)$-action on M. If n is not prime then the link has two components with different branching orders $a \neq b$, and thus only the subgroup $\mathbf{Z}_2$ of $\mathbf{D}_2$ preserving these components lifts to a $\mathbf{Z}_2$-action on M which, together with the covering group, gives a $\mathbf{D}_n$-action on M.

The $\mathbf{D}_2$-symmetry of the tangle preserves both standard unknotting tunnels of the link (see [2]). Consider the boundary of a $\mathbf{D}_2$-invariant regular neighborhood of one of these two tunnels; this is a 2-sphere meeting the link in four points, which defines a Heegaard decomposition (isotopic to the original one) of the orbifold $M/\mathbf{Z}_n$ into two handlebody orbifolds. Its preimage in M gives a Heegaard decomposition of M, isotopic to the original one, into two handlebodies invariant under the $\mathbf{D}_n$-action (resp. $(\mathbf{D}_n \times \mathbf{Z}_2)$-action). Hence M is a minimal $\mathbf{D}_n$-manifold and, if n is prime, also a minimal $(\mathbf{D}_n \times \mathbf{Z}_2)$-manifold.

Now let M be a minimal $(\mathbf{D}_n \times \mathbf{Z}_2)$-manifold. Then M has a decomposition into two handlebodies V_i invariant under the action of $\mathbf{D}_n \times \mathbf{Z}_2$, $i = 1$ and 2. Let $\mathbf{Z}_n$ be the unique cyclic subgroup of order n of $\mathbf{D}_n \times \mathbf{Z}_2$. The quotient $V_i/\mathbf{Z}_n$ is an orbifold all of whose possible singular points belong to branching axes of order n. It follows that $V_i/\mathbf{Z}_n$ is the handlebody orbifold in Figure 3a, with $a = b = n$. The quotient $M/\mathbf{Z}_n$ is an orbifold obtained by identifying two such orbifolds along their boundaries. The result is the 3-sphere with a 2-bridge link as singular set, and consequently M is an n-fold cyclic branched covering of such a link. Moreover, since the induced $\mathbf{D}_2$-action on $M/\mathbf{Z}_n$ lifts to M, it follows easily that this cyclic branched covering is uniform.

Conversely, let L be an oriented 2-bridge link and M a uniform n-fold branched covering of L. Then, as above, the $\mathbf{D}_2$-symmetry of L lifts to a $\mathbf{D}_2$-action on M and we obtain again an action of the group $\mathbf{D}_n \times \mathbf{Z}_2$ on M that leaves invariant a Heegaard decomposition of M of genus $n - 1$. Thus M is a minimal $(\mathbf{D}_n \times \mathbf{Z}_2)$-manifold.

Finally, it follows from the results in [15] that the order $4n$ of $\mathbf{D}_n \times \mathbf{Z}_2$ is maximal for finite group actions on handlebodies of genus $g = n - 1$, for infinitely many values of g (also, for infinitely many values of g it is not maximal). □

Now we consider 3-manifolds of Heegaard genus 2. It is well known that the hyperelliptic involution of a Heegaard surface of genus 2 of such a manifold extends to both handlebodies of the Heegaard decompositon. The quotient of each handlebody by the extended involution is the handlebody orbifold in Figure 3b, and consequently M is a 2-fold branched covering of a 3-bridge link (see e.g. [3, Cor. 11.9]). By the next corollary, the 3-manifolds of genus 2 can be divided into four classes.

COROLLARY. *Let M be a closed 3-manifold of genus 2. Then M is a G-manifold of genus 2, where G is one of the four groups $\mathbf{Z}_2$, $\mathbf{D}_2$, $\mathbf{D}_4$, or $\mathbf{D}_6$. The $\mathbf{D}_6$-manifolds (which are maximally symmetric: in fact, $\mathbf{D}_6$ is the unique maximal handlebody group of smallest possible order 12) are exactly the 3-fold cyclic branched coverings of the 2-bridge links. The $\mathbf{D}_4$-manifolds are exactly the 4-fold cyclic branched coverings of 2-bridge links with two components of branching orders 2 and 4.*

Proof. By the remarks preceding the corollary, each 3-manifold M of genus 2 is a $\mathbf{Z}_2$-manifold of genus 2. By [11, Thm. 8.2], the finite groups that act on the handlebody of genus 2 are the subgroups of $\mathbf{D}_4$ and $\mathbf{D}_6 \cong \mathbf{D}_3 \times \mathbf{Z}_2$, that is, the cyclic groups of orders 2, 3, 4, 6 and the dihedral groups of orders 4, 6, 8, 12. By Proposition 6, for $n > 2$ every minimal $\mathbf{Z}_n$-manifold is also a minimal $\mathbf{D}_n$-manifold, and every minimal $\mathbf{D}_3$-manifold is a minimal $\mathbf{D}_6$-manifold. Hence there remain exactly the four cases of the corollary. The remaining part of the corollary is also a consequence of Proposition 6. □

EXAMPLES. (a) The following example is from [12]. It shows that finite symmetry or isometry groups G of hyperbolic 3-manifolds of Heegaard genus 2 can be arbitrarily large, and consequently also the equivariant Heegaard genus of such group actions.

Let K be the figure-8 knot and M_n the n-fold cyclic branched covering of K. Then, as in the proof of Proposition 6, M_n has a finite symmetry group $\mathbf{D}_n \times \mathbf{Z}_2$. The knot K has a cyclic symmetry τ of order 2, and the projection of K and the axis A of τ to the quotient S^3/τ (which is again the 3-sphere S^3) is an exchangeable link in S^3 with two components $\bar{K}$ and $\bar{A}$ that are trivial knots. The n-fold cyclic branched covering of the trivial knot $\bar{K}$ is the 3-sphere, and the preimage of $\bar{A}$ is a turk's head link A_n of bridge number 3. The 2-fold branched covering of A_n is again the manifold M_n, and consequently the Heegaard genus of M_n is equal to 2 (for $n \geq 3$), see for example [3, Prop. 11.4]. On the other hand, the minimal $(\mathbf{D}_n \times \mathbf{Z}_2)$-manifold M_n has genus $n - 1$. The fundamental groups of the manifolds M_n, which are hyperbolic for $n > 3$, are the Fibonacci groups $F(2, 2n)$; see [14].

(b) The ten closed hyperbolic 3-manifolds of smallest known volumes can all be obtained by Dehn surgery on the Whitehead link; see [6] and [12]. Following [12] we denote these manifolds by $\mathcal{M}_i$, $1 \leq i \leq 10$, in order of increasing volume; thus $\mathcal{M}_1$ is the hyperbolic 3-manifold of smallest known volume. All these manifolds are 2-fold branched coverings of certain 3-bridge links that have been identified explicitly in [12]. For example, in the notation of the tables in [3] and [17], $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_4$ are (respectively) the 2-fold branched coverings of the hyperbolic 3-bridge knots 9_{49}, 10_{161}, 10_{155}. The symmetry or isometry groups of these links are respectively $\mathbf{D}_3$, $\mathbf{D}_1 \cong \mathbf{Z}_2$, and $\mathbf{D}_2$; see the lists in [5] and [7]. By the pictures in [3, p. 265] and [17, pp. 414–415] for each of the three knots, there exists a 2-sphere in S^3 defining a 3-bridge presentation of the knot which is invariant under the action of its symmetry group. Lifting everything to the 2-fold branched covering, we see that $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_4$ are respectively $\mathbf{D}_6$-, $\mathbf{D}_2$-, $\mathbf{D}_4$-manifolds of genus 2. Moreover, by [6] these groups are also the full isometry groups of the three hyperbolic 3-manifolds.

Examples of $\mathbf{Z}_2$-manifolds of genus 2 that are not $\mathbf{D}_n$-manifolds of genus 2 (for $n = 2$, 3 or 6) are hyperbolic manifolds M that are 2-fold branched coverings of 3-bridge knots and links with trivial symmetry group—for example, the π-hyperbolic knots 9_{32} and 9_{33}. To see that such a manifold M is not a $\mathbf{D}_n$-manifold of genus 2, one can use the main result of [26], which implies that M is a 2-fold branched covering of a unique link.

4. Maximally Symmetric 3-Manifolds of Small Genus

As noted in Section 2, each maximal handlebody group of order $12(g-1)$ is a G_i-group for $i = 2$, 3, 4, or 5. The G_2-groups are also exactly the maximal bounded surface groups; see [15]. For the G_2-groups of small order see [9]; also, it has been noted in [24] that, for $i > 2$, the G_i-groups of order less than 96 are $\mathbf{S}_4$, $\mathbf{S}_4 \times \mathbf{Z}_2$, and $\mathbf{A}_5$. This yields the following.

PROPOSITION 7. *The maximal handlebody groups G of genus $g \leq 5$ are exactly the following groups:* $\mathbf{D}_6$ *of genus* 2, $\mathbf{S}_4$ *of genus* 3, $\mathbf{D}_3 \times \mathbf{D}_3$ *of genus* 4, *and* $\mathbf{S}_4 \times \mathbf{Z}_2$ *of genus* 5. *Other maximal handlebody groups of small order are* $\mathbf{A}_5$ *of genus* 6 *and* $\mathbf{A}_5 \times \mathbf{Z}_2$ *of genus* 11.

The maximally symmetric $\mathbf{D}_6$-manifolds were dicussed in Section 3. In a similar way, for the maximally symmetric $(\mathbf{D}_3 \times \mathbf{D}_3)$-manifolds we have our next proposition.

PROPOSITION 8. *The maximally symmetric* $(\mathbf{D}_3 \times \mathbf{D}_3)$*-manifolds are exactly the regular branched* $(\mathbf{Z}_3 \times \mathbf{Z}_3)$*-coverings of* 2*-bridge links with two components.*

Proof. Let M be a minimal $(\mathbf{D}_3 \times \mathbf{D}_3)$-manifold. As before, M has a decomposition into two handlebodies V_i invariant under the action of $\mathbf{D}_3 \times \mathbf{D}_3$, $i = 1$ and 2. Considering the subgroup $\mathbf{Z}_3 \times \mathbf{Z}_3$ of $\mathbf{D}_3 \times \mathbf{D}_3$, the quotient $V_i/(\mathbf{Z}_3 \times \mathbf{Z}_3)$ is the orbifold in Figure 3a with $a = b = 3$, and consequently M is a regular $(\mathbf{Z}_3 \times \mathbf{Z}_3)$-covering of a 2-bridge link with two components. □

In the remaining part of this section we will construct, for the other groups G in Proposition 6, the "simplest" hyperbolic maximally symmetric G-manifolds. Recall from Section 2 that each m.s. G-manifold is defined by some admissible epimorphism from $\pi_1\mathcal{O}(\sigma, i, j)$ onto G, for some of the minimal orbifolds $\mathcal{O}(\sigma, i, j)$ of largest negative Heegaard number $-1/12$. The easiest of these are the tetrahedral orbifolds whose singular set is the 1-skeleton of a tetrahedron. Accordingly, the simplest m.s. G-manifolds will be uniformized by normal subgroups of small index, operating without fixed points, in one of the spherical, Euclidean, or hyperbolic *tetrahedral groups*.

By a *Coxeter tetrahedron* we understand a bounded spherical, Euclidean, or hyperbolic tetrahedron T all of whose dihedral angles are submultiples π/n of π, where $n > 1$. The *Coxeter group* associated to the tetrahedron is the group $C(T)$ of isometries of spherical, Euclidean, or hyperbolic 3-space $\mathbf{X}^3$ generated by the reflections in its faces, and the corresponding *tetrahedral group* $G(T)$ is the subgroup of index 2 of orientation-preserving elements. The quotient $\mathbf{X}^3/G(T)$ is a closed 3-orbifold $\mathcal{O}(T)$ whose topological space is the 3-sphere S^3 and whose singular set is the 1-skeleton of the tetrahedron T, where the branching order of an edge with dihedral angle π/n is n. Also, the orbifold fundamental group $\pi_1\mathcal{O}(T)$ is isomorphic to $G(T)$ (the universal covering group of the orbifold).

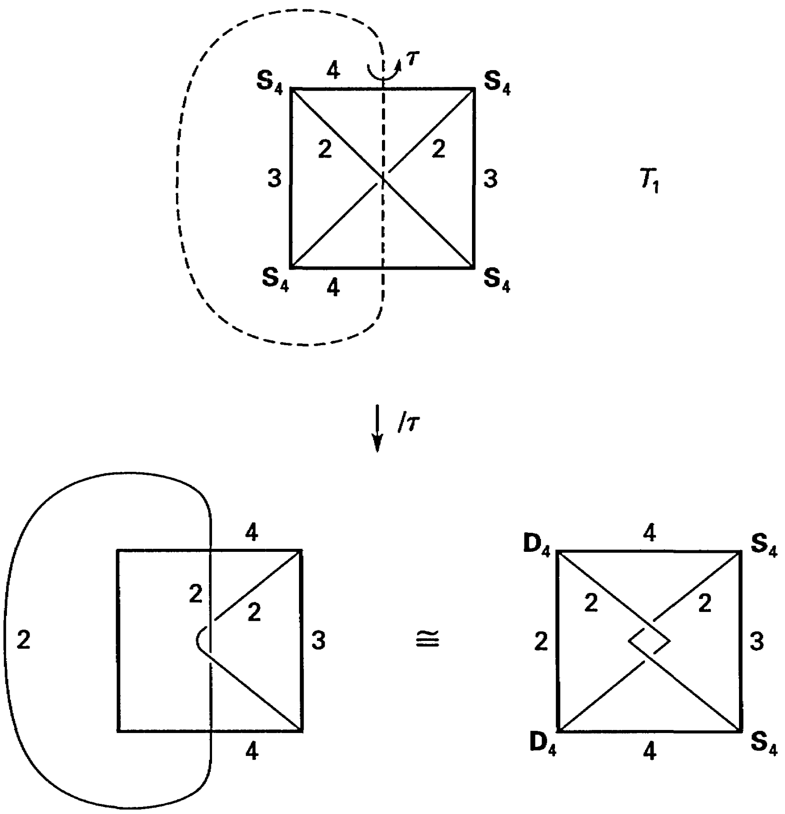

Figure 4

There are exactly nine hyperbolic tetrahedra (the Lanner tetrahedra; see e.g. [1; 19]). One of these is the tetrahedron T_1 in Figure 4, where a number n at an edge denotes a dihedral angle π/n at that edge; the vertex groups indicated in Figure 4 are the local groups of the orbifold $\mathcal{O}(T_1)$ belonging to these points. Note that the orbifold $\mathcal{O}(T_1)$, of Heegaard number $-1/6$, is not of minimal type; however, as indicated in Figure 4, $\mathcal{O}(T_1)$ has a symmetry τ of order 2, and the quotient orbifold $\bar{\mathcal{O}}(T_1) := \mathcal{O}(T_1)/\tau$ has Heegaard number $-1/12$ and is of minimal type. In the notation of Proposition 5, the orbifold fundamental groups are as follows (with

$\sigma(a) = a$, $\sigma(b) = bxb^{-1}$, and $\sigma(x) = b$ in the first case and with $\sigma(a) = a$, $\sigma(b) = bxbx^{-1}b^{-1}$, and $\sigma(x) = bxb^{-1}$ in the second):

$$\pi_1\mathcal{O}(T_1) \cong G(T_1) \cong \mathbf{S}_4 *_{\mathbf{Z}_4} \mathbf{S}_4/\langle (by)^4\rangle, \quad \pi_1\bar{\mathcal{O}}(T_1) \cong \mathbf{D}_4 *_{\mathbf{Z}_4} \mathbf{S}_4/\langle (bxb^{-1}y)^4\rangle.$$

Now, by some easy computations in $\mathbf{S}_4$ (which we omit), the next lemma follows.

LEMMA 1. (a) *Up to automorphisms of* $\mathbf{S}_4 \times \mathbf{Z}_2$ *and* $\mathbf{S}_4$, *there are exactly two admissible surjections from* $\pi_1\bar{\mathcal{O}}(T_1)$ *onto* $\mathbf{S}_4 \times \mathbf{Z}_2$, *and also onto* $\mathbf{S}_4$ (*induced by the first ones*).

(b) *Up to conjugation in* $\mathbf{S}_4$, *there are exactly two admissible surjections from* $G(T_1) \cong \pi_1\mathcal{O}(T_1)$ *onto* $\mathbf{S}_4$ (*and no admissible surjection onto* $\mathbf{S}_4 \times \mathbf{Z}_2$); *these are the restrictions of the surjections from* (a).

Let K be the kernel of one of the two surjections from $G(T_1)$ onto $\mathbf{S}_4$ (or, equivalently, of its extension to a surjection from $\pi_1\bar{\mathcal{O}}(T_1)$ onto $\mathbf{S}_4 \times \mathbf{Z}_2$). Then K acts on hyperbolic 3-space $\mathbf{H}^3$, and $G(T_1)$ is the semidirect product $K \ltimes \mathbf{S}_4$. The quotient $M_1 := \mathbf{H}^3/K$ is a closed hyperbolic 3-manifold with an $\mathbf{S}_4$-action. Equivalently, M_1 is the regular covering of the orbifold $\mathcal{O}(T_1)$ corresponding to the subgroup K of $\pi_1\mathcal{O}(T_1)$. The surjection from $\pi_1\bar{\mathcal{O}}(T_1)$ onto $\mathbf{S}_4 \times \mathbf{Z}_2$ induces a surjection from $\pi_1\bar{\mathcal{O}}(T_1)$ onto $\mathbf{S}_4$; let $\bar{M}_1$ be the hyperbolic 3-manifold which is the regular covering of the orbifold $\bar{\mathcal{O}}(T_1)$ corresponding to the kernel of this induced surjection. Note that M_1 is a 2-fold covering of $\bar{M}_1$. Also, M_1 is a m.s. $(\mathbf{S}_4 \times \mathbf{Z}_2)$-manifold and $\bar{M}_1$ a m.s. $\mathbf{S}_4$-manifold.

For the manifold M_1 we have the following nice and simple geometric construction. The tetrahedral group $G(T_1)$ has two copies $T_1 \cup r(T_1)$ of T_1 as a fundamental region in $\mathbf{H}^3$, where r is a reflection in one of the faces of T_1. We may assume that one of the four vertices of the tetrahedron T_1 is the origin 0 in the Poincaré 3-ball model of $\mathbf{H}^3$. Then the stabilizer of 0 in $G(T_1)$ is the octahedral group $\mathbf{S}_4$, and $\mathcal{P}_1 := \mathbf{S}_4(T_1 \cup r(T_1))$ is a polyhedron in $\mathbf{H}^3$ (consisting of 48 copies of T_1) that is a fundamental region for K in $\mathbf{H}^3$. Combinatorially, $\mathcal{P}_1$ is a cube with each of its six faces divided into four quadrangles (see Figure 5); a number n at an edge of

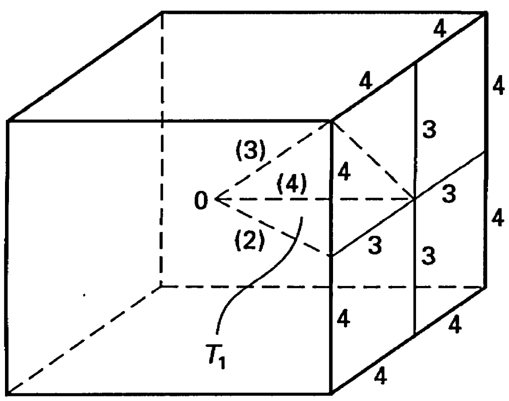

Figure 5

such a quadrangle denotes a dihedral angle $2\pi/n$ of $\mathcal{P}_1$ along that edge, a number (n) an axis of $\mathbf{S}_4$ of order n.

The manifold M_1 is obtained by identification in pairs (reversing the orientation of the faces because M_1 is orientable) of the 24 faces of $\mathcal{P}_1$; these identifications are induced by elements of K. The normality of K in the semidirect product $G(T_1) = K \ltimes \mathbf{S}_4$ has the following consequence.

LEMMA 2. *The pattern of identification of the faces of $\mathcal{P}_1$ is $\mathbf{S}_4$-equivariant, that is, equivariant with respect to the action of the octahedral group $\mathbf{S}_4$ on $\mathcal{P}_1$. Two identified faces of $\mathcal{P}_1$ are related by an involution in $\mathbf{S}_4$.*

Proof. It is clear that normality of K in $G(T_1) = K \ltimes \mathbf{S}_4$ implies that the pattern of identifications of the faces of $\mathcal{P}_1$ is $\mathbf{S}_4$-equivariant; that is, if a face F is identified with a face F' then, for every $f \in \mathbf{S}_4$, the face $f(F)$ is identified with $f(F')$. Choose the unique $k \in \mathbf{S}_4$ such that $k(F) = F'$. It follows that $k(F) = F'$ is identified with $k(F') = k^2(F)$, which must be F again; therefore, k^2 is the identity. □

Lemma 2 makes it easy to find the different possibilities of $\mathbf{S}_4$-equivariant identifications of the faces of $\mathcal{P}_1$ such that the lengths of the edge cycles of such an identification are 3 or 4, corresponding to a dihedral angle $2\pi/3$ or $2\pi/4$ at an edge (see [8] for the notion of an edge cycle and Poincaré's theorem on fundamental polyhedra). The result is as follows.

LEMMA 3. *There exist exactly the two $\mathbf{S}_4$-equivariant identifications of $\mathcal{P}_1$ indicated in Figure 6 that are in accordance with its dihedral angles. These differ by a reflection of $\mathcal{P}_1$ (an element of the Coxeter group $C(T_1)$) and therefore define the same hyperbolic 3-manifold M_1.*

It is already implied by Lemma 1 that there must exist exactly two $\mathbf{S}_4$-equivariant identifications of $\mathcal{P}_1$, so for the proof of Lemma 3 one need only check that the two identifications in Figure 6 have the right lengths of edge cycles. Collecting our results yields the following.

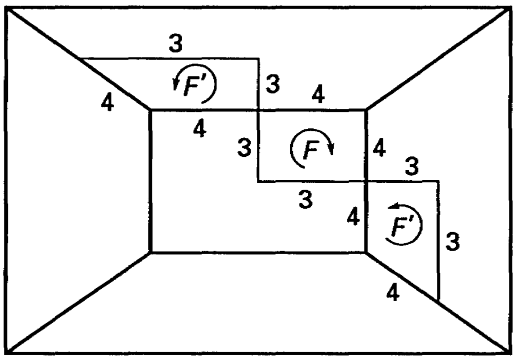

Figure 6

PROPOSITION 9. (a) *Up to automorphisms of the preimage and image groups, there exists a unique admissible surjection from* $G(T_1) \cong \pi_1\mathcal{O}(T_1)$ *onto* $\mathbf{S}_4$ *and from* $\pi_1\bar{\mathcal{O}}(T_1)$ *onto* $\mathbf{S}_4 \times \mathbf{Z}_2$. *The covering corresponding to the kernel of these surjections is the hyperbolic maximally symmetric* $(\mathbf{S}_4 \times \mathbf{Z}_2)$*-manifold* M_1 *of genus* 5.

(b) *Up to automorphisms, there exists a unique admissible surjection from* $\pi_1\bar{\mathcal{O}}(T_1)$ *onto* $\mathbf{S}_4$; *the covering corresponding to the kernel of this surjection is the hyperbolic maximally symmetric* $\mathbf{S}_4$*-manifold* $\bar{M}_1$ *of genus* 4, *which is a* 2*-fold covering of* M_1.

Proof. A reflection of $\mathcal{P}_1$ in a suitable face of the tetrahedron T_1 lies in the Coxeter group $C(T_1)$ and transforms one of the two identifications of Lemma 3 into the other; therefore the automorphism of $G(T_1)$ induced by it (by conjugation) does the same for the two surjections from $G(T_1) \cong \pi_1\mathcal{O}(T_1)$ onto $\mathbf{S}_4$. This reflection, projected to the tetrahedral orbifold $\mathcal{O}(T_1)$, commutes with the rotation τ of Figure 4, and therefore induces also a reflection of the orbifold $\bar{\mathcal{O}}(T_1)$ and an automorphism of $\pi_1\bar{\mathcal{O}}(T_1)$ that transforms one of the two surjections from $\pi_1\bar{\mathcal{O}}(T_1)$ onto $\mathbf{S}_4 \times \mathbf{Z}_2$ or $\mathbf{S}_4$ into the other. □

The preceding constructions apply to some others of the nine hyperbolic tetrahedra. Let T_3 (resp. T_5) be the two hyperbolic tetrahedra in Figure 7, which we consider simultaneously. Again we have an involution τ of $\mathcal{O}(T_i)$ and denote the quotient by $\bar{\mathcal{O}}(T_i)$, $i = 3$ (resp. $i = 5$).

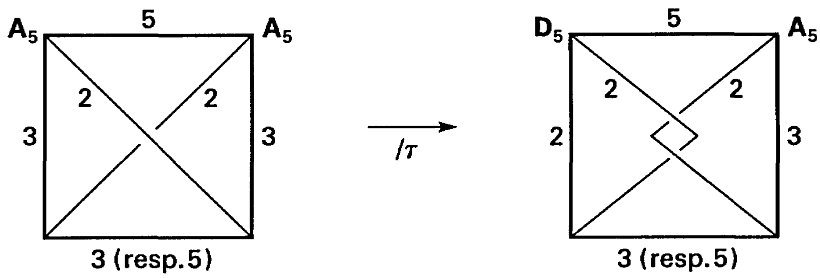

Figure 7

LEMMA 4. (a) *Up to automorphisms of the image groups, there exist exactly two* (*resp. three*) *admissible surjections from* $\pi_1\bar{\mathcal{O}}(T_3)$ (*resp.* $\pi_1\bar{\mathcal{O}}(T_5)$) *onto both* $\mathbf{A}_5 \times \mathbf{Z}_2$ *and* $\mathbf{A}_5$.

(b) *Up to conjugation in* $\mathbf{S}_5$, *there are exactly two* (*resp. three*) *admissible surjections from* $G(T_3) \cong \pi_1\mathcal{O}(T_3)$ (*resp.* $G(T_5) \cong \pi_1\mathcal{O}(T_5)$) *onto* $\mathbf{A}_5$ *and no admissible surjection onto* $\mathbf{A}_5 \times \mathbf{Z}_2$; *these are the restrictions of the surjections from* (a).

As in the case of T_1, we take two copies of T_3 (resp. T_5) and apply the dodecahedral group A_5 to them. We get polyhedra $\mathcal{P}_3$ (resp. $\mathcal{P}_5$); combinatorially, these are obtained by subdividing each face of a dodecahedron into five quadrangles; see Figure 8.

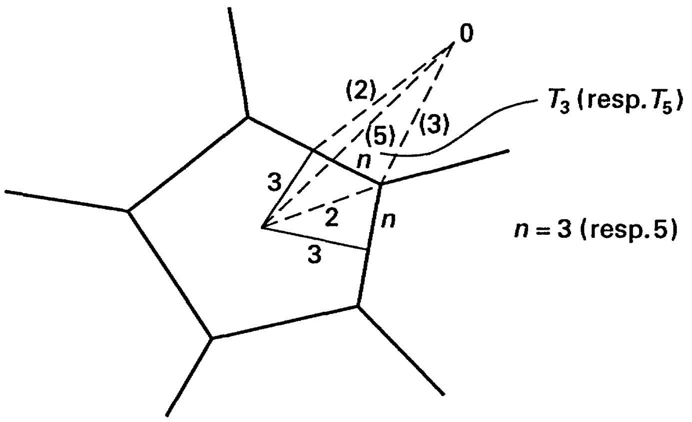

Figure 8

PROPOSITION 10. (a) *There are exactly the two* $\mathbf{A}_5$*-equivariant identifications of* $\mathcal{P}_3$ *indicated in Figure 9, in accordance with the dihedral angles. These differ by a reflection in* $C(T_3)$ *and therefore define the same hyperbolic maximally symmetric* $(\mathbf{A}_5 \times \mathbf{Z}_2)$*-manifold* M_3 *of genus* 11.

(b) *There are exactly the three* $\mathbf{A}_5$*-equivariant identifications of* $\mathcal{P}_5$ *indicated in Figure 9, in accordance with the dihedral angles. Two of these differ by a reflection in* $C(T_5)$ *and therefore define the same hyperbolic maximally symmetric* $(\mathbf{A}_5 \times \mathbf{Z}_2)$*-manifold* M_5 *of genus* 11. *The third identification is invariant under the action of the extended dodecahedral group* $\bar{\mathbf{A}}_5$*, and defines a hyperbolic maximally symmetric* $(\mathbf{A}_5 \times \mathbf{Z}_2)$*-manifold* N_5 *whose universal covering group is normal in the Coxeter group* $C(T_5)$.

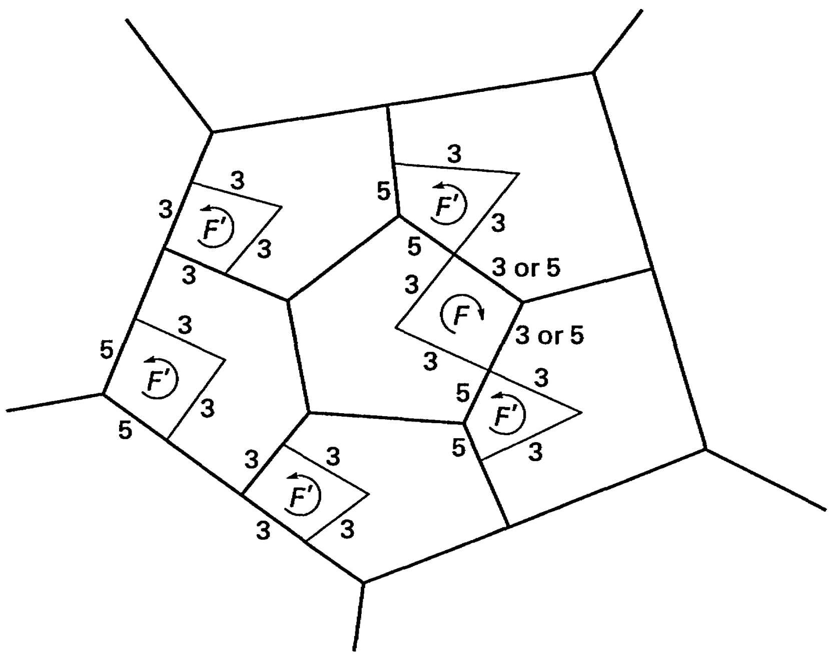

Figure 9

(c) The manifolds M_3, M_5, N_5 are (respectively) 2-fold coverings of hyperbolic maximally symmetric $\mathbf{A}_5$-manifolds $\bar{M}_3$, $\bar{M}_5$, $\bar{N}_5$ of genus 6 corresponding to the kernels of the one (resp. two) surjections, up to automorphisms of image and preimage groups, from $\pi_1\bar{\mathcal{O}}(T_3)$ (resp. $\pi_1\bar{\mathcal{O}}(T_5)$) onto $\mathbf{A}_5$.

For some further properties of the manifold N_5, see [25].

The last hyperbolic tetrahedron for which a similar construction works is the tetrahedron T_2 in Figure 10a; we consider it together with the spherical tetrahedron T_0 in Figure 10b.

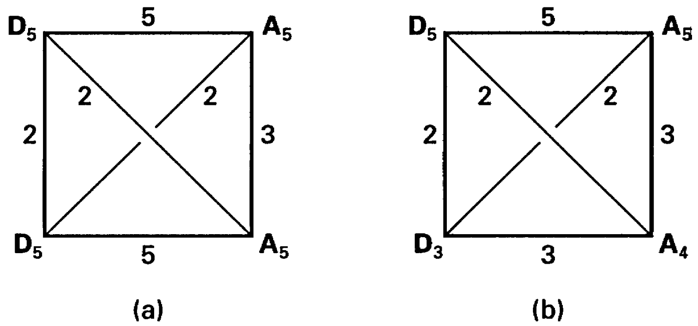

Figure 10

PROPOSITION 11. *Up to conjugation in* $\mathbf{S}_5$, *there are exactly two admissible surjections from* $G(T_2) \cong \pi_1\mathcal{O}(T_2)$ (*resp.* $G(T_0) \cong \pi_1\mathcal{O}(T_0)$) *onto* $\mathbf{A}_5$. *The associated polyhedra* $\mathcal{P}_2$ (*resp.* $\mathcal{P}_0$) *are the regular hyperbolic* $2\pi/5$*-dodecahedron having dihedral angles* $2\pi/5$ (*resp. the spherical* $2\pi/3$*-dodecahedron*). *Accordingly, in both cases there are exactly two* A_5*-equivariant identifications compatible with the dihedral angles: after a rotation by* $\pm\pi/10$ (*resp.* $\pm 3\pi/10$), *each of the twelve faces of the dodecahedron is identified with its opposite face. These identifications differ by a reflection and thus define the same maximally symmetric* $\mathbf{A}_5$*-manifold, which is the Seifert–Weber hyperbolic dodecahedral space* (*resp. the spherical Poincaré homology* 3*-sphere*).

The tetrahedral groups of the remaining five hyperbolic tetrahedra do not admit an admissible surjection onto a vertex group (i.e., do not split over a vertex group). However, the construction works for some of the other spherical and Euclidean Coxeter tetrahedra.

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