Spectral Synthesis of Ideals in Zygmund Algebras: The Asymptotic Cauchy Problem Approach

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0. Introduction

In the present work, we study the structure of closed ideals in algebras of functions f on the cube $Q_0 := [-1, 1]^n$ satisfying, for a given majorant ω, the Zygmund condition $|f(x+h) - 2f(x) + f(x-h)| \leq C\omega(|h|)$. It is assumed that

$$\int_0^1 \frac{\omega(t)}{t^2}\, dt = +\infty \tag{0}$$

or equivalently that Λ^ω is not contained in $C^1(Q_0)$. The main result of the paper is a theorem on spectral synthesis of ideals in Zygmund algebras Λ^ω (Theorem 2) claiming that, for regular majorants ω (i.e., those subject to regularity conditions (R1) and (R2) below), every closed proper ideal in the algebra Λ^ω is an intersection of closed primary ideals. Assertions of such type go back to the classical algebraic works of E. Noether and E. Lasker on ideals in Noetherian rings. Later, theorems on spectral synthesis of ideals were proved (or disproved) for various algebras of smooth functions. For more extensive discussion, see Section 1.

Our proof of Theorem 2 depends on two major results: on an abstract spectral synthesis theorem for a class of functional Banach algebras, called the class of D-algebras and defined in terms of point derivations (Theorem 0) [H2; H4]; and on a special extension theorem for Zygmund functions (Theorem 1) which implies that the algebras Λ^ω are D-algebras.

The main difficulty in proving Theorem 1 arises from the absence of an intrinsic description of traces of Zygmund functions to general sets in $\mathbb{R}^n$ for $n > 2$. Using such descriptions for $n = 1, 2$ [Shv; H4; H5], direct proofs of Theorem 1 (and thereby of the spectral synthesis theorem) for algebras Λ^ω with *arbitrary* majorants ω satisfying condition (0) were obtained in [H3; H5]. An alternative proof of Theorem 1 in the case $n = 1$, $\omega(t) = t$, is presented in [H4].

Our proof of Theorem 1 is based on the method of quasiharmonic extensions of smooth functions [D1; D2]. This method works for any dimension n; however, it presupposes certain regularity of majorants ω.

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Results of this work are related with the classical problem of spectral approximation. The latter consists of describing, for a certain space of functions (not necessarily an algebra with respect to pointwise operations), the closure J_E of the set of functions vanishing in a neighborhood of a given closed set E. Stimulated by the classical theorem of Malliavin [Ma] on the lack of spectral synthesis in Wiener algebras for noncompact abelian groups, the problem of spectral approximation was studied originally in the framework of harmonic analysis. The most interesting results in the nonharmonic setting are exemplified by the solution of the spectral approximation problem for Sobolev spaces [He; HW], and for Besov and Lizorkin–Triebel spaces [N]. For every function in J_E, its derivatives (whatever they are) vanish in an appropriate sense on E. This renders the problem of spectral approximation for many natural classes of smooth functions (in particular, for nonseparable Zygmund spaces Λ^ω) too restrictive. For functional Banach algebras, a more general approach based on the property of spectral synthesis of ideals was suggested by Shilov [Shi]. In this case, J_E turns out to be the minimal closed ideal with the cospectrum E.

As a corollary of Theorem 1 we obtain a solution of the spectral approximation problem for Zygmund spaces Λ^ω with regular majorants ω. Also, we show that every closed ideal in the corresponding "small" Zygmund algebra λ^ω is completely determined by its cospectrum (Theorem 3). In the cases $n = 1$ and $n = 2$, these two facts are valid for any majorant ω subject to condition (0) (see [H4; H5]). For arbitrary n and $\omega(t) = t$, Theorem 3 was obtained in [D2].

In Section 1, we give the main definitions, examples, and results related to spectral synthesis of ideals, including the concept of D-algebras. Notation and some preliminary facts are outlined in Section 2. In Section 3, we recall the definition of Zygmund spaces and state some of their properties. Main results of the paper are formulated in Section 4 where we deduce spectral synthesis theorems for "big" and "small" Zygmund algebras and the theorem on spectral approximation from the special extension theorem. The proof of the latter is presented in Section 5.

1. Spectral Synthesis of Ideals: Definitions and Examples

Let X be a locally compact Hausdorff space, and let $\mathcal{A}$ be a Shilov regular Banach algebra of real or complex continuous functions on X with respect to pointwise operations. We assume that the space of maximal ideals of algebra $\mathcal{A}$ coincides with X.

For every ideal I in $\mathcal{A}$, we define a closed subset in X, $\sigma(I) := \bigcap\{f^{-1}(0) : f \in I\}$, which is called the *cospectrum* of I. An ideal I is called *primary* at a point $x \in$ X if $\sigma(I) = \{x\}$. We associate with every closed subset E in X the set M_E of all functions in $\mathcal{A}$ vanishing on E, and the closure J_E of the set of functions in $\mathcal{A}$ vanishing in a neighborhood of E. It is well known (see e.g. [GRS, Sec. 36]) that M_E and J_E are the maximal and the minimal closed ideals in $\mathcal{A}$ with cospectrum E (in particular, M_x is the maximal ideal at x and J_x is the minimal closed primary ideal at x). Thus, for every closed ideal I in $\mathcal{A}$ with cospectrum E, we have $J_E \subset I \subset M_E$.

The *primary component* I_x of an ideal I at a point $x \in \mathrm{X}$ is defined to be the smallest closed primary ideal at x containing I. It is easy to see that $I_x = \mathrm{clos}_{\mathcal{A}}(I + J_x)$.

We say that the *algebra* $\mathcal{A}$ *admits spectral synthesis of ideals* (notation: $\mathcal{A} \in$ Synt) if, for every closed proper ideal I in $\mathcal{A}$,

$$I = \bigcap \{ I_x : x \in \sigma(I) \}. \tag{1.1}$$

In other words, $\mathcal{A} \in$ Synt if every closed proper ideal in $\mathcal{A}$ is an intersection of closed primary ideals. If $J_x = M_x$ for all $x \in \mathrm{X}$, then (1.1) becomes

$$I = M_E, \quad E = \sigma(I), \tag{1.2}$$

and in this case we write $\mathcal{A} \in$ synt.

Here are a few basic examples of algebras of smooth functions admitting spectral synthesis of ideals.

(1) The algebra $C(\mathrm{X})$ of all continuous functions on a compact Hausdorff space X equipped with the norm $\|f\|_{\mathrm{X}} := \sup\{ |f(x)| : x \in \mathrm{X}\}$. As shown in [S; Shi], $C(\mathrm{X}) \in$ synt. More generally, this is true for the algebra $C_0(\mathrm{X})$ of continuous functions on a locally compact Hausdorff space X "vanishing at infinity".

(2) The "small" Lipschitz algebra $\mathrm{lip}(K, \rho)$ of functions on a compact metric space (K, ρ) with the finite norm $\|f\|_{K,\rho} := \max\{\|f\|_K, |f|_{K,\rho}\}$, where

$$|f|_{K,\rho} := \sup\left\{ \frac{|f(x) - f(y)|}{\rho(x, y)} : x, y \in K,\ x \neq y \right\},$$

and satisfying the condition $\lim_{\rho(x,y)\to 0}(f(x) - f(y))/\rho(x, y) = 0$. The algebra $\mathrm{lip}(K, \rho)$ possesses the spectral synthesis property (1.2); see [She].

(3) The "big" Lipschitz algebra of all functions on a compact metric space (K, ρ) with the finite norm $\|\cdot\|_{K,\rho}$ [Wa]. (The particular case $K \subset \mathbb{R}^n$, $\rho(x, y) = |x - y|^\alpha$, $0 < \alpha \le 1$, was treated in [G].)

(4) The algebra $C^m(Q)$ of m times continuously differentiable functions on a closed cube Q in $\mathbb{R}^n$ [W2; M] (for $m = 1$, this was independently established in [Sn]). Also, $C^\infty(Q) \in$ Synt [M].

(5) The algebra $C^m \,\mathrm{lip}\, \omega(Q)$ of C^m-functions on Q with higher-order derivatives in $\mathrm{lip}\,\omega(Q)$ [H1]. Here ω is any nondecreasing function on $\mathbb{R}_+$ such that $\omega(0) = \omega(0+) = 0$, $\omega(t) > 0$ for $t > 0$, and $\lim_{t\to 0+} \omega(t)/t = +\infty$.

(6) The algebra $C^m \,\mathrm{Lip}\, \omega(Q)$ for $n = 1$ [H5] with a majorant ω satisfying the above conditions except possibly for the last one.

(7) The Sobolev algebras $W_p^l(\mathbb{R}^n)$ for $n = 1 \le p < +\infty$ and $2 \le n < p < +\infty$ [H1; H5]. For all other possible values of parameters p, l, n, Sobolev algebras fail to admit spectral synthesis of ideals.

The proofs of these results for every particular algebra are very specific. In [H2; H4] the second author developed a unified approach to spectral synthesis of ideals in a certain class of Banach algebras which we shall briefly describe.

Let $\mathcal{A}$ be a Shilov regular Banach algebra of continuous functions on a compact Hausdorff space X containing the unity function and having the following inversion property: if $f \in \mathcal{A}$ and $f(x) \neq 0$ for all $x \in \mathrm{X}$ then $1/f \in \mathcal{A}$. Hence, the space of maximal ideals of $\mathcal{A}$ coincides with X.

DEFINITION 1. A bounded linear functional $D \in \mathcal{A}^*$ is called a *point derivation* of algebra $\mathcal{A}$ at a point $x \in \mathrm{X}$ if

$$D(fg) = f(x)Dg + g(x)Df \quad \text{for all } f, g \in \mathcal{A}.$$

Let $\mathcal{D}_x$ be the linear space of all point derivations of $\mathcal{A}$ at a point x. Observe that, due to the regularity of $\mathcal{A}$, $D(J_x) = \{0\}$ for all $D \in \mathcal{D}_x$. For a closed subset E in X, define $K_E := \{ D : D \in \mathcal{D}_x, \ x \in E, \ \|D\| \leq 1 \}$. Obviously, K_E is compact in the weak* topology on $\mathcal{A}^*$.

To each function $f \in \mathcal{A}$, we associate a function $\hat{f} \in C(K_E)$ by setting $\hat{f}(D) := Df, \ D \in K_E$. This formula determines a linear mapping $d_E : \mathcal{A}/J_E \to C(K_E)$ such that $\|\hat{f}\|_{K_E} \leq \|\dot{f}\|_E$ for all $f \in \mathcal{A}$, where $\|\dot{f}\|_E$ stands for the quotient norm of the class $\dot{f} \in \mathcal{A}/J_E$ containing f.

DEFINITION 2. An algebra $\mathcal{A}$ is called a *D-algebra* if, for every closed set $E \subset \mathrm{X}$, there is a constant $A(E)$ such that

$$\|\dot{f}\|_E \leq A(E)\|\hat{f}\|_{K_E} \quad \text{for all } f \in M_E. \tag{1.3}$$

Condition (1.3) is a kind of an extension theorem which implies in particular that the "trace" of a function in a D-algebra to a closed set $E \subset \mathrm{X}$ is completely determined by values of the function on E and those of its point derivations at points of E.

Introducing the class of D-algebras is justified by the following result.

THEOREM 0 [H2; H4]. *Every D-algebra admits spectral synthesis of ideals.*

The most important examples of D-algebras are as follows.

(1) Algebras satisfying condition (1.2) and hence having only trivial point derivations. This is the case for algebras $C(\mathrm{X})$, $C_0(\mathrm{X})$, $\operatorname{lip}(K, \rho)$, and $W_p^1(\mathbb{R}^n)$ with $n = 1 \leq p < +\infty$ and $2 \leq n < p < +\infty$.
(2) Lipschitz algebras $\operatorname{Lip}(K, \rho)$. In the real case, inequality (1.3) is satisfied with $A(E) = 1$ and thus turns into equality (see [She; Wa] for substantiation).
(3) Algebras $C^1(Q)$ on a closed cube Q in $\mathbb{R}^n$. In these algebras, every point derivation at a point x is of the form $f \mapsto \nabla f(x)\nu$ for some vector $\nu \in \mathbb{R}^n$, that is, is a directional derivative. Hence, (1.3) follows from the Whitney extension theorem [W1; M] with a constant depending only on n. It is worth noting that the algebras $C^1 \operatorname{Lip} \omega(Q)$ fail to be D-algebras [H5].
(4) In the present contribution we show that Zygmund algebras Λ^ω are D-algebras for majorants ω subject to (0) and satisfying, for all $t \in (0, 1/2]$ and for some constant $C > 0$, the following regularity conditions:

$$\int_0^t \frac{\omega(s)}{s}\, ds \leq C\omega(t), \tag{R1}$$

$$t^2 \int_t^1 \frac{\omega(s)}{s^3}\, ds \le C\omega(t). \tag{R2}$$

Observe that in the case of power majorants $\omega(t) = t^\alpha$, conditions (R1) and (R2) are satisfied for $0 < \alpha < 2$ while condition (0) is fulfilled for $0 < \alpha \le 1$. Note also that if $0 < \alpha < 1$ then $\Lambda^\alpha = \operatorname{Lip}\alpha$. Thus, in the power scale, the only case of interest is $\omega(t) = t$, that is, that of the classical Zygmund space Λ [Z].

2. Preliminary Observations and Notation

We denote by $|x|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$, and set $x^2 := |x|^2$. For $a \in \mathbb{R}^n$ and $r > 0$, we set $B(a, r) := \{x \in \mathbb{R}^n : |x - a| \le r\}$ (if $a = 0$, we write $B(r)$). An interval in $\mathbb{R}^n$ is a set of the form $[a, b] := \{a + t(b - a) : 0 \le t \le 1\}$, where $a, b \in \mathbb{R}^n$. A set $S = Q(a, d) = \{x \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i - a_i| \le d\}$ with $a \in \mathbb{R}^n$ and $0 \le d \le +\infty$ will be called a *cube*, and we will write $c_S = a$, $d_S = d$, and $\operatorname{diam} S = 2\sqrt{n}d$.

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we denote by D^α the corresponding partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Partial derivatives of the first order will be written also in the form $\partial/\partial x_i$, $i = 1, \ldots, n$. Symbols ∇, ∇^2, and Δ stand for the gradient, the Hesse matrix, and the Laplacian, respectively. Support of a function will be denoted by supp.

In the sequel, C will stand for various positive constants that may depend only on n and may differ even in the same chain of estimates. Every cube in $\mathbb{R}^n$ of the form $Q(0, C)$ will be denoted by Q.

For a function f defined on a cube S in $\mathbb{R}^n$ and for all admissible x and h, we set $\Delta_h^1 f(x) := f(x + h) - f(x)$, $\Delta_h^2 f(x) := f(x + h) - 2f(x) + f(x - h)$,

$$\omega_2(f; S; t) := \sup\{|\Delta_h^2 f(x)| : x \pm h \in S, |h| \le t\}, \quad t \ge 0,$$

and $\|f\|_S := \sup\{|f(x)| : x \in S\}$ (if $S = \mathbb{R}^n$ we write $\|f\|_\infty$). Let $\mathcal{P}_1$ be the set of polynomials in n variables of degree not greater than 1. For a bounded function f defined on a set F in $\mathbb{R}^n$, we denote by $E_1(f; F) := \inf\{\|f - P\|_F : P \in \mathcal{P}_1\}$ the uniform best polynomial approximation to f on F of order 1. It is well known [B] that, for every cube S in $\mathbb{R}^n$,

$$E_1(f; S) \le C\omega_2(f; S; d_S) \tag{2.1}$$

(the converse inequality $\omega_2(f; S; d_S) \le 4E_1(f; S)$ is obvious).

The Euclidean distance from a point $x \in \mathbb{R}^n$ to a set $E \subset \mathbb{R}^n$ will be denoted by $d(x, E)$, and $d(F, E) = \inf\{d(x, E) : x \in F\}$ will stand for the distance between sets F and E. For $\delta > 0$, we put $E_\delta = \{x \in \mathbb{R}^n : d(x, E) \le \delta\}$. As usual, χ_E denotes the characteristic function of a set E.

For a closed set E in $\mathbb{R}^n$, we denote by W_E the *Whitney decomposition* of $\mathbb{R}^n \setminus E$ [M], that is, a collection of cubes with disjoint interiors such that:

(i) $\mathbb{R}^n \setminus E = \bigcup\{K : K \in W_E\}$, and the multiplicity of this covering is uniformly bounded by a constant depending only on n;

(ii) $\operatorname{diam} K \le d(K, E) \le 4 \operatorname{diam} K$, $K \in W_E$.

Associated with the Whitney decomposition W_E is a partition of unity by functions ψ_K, $K \in W_E$, possessing the following properties:

(i) $\sum_{K \in W_E} \psi_K(x) = 1$, $x \in \mathbb{R}^n \setminus E$;
(ii) $\operatorname{supp} \psi_K \subset Q(c_K, 2d_K)$;
(iii) for every $K \in W_E$, $\psi_K \in C^\infty(\mathbb{R}^n)$ and $\|D^\alpha \psi_K\|_\infty \le C d_K^{-|\alpha|}$, $|\alpha| \le 2$.

The function $\rho(x, E) := \sum_{K \in W_E} d(K, E)\psi_K(x)$ is called the *regularized distance* from a point x to the set E. An easy calculation shows that

$$\tfrac{1}{2}\rho(x, E) \le d(x, E) \le \tfrac{5}{2}\rho(x, E), \quad x \in \mathbb{R}^n. \tag{2.2}$$

Besides this, $\rho \in C^\infty(\mathbb{R}^n)$ and

$$|D^\alpha \rho(x, E)| \le C d(x, E)^{1-|\alpha|}, \quad 1 \le |\alpha| \le 2. \tag{2.3}$$

Vectors $z \in \mathbb{R}^{n+1}$ will be represented in the form $z = (x, y)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. To distinguish n-dimensional cubes, balls, and neighborhoods from their $(n+1)$-dimensional counterparts, we supply the latter with the symbol "$\tilde{\ }$". We recall that the Poisson kernel for $\mathbb{R}^{n+1}_+$ is the function $P(x, y) := c_n y/(x^2 + y^2)^{(n+1)/2}$ ($x \in \mathbb{R}^n$, $y \in \mathbb{R}$), where c_n is defined by

$$\int_{\mathbb{R}^n} P(x, y)\, dx = 1, \quad y > 0. \tag{2.4}$$

A nondecreasing function ω on $\mathbb{R}_+$ such that $\omega(0) = \omega(0+) = 0$, $\omega(t) > 0$ for $t > 0$ and $\omega(t) = \omega(1) = 1$ for $t > 1$ will be called a *majorant*. It can be easily checked that every majorant has the following properties:

$$s \int_s^1 \frac{\omega(v)}{v^2}\, dv \le Ct \int_t^1 \frac{\omega(v)}{v^2}\, dv, \quad 0 < s \le t \le 1/2, \tag{2.5}$$

$$t \int_t^1 \frac{\omega(v)}{v^2}\, dv \to 0 \quad \text{as } t \to 0+. \tag{2.6}$$

3. Zygmund Algebras: Definitions and Main Properties

For a given majorant ω, the Zygmund space $\Lambda^\omega = \Lambda^\omega(Q_0)$ is defined as the set of all bounded functions f on the cube $Q_0 = [-1, 1]^n$ satisfying, for all admissible x and h and for some constant $A \ge 0$, the (generalized) Zygmund condition $|\Delta_h^2 f(x)| \le A\omega(|h|)$. Everywhere below we will assume without loss of generality that the function

$$\omega(t)/t^2 \text{ is nonincreasing for } \quad t > 0. \tag{R}$$

This implies

$$\omega(\alpha t) \le \alpha^2 \omega(t), \quad \alpha \ge 1,$$

an inequality that will be (sometimes tacitly) exploited throughout.

The space Λ^ω is supplied with the norm $\|f\|_{\Lambda^\omega} := \max\{\|f\|_{Q_0}, |f|_{\Lambda^\omega}\}$, where the Zygmund seminorm $|\cdot|_{\Lambda^\omega}$ is the infimum of all constants A involved in the above Zygmund condition. We define the "small" Zygmund space λ^ω to be the set of all functions $f \in \Lambda^\omega$ satisfying the condition $\omega_2(f; Q_0; t)/\omega(t) \to 0$ as $t \to 0$. A standard argument shows that λ^ω is a linear closed separable subspace in Λ^ω. Similarly, one can define the space $\Lambda^\omega(\mathbb{R}^n)$.

Note that by the Marchaud inequality [Mar], for every function f defined on an interval $L = [a-d, a+d]$ and for all x and $h \neq 0$ such that $x, x+h \in L$,

$$|\Delta_h^1 f(x)| \le C|h| \left[\int_{|h|}^{d} \frac{\omega_2(f; L; t)}{t^2}\, dt + \frac{\|f\|_L}{d} \right]. \tag{3.1}$$

Hence, in view of (2.6), $\Lambda^\omega \subset C(Q_0)$. Also, the space Λ^ω is imbedded in $C^1(Q_0)$ under the following condition.

PROPOSITION. *For any majorant subject to* (R),

$$\Lambda^\omega \subset C^1(Q_0) \quad \textit{iff} \quad \int_0^1 \frac{\omega(t)}{t^2}\, dt < \infty.$$

Furthermore, this condition implies

$$\Lambda^\omega \subset C^1 \operatorname{Lip} \gamma(Q_0) \quad \textit{with} \quad \gamma(t) := \int_0^t \frac{\omega(s)}{s^2}\, ds.$$

The proof of the Proposition can be found in [H3, H5].

The Zygmund space Λ^ω is a Banach algebra with respect to pointwise multiplication. In fact, we apply (3.1) and the inequality

$$\left[t \int_t^1 \frac{\omega(s)}{s^2}\, ds \right]^2 \le C\omega(t), \quad 0 < t \le 1/2, \tag{3.2}$$

(for the proof of (3.2), see [H5]) to the identity

$$\Delta_h^2(fg)(x) = \Delta_h^2 f(x) g(x+h) + 2\Delta_h^1 f(x-h)\Delta_h^1 g(x) + f(x-h)\Delta_h^2 g(x) \tag{3.3}$$

to obtain $\|fg\|_{\Lambda^\omega} \le C\|f\|_{\Lambda^\omega}\|g\|_{\Lambda^\omega}$. The algebra Λ^ω obviously meets all the requirements of Section 1.

If $\lim_{t\to 0+} \omega(t)/t^2 = +\infty$, that is, for all majorants $\omega(t)$ except for those equivalent to t^2, then inequality (3.2) can be strengthened—namely, in this case,

$$\lim_{t\to 0+} \frac{1}{\omega(t)} \left[t \int_t^1 \frac{\omega(s)}{s^2}\, ds \right]^2 = 0 \tag{3.4}$$

(see [H5]).

4. Main Results

The ensuing result is the main contribution of this work.

THEOREM 1. *Suppose ω is a majorant satisfying conditions* (R1), (R2), *and* (0). *Let E be a closed subset in Q_0, let f be a function in $\Lambda^\omega(\mathbb{R}^n)$ with support in $Q(0, C)$ such that $f|_E \equiv 0$, and set*

$$M := \limsup_{d(x,E)\to 0,\, h\to 0} \frac{|\Delta_h^2 f(x)|}{\omega(|h|)}.$$

Then for every $\varepsilon > 0$ there exist $\delta > 0$ and a function $g \in \Lambda^\omega(\mathbb{R}^n)$ with $g|_{E_\delta} = f|_{E_\delta}$ and $\|g\|_{\Lambda^\omega} \le C(M+\varepsilon)$.

Theorem 1 will be proved in Section 5.

Applying the argument developed for the case $\omega(t) = t$ in [H4, Sec. 2] (for the general case, see [H5]), by invoking (3.4) we show that $M = \|\hat{f}\|_{K_E}$. Therefore, Theorem 1 implies that Λ^ω is a D-algebra. Together with Theorem 0 this leads us to the following spectral synthesis theorem for Zygmund algebras.

THEOREM 2. *For majorants ω satisfying conditions* (R1), (R2), *and* (0), $\Lambda^\omega \in$ Synt.

Using regularization, we extract from Theorem 1 the following corollary; see [H4, Sec. 5] and [H5] for further details.

THEOREM 3. *Let majorant ω satisfy* (R1), (R2), *and* (0). *Then* $\lambda^\omega \in$ synt.

Another result that can be derived immediately from Theorem 1 is the following theorem on spectral approximation. We recall that, given a closed set $E \subset Q_0$, J_E is the closure in Λ^ω of the set of functions in Λ^ω vanishing in a (relative) neighborhood of E.

THEOREM 4. *If ω is a majorant with properties* (R1), (R2), *and* (0), *then for any closed set E in Q_0 we have*

$$J_E = \left\{ f \in \Lambda^\omega : f|_E \equiv 0,\ \lim_{d(x,E)\to 0,\, h\to 0} \frac{\Delta_h^2 f(x)}{\omega(|h|)} = 0 \right\}.$$

5. Proof of Theorem 1

5.1. Auxiliary Results

Collected in Section 5.1 are a few results of analytical nature used in our proof of Theorem 1. A crucial role in this proof is played by the quasiharmonic extensions of functions that appear in the context of the following assertion.

LEMMA 1. *Let ω be an arbitrary majorant, and let $f \in \Lambda^\omega(\mathbb{R}^n)$. Then there exists a function $u \in C^1(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1} \setminus \mathbb{R}^n)$ such that:*

(i) $u(x,-y) = -u(x,y)$ *for* $(x,y) \in \mathbb{R}^{n+1}$ (*in particular,* $u(x,0) = 0$ *for* $x \in \mathbb{R}^n$);
(ii) $\frac{\partial}{\partial y} u(x,0) = f(x)$, $x \in \mathbb{R}^n$;
(iii) $|\Delta u(x,y)| \le C|f|_{\Lambda^\omega}\omega(|y|)/|y|$, $x \in \mathbb{R}^n$, $y \ne 0$;
(iv) *if* $\operatorname{supp} f \subset B(r)$, *then* $\operatorname{supp} u \subset \tilde{B}(Cr)$.

Moreover, this function u has the following local properties.

LEMMA 2. *Suppose the majorant ω satisfies condition* (R1). *Let $f \in \Lambda^{\omega}(\mathbb{R}^n)$, let u be the function from Lemma 1, and let E be a closed subset in $\mathbb{R}^n$. Suppose that $f|_E \equiv 0$ and that for some $\delta \in (0, 1/2]$ and $A \geq 0$, $|\Delta_h^2 f(x)| \leq A\omega(|h|)$ whenever $d(x, E) \leq \delta$ and $|h| \leq \delta$. Then, for every $z = (x, y) \in \mathbb{R}^{n+1}$ with $d(z, E) \leq \delta$, the following conditions hold:*

(i) $$|\Delta u(z)| \leq CA\omega(|y|);$$

(ii) $$|\nabla u(z)| \leq C\|f\|_{\Lambda^\omega} d(z, E) \int_{d(z,E)}^{1} \frac{\omega(t)}{t^2}\, dt;$$

(iii) $$|u(z)| \leq C\|f\|_{\Lambda^\omega} d^2(z, E) \int_{d(z,E)}^{1} \frac{\omega(t)}{t^2}\, dt.$$

For the proof of Lemma 1 and of assertion (i) in Lemma 2 we refer the reader to Theorem 1 in [D2], which discusses the case of power majorants ω; however, the proof is the same for any majorant. Bounds (ii) and (iii) in Lemma 2 are established in [D2, Lemma 4] for $\omega(t) = t$, but the proof carries over without any difficulty to any majorant with the property (R1).

LEMMA 3. *Let G be a closed set contained in a ball $B = B(r) \subset \mathbb{R}^n$, and let ω be a majorant satisfying condition* (R1). *Then, for every harmonic function F in $2\tilde{B} \setminus G$,*

$$\int_{\tilde{B}\setminus G} |F(z)| \frac{\omega(|y|)}{|y|}\, dz \leq C \int_{2\tilde{B}\setminus G} |F(z)| \frac{\omega(d(z))}{d(z)}\, dz,$$

where $d(z) := d(z, G)$, $z \in \mathbb{R}^{n+1}$.

Proof. Observe that if $z \in \tilde{B} \setminus G$, $\zeta \in \mathbb{R}^{n+1}$, and $|\zeta - z| \leq d(z)/2$ then $\zeta \in 2\tilde{B}$ and $\frac{2}{3} d(\zeta) \leq d(z) \leq 2d(\zeta)$. By the mean value theorem for harmonic functions, we have

$$F(z) = \frac{C}{d^{n+1}(z)} \int_{\tilde{B}(z,d(z)/2)} F(\zeta)\, d\zeta, \quad z \in \tilde{B} \setminus G.$$

Hence,

$$\begin{aligned} \int_{\tilde{B}\setminus G} |F(z)| \frac{\omega(|y|)}{|y|}\, dz &\leq C \int_{\tilde{B}\setminus G} \frac{\omega(|y|)}{|y|} \frac{1}{d^{n+1}(z)} \int_{\tilde{B}(z,d(z)/2)} F(\zeta)\, d\zeta\, dz \\ &\leq C \int_{2\tilde{B}\setminus G} |F(\zeta)| \frac{1}{d^{n+1}(\zeta)} \int_{\tilde{B}(\zeta,d(\zeta))} \frac{\omega(|y|)}{|y|}\, dz\, d\zeta. \end{aligned}$$

For every point $\zeta = (\xi, \eta)$ we have $|\eta| \leq d(\zeta)$. Therefore, $\tilde{B}(\zeta, d(\zeta)) \subset B(\xi, d(\zeta)) \times [-2d(\zeta), 2d(\zeta)]$. Now we continue the previous estimates and make use of (R1) to obtain

$$\begin{aligned} \int_{\tilde{B}\setminus G} |F(z)| \frac{\omega(|y|)}{|y|}\, dz &\leq C \int_{2\tilde{B}\setminus G} |F(\zeta)| \frac{1}{d(\zeta)} \int_0^{2d(\zeta)} \frac{\omega(t)}{t}\, dt \\ &\leq C \int_{2\tilde{B}\setminus G} |F(\zeta)| \frac{\omega(d(\zeta))}{d(\zeta)}\, d\zeta, \end{aligned}$$

which completes the proof. □

Our next result is a generalization of the corresponding statement in [D2], which was established there for power majorants and proved via a different method.

LEMMA 4. *Suppose the majorant* ω *meets conditions* (R1) *and* (R2). *Let* ϕ *be a measurable function on* $\mathbb{R}^{n+1}$ *with* $\operatorname{supp}\phi \subset \tilde{Q}$ *such that, for some constant* $A \geq 0$,

$$|\phi(x, y)| \leq A\frac{\omega(|y|)}{|y|}, \quad y \neq 0,$$

and let

$$g(x) := \int_{\mathbb{R}^{n+1}} \phi(x+t, y)P(t, y)\,dt\,dy, \quad x \in \mathbb{R}^n. \tag{5.1}$$

Then $g \in \Lambda^\omega$ *and* $\|g\|_{\Lambda^\omega} \leq CA$.

Proof. Due to (2.4) and (R1), the integral in (5.1) converges absolutely. In view of $\omega(1) = 1$ we find that

$$\|g\|_\infty \leq 2A\int_0^C \frac{\omega(s)}{s}\,ds \leq CA.$$

While estimating the second difference $\Delta_h^2 g(x)$ we may assume that $|h| \leq 1/2$. We have

$$\begin{aligned}\Delta_h^2 g(x) &= \int_{|y|\leq|h|}\int_{\mathbb{R}^n} \Delta_h^2\phi(x+t, y)P(t, y)\,dt\,dy\\ &+ \int_{|y|>|h|}\int_{|t|\leq 2|y|} \phi(x+t, y)\Delta_h^2 P(t, y)\,dt\,dy\\ &+ \int_{|y|>|h|}\int_{|t|>2|y|} \phi(x+t, y)\Delta_h^2 P(t, y)\,dt\,dy = I_1 + I_2 + I_3.\end{aligned}$$

Proceeding from the trivial estimate $\|\Delta_h^2\phi\|_\infty \leq 4\|\phi\|_\infty$ and using (2.4) and (R1), we conclude that

$$|I_1| \leq CA\int_0^{|h|} \frac{\omega(s)}{s}\,ds \leq CA\omega(|h|).$$

Note that, for every function $f \in C^2$,

$$|\Delta_h^2 f(x)| \leq C|h|^2 \max_{t\in[x-h,\, x+h]} |\nabla^2 f(t)|.$$

For the Poisson kernel, for $t \in \mathbb{R}^n$ and $y \in \mathbb{R}$ we have

$$|\Delta_t^2 P(x, y)| \leq C|y|(t^2 + y^2)^{-(n+3)/2}. \tag{5.2}$$

Hence, $|\Delta_h^2 P(t, y)| \leq C|h|^2|y|^{-(n+2)}$. Using (R2), for the integral I_2 we obtain the following estimate:

$$|I_2| \leq CA|h|^2\int_{|h|}^C \frac{\omega(y)}{y^3}\,dy \leq CA\omega(|h|).$$

If $|t| > 2|y|$ then (5.2) implies $|\Delta_h^2 P(t, y)| \le C|h|^2|y||t|^{-(n+3)}$. Therefore,

$$|I_3| \le CA|h|^2 \int_{|h|}^{C} \omega(y) \int_{|t|>2y} |t|^{-(n+3)}\, dt\, dy$$
$$\le CA|h|^2 \int_{|h|}^{C} \frac{\omega(y)}{y^3}\, dy \le CA\omega(|h|).$$

Thus we conclude that $|\Delta_h^2 g(x)| \le CA\omega(|h|)$. Consequently, $g \in \Lambda^\omega$ and $\|g\|_{\Lambda^\omega} \le CA$. Lemma 4 is proved. □

Our next assertion provides a formula for the integral $\int_{\mathbb{R}^{n+1}} \Delta v(z) P(z)\, dz$, where $v \in C^2(\mathbb{R}^{n+1} \setminus \mathbb{R}^n)$ and P is the Poisson kernel. Formally, the classical Green's formula does not apply in this case. However, the result is the same if we assume that $v \in C^1(\mathbb{R}^{n+1})$ and understand the integral as the limit as $\delta \to 0+$ of the integrals over $\mathbb{R}^n \times ((-\infty, -\delta] \cup [\delta, +\infty))$.

LEMMA 5. *Let $v \in C^1(\mathbb{R}^{n+1}) \cap C^2(\mathbb{R}^{n+1} \setminus \mathbb{R}^n)$ be a function with compact support such that $v(x, 0) = 0$ for all $x \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^{n+1}} \Delta v(x + t, y) P(t, y)\, dt\, dy = -2\frac{\partial v}{\partial y}(x, 0), \qquad x \in \mathbb{R}^n.$$

Proof. Clearly, it suffices to show that

$$\int_{\mathbb{R}_+^{n+1}} \Delta v(t, y) P(t, y)\, dt\, dy = -\frac{\partial v}{\partial y}(0, 0).$$

Integrating by parts and taking into account harmonicity of the Poisson kernel, for all $\delta > 0$ we have

$$\int_{\mathbb{R}^n} \int_\delta^\infty \Delta v(t, y) P(t, y)\, dt\, dy = -\int_{\mathbb{R}^n} \frac{\partial v}{\partial y}(t, \delta) P(t, \delta)\, dt + \int_{\mathbb{R}^n} v(t, \delta) \frac{\partial P}{\partial y}(t, \delta)\, dt. \tag{5.3}$$

Observe that

$$\frac{\partial P}{\partial y}(t, \delta) = c_n \frac{t^2 - n\delta^2}{(t^2 + \delta^2)^{(n+3)/2}}.$$

Hence,

$$a_n := -\int_{|t| \le \sqrt{n}\delta} \delta \frac{\partial P}{\partial y}(t, \delta)\, dt = \int_{|t| > \sqrt{n}\delta} \delta \frac{\partial P}{\partial y}(t, \delta)\, dt < +\infty$$

and does not depend on δ. Setting

$$K_\delta^0(t) := -a_n^{-1} \delta \frac{\partial P}{\partial y}(t, \delta) \chi_{B(\sqrt{n}\delta)}(t), \qquad K_\delta^\infty(t) := a_n^{-1} \delta \frac{\partial P}{\partial y}(t, \delta) \chi_{\mathbb{R}^n \setminus B(\sqrt{n}\delta)}(t),$$

and recalling that $v|_{\mathbb{R}^n} \equiv 0$, we rewrite (5.3) in the form

$$\int_{\mathbb{R}^n}\int_\delta^\infty \Delta v(t,y)P(t,y)\,dt\,dy = -\int_{\mathbb{R}^n}\frac{\partial v}{\partial y}(t,\delta)P(t,\delta)\,dt$$
$$-a_n\int_{\mathbb{R}^n}\frac{v(t,\delta)-v(t,0)}{\delta}K_\delta^0(t)\,dt$$
$$+a_n\int_{\mathbb{R}^n}\frac{v(t,\delta)-v(t,0)}{\delta}K_\delta^\infty(t)\,dt$$
$$= -I_1(\delta)-a_nI_2(\delta)+a_nI_3(\delta).$$

It is easily seen that if $\{K_\delta\}_{\delta>0}$ is a family of nonnegative measurable functions on $\mathbb{R}^n$ with the properties

(i) $\int_{\mathbb{R}^n} K_\delta(t)\,dt = 1$ for each δ and

(ii) $\int_{|t|>r} K_\delta(t)\,dt \to 0$ as $\delta\to 0$ for each $r>0$,

then for every continuous function f on $\mathbb{R}_+^{n+1}$ with compact support,

$$\int_{\mathbb{R}^n} f(t,\delta)K_\delta(t)\,dt \to f(0,0) \quad \text{as } \delta\to 0.$$

Applying this argument to the above three integrals, we readily see that $I_k(\delta)\to \frac{\partial}{\partial y}v(0,0)$ as $\delta\to 0$, $k=1,2,3$, which leads immediately to the required conclusion. □

5.2. *Proof of Theorem 1*

By definition of M, for given f and $\varepsilon>0$ we choose $\delta_0\in(0,1/2]$ so that

$$|\Delta_h^2 f(x)| \le (M+\varepsilon)\omega(|h|) \quad \text{for } x,h\in\mathbb{R}^n \text{ with } d(x,E)\le\delta_0,\ |h|\le\delta_0. \tag{5.4}$$

For the function u provided by Lemma 1, we have $\operatorname{supp} u\subset \tilde{B}(r)$ with $r\le C$. Indeed, we may assume that $B(r)\supset E_{\delta_0}$.

Let $\tilde{Q}$ be a cube in $\mathbb{R}^n$ containing $\tilde{B}(2r)$. For $\delta\in(0,\delta_0/20)$, define the space $\mathcal{F}_\delta$ of all harmonic functions F in $\tilde{Q}\setminus E_\delta$ with the finite norm

$$\|F\|_{\mathcal{F}_\delta} := \int_{\tilde{Q}\setminus E_\delta}|F(z)|\frac{\omega(d(z,E_\delta))}{d(z,E_\delta)}\,dz.$$

Clearly, $\mathcal{F}_\delta$ is a linear subspace in $L^1(\tilde{Q}\setminus E_\delta,\mu)$ with

$$d\mu := \frac{\omega(d(z,E_\delta))}{d(z,E_\delta)}\,dz.$$

The mapping

$$\Phi\colon F\mapsto \int_{\tilde{Q}\setminus E_\delta} F(z)\Delta u(z)\,dz, \qquad F\in\mathcal{F}_\delta,$$

defines a linear functional Φ on $\mathcal{F}_\delta$. In view of property (iii) from Lemma 1 and according to Lemma 3 with $G=E_\delta$, its norm $\|\Phi\|_\delta$ is finite and depends on the global estimate of Δu. A more detailed analysis shows, however, that the norm of the functional Φ is bounded by a constant depending on the estimate of Δu in E_{δ_0}

only (i.e., near the essential part of the boundary of the harmonicity domain for functions in $\mathcal{F}_\delta$) whereas the latter estimate depends in turn on (5.4) (see claim (i) in Lemma 2). This fact is a matter of the following assertion, which is one of the driving forces of the proof.

LEMMA 6. *There exists a $\delta \in (0, \delta_0/20)$ such that $\|\Phi\|_\delta \le C(M+\varepsilon)$.*

The proof of Lemma 6 is given in Section 5.3, so we continue with the proof of Theorem 1.

We fix δ from Lemma 6 and denote hereafter $G := E_\delta$, $d(z) := d(z, G)$. Using the Hahn–Banach theorem, we extend Φ to a linear functional on $L^1(\tilde{Q} \setminus G, \mu)$ with the same norm and thus get a function $\psi \in L^\infty(\tilde{Q} \setminus G, \mu)$ satisfying

$$\int_{\tilde{Q}\setminus G} F(z)\Delta u(z)\,dz = \int_{\tilde{Q}\setminus G} \psi(z)F(z)\frac{\omega(d(z))}{d(z)}\,dz, \quad F \in \mathcal{F}_\delta, \tag{5.5}$$

and, owing to Lemma 6, such that $A := \|\psi\|_\infty \le C(M+\varepsilon)$.

For $t \in \mathbb{R}^n$, we define the function $F_t(z) := -P(x-t, y)/2$, plug it into both sides of (5.5), and study the two functions of t thus obtained. First, due to Lemma 5,

$$\int_{\tilde{Q}\setminus G} F_t(z)\Delta u(z)\,dz = f(t), \quad t \in \mathbb{R}^n. \tag{5.6}$$

Further, let

$$g_0(t) := \int_{\mathbb{R}^{n+1}\setminus G} \psi(z)F_t(z)\frac{\omega(d(z))}{d(z)}\,dz, \quad t \in \mathbb{R}^n. \tag{5.7}$$

In the simplest case $\omega(s) = s$, we finish up the proof in the following way. For $t \in G$, the function $F_t(z)$ is harmonic in $\mathbb{R}^{n+1} \setminus G$, and

$$F_t \in \mathcal{F}_\delta, \quad t \in G. \tag{5.8}$$

Then, combining (5.5), (5.6), and (5.7), we see that g_0 coincides with f on G. Also, by Lemma 4, $\|g_0\|_{\Lambda^\omega} \le CA$. Thus, in the case under study, g_0 is the function desired.

Now pass to general majorants ω. As we shall see later, (5.8) remains true; hence, $g_0|_G = f|_G$. However, g_0 does not necessarily belong to Λ^ω! To overcome this difficulty, we need an additional technical step. First, we extract a "good" part, g_1, of function g_0 with

$$\|g_1\|_{\Lambda^\omega} \le CA. \tag{5.9}$$

Then we subject the remaining "bad" part, $g_2 := g_0 - g_1$, to a special nonhomogeneous averaging and thus obtain function $\tilde{g}_2$ such that $\tilde{g}_2 = g_2$ on G, and

$$\|\tilde{g}_2\|_{\Lambda^\omega} \le CA. \tag{5.10}$$

Setting finally $g := g_1 + \tilde{g}_2$, we see that $\|g\|_{\Lambda^\omega} \le CA \le C(M+\varepsilon)$ and $g|_G = g_0|_G = f|_G$. Therefore, g is the function required in Theorem 1.

To complete the proof, we must establish the relations (5.8)–(5.10). Observe that if $d(x) \le |y|$ (in particular, if $x \in G$) then $|y| \le d(z) \le \sqrt{2}|y|$. Hence we may apply Lemma 4 to the function

$$g_1(t) := \int_{\mathbb{R}^{n+1}\setminus G} \psi(z)\chi_{\{d(x)\le|y|\}} F_t(z)\frac{\omega(d(z))}{d(z)}\,dz$$

and obtain the estimate (5.9).

Now turn to the function $g_2 = g_0 - g_1$, which has the form

$$g_2(t) = \int_{\mathbb{R}^n\setminus G}\int_{|y|<d(x)} \tau(x,y)\frac{\omega(d(x))}{d(x)}P(x-t,y)\,dy\,dx, \quad t\in\mathbb{R}^n,$$

where $\operatorname{supp}\tau \subset \tilde{Q}$ and $\|\tau\|_\infty \le CA$. We claim that, for every such τ,

$$\|g_2\|_\infty \le CA \tag{5.11}$$

and

$$|\Delta_h^2 g_2(t)| \le CA\omega(\max\{|h|, d(t)\}). \tag{5.12}$$

We start the proof of our claim by observing that, for any point $a\in G$ and for each $r>0$,

$$\left|\int_{B(a,r)\setminus G}\int_{|y|<d(x)} \tau(x,y)\frac{\omega(d(x))}{d(x)}P(x-s,y)\,dy\,dx\right| \le CA\omega(r), \quad s\in B(a,r). \tag{5.13}$$

To prove (5.13) we consider the following two cases, in which the corresponding integrals are denoted by I_1 and I_2.

(1) $|x-s|\le d(s)/2$. In this case, $\frac{1}{2}d(s)\le d(x)\le \frac{3}{2}d(s)$ and hence, by (2.5),

$$|I_1| \le CA\frac{\omega(d(s))}{d(s)}\int_0^{2d(s)}\int_{\mathbb{R}^n} P(x-s,y)\,dx\,dy \le CA\omega(d(s)) \le CA\omega(r).$$

(2) $|x-s| > d(s)/2$, in which case $d(x) < 3|x-s|$. Therefore, using the estimate $P(t,y)\le C/|t|^n$ and (R1), we have

$$\begin{aligned}|I_2| &\le CA\int_{B(a,r)\setminus G}\frac{\omega(d(x))}{|x-s|^n}\,dx \le CA\int_{B(s,2r)}\frac{\omega(|x-s|)}{|x-s|^n}\,dx\\ &\le CA\int_0^{2r}\frac{\omega(v)}{v}\,dv \le CA\omega(r).\end{aligned}$$

Relation (5.11) is now derived directly from (5.13). Together with $\|g_1\|_\infty \le CA$ (see (5.9)), this yields $\|g_0\|_\infty \le CA$. Indeed, the latter estimate holds true for every function $\psi\in L^\infty(\mathbb{R}^{n+1})$ with $\operatorname{supp}\psi\subset\tilde{Q}$ involved in (5.7) (recall that $A=\|\psi\|_\infty$). By setting $\psi(z) = -\operatorname{sign} y\chi_{\tilde{Q}} = \operatorname{sign} F_t(z)\chi_{\tilde{Q}}$ in (5.7), we obtain (5.8).

To verify (5.12), denote a point in G closest to t by a and set $\rho := \max\{|h|, d(t)\}$. We invoke (5.13) with $s=t,\ t\pm h$, and $r=3\rho$ to get

$$\left|\int_{B(a,3\rho)\setminus G}\int_{|y|\le d(x)} \tau(x,y)\frac{\omega(d(x))}{d(x)}\Delta_h^2 P(x-t,y)\,dy\,dx\right| \le CA\omega(\rho).$$

For $|x-a|\ge 3\rho$ we have $|x-t|\ge 2|h|$. Note also that if $|s|\ge 2|h|$ then by (5.2) $|\Delta_h^2 P(s,y)| \le C|h|^2 s^{-(n+2)}$. Using (R2), we estimate the remaining part of the integral,

$$\left|\int_{\mathbb{R}^n\setminus(B(a,3\rho)\cup G)}\int_{|y|<d(x)}\tau(x,y)\frac{\omega(d(x))}{d(x)}\Delta_h^2 P(x-t,y)\,dy\,dx\right|$$
$$\leq CA|h|^2\int_{\tilde{Q}\setminus B(a,3\rho)}\frac{\omega(d(x))}{|x-t|^{n+2}}\,dx \leq CA|h|^2\int_{\tilde{Q}\setminus B(a,2\rho)}\frac{\omega(|x-t|)}{|x-t|^{n+2}}\,dx$$
$$\leq CA|h|^2\int_{2\rho}^{C}\frac{\omega(v)}{v^3}\,dv \leq CA|h|^2\frac{\omega(\rho)}{\rho^2} \leq CA\omega(\rho),$$

and (5.12) follows.

Let ϕ_0 be a C^∞-function on $\mathbb{R}$ such that $\operatorname{supp}\phi_0 \subset [-1,1]$, $\int_{\mathbb{R}}\phi_0(s)\,ds = 1$, and $\int_{\mathbb{R}}\phi_0(s)s\,ds = 0$. We set $\phi(v) := \prod_{i=1}^n \phi_0(v_i)$, $v \in \mathbb{R}^n$. Obviously, $\int_{\mathbb{R}^n}\phi(v)\,dv = 1$ and $\int_{\mathbb{R}^n} v_i\phi(v)\,dv = 0$, $i = 1,\dots,n$. Now define

$$\tilde{g}_2(t) := \int_{\mathbb{R}^n} g_2(t+\rho(t)v)\phi(v)\,dv, \quad t \in \mathbb{R}^n$$

(we recall that $\rho(t)$ is the regularized distance from a point $t \in \mathbb{R}^n$ to the set G). Indeed, $\tilde{g}_2(t) = g_2(t)$ for all $t \in G$.

Relation (5.11) implies

$$\|\tilde{g}_2\|_\infty \leq CA. \tag{5.14}$$

To estimate the second difference $\Delta_h^2\tilde{g}_2(t)$, consider the following two cases.

(1) $d(t) \leq 2|h|$. Set $K_1 := Q(t, 7|h|)$ and let $P \in \mathcal{P}_1$ be the polynomial of best uniform approximation to $\tilde{g}_2$ on K_1. It follows from (2.1) and (5.12) that $\|g_2 - P_1\|_{K_1} \leq CA\omega(|h|)$. For $s = t$, $t \pm h$, we derive from (2.2) the inclusion $s + \rho(s)Q_0 \subset K_1$. Hence, from the identity

$$\tilde{g}_2(s) = P_1(s) + \int_{\mathbb{R}^n}[g_2(s+\rho(s)v) - P_1(s+\rho(s)v)]\phi(v)\,dv$$

we conclude that $|\Delta_h^2\tilde{g}_2(t)| \leq CA\|g_2 - P_1\|_{K_1} \leq CA\omega(|h|)$.

(2) $d(t) > 2|h|$. Let $P_2 \in \mathcal{P}_1$ be the polynomial of best uniform approximation to g_2 on $K_2 := Q(t, 4d(t))$. Clearly, if $s = t$, $t \pm h$ then $s + \rho(s)Q_0 \subset K_2$. Using the identity

$$\tilde{g}_2(s) = P_2(s) + \int_{\mathbb{R}^n}[g_2(v) - P_2(v)]\phi\left(\frac{v-s}{\rho(s)}\right)\rho^{-n}(s)\,dv,$$

we find that

$$|\Delta_h^2\tilde{g}_2(t)|$$
$$\leq CA\|g_2 - P_2\|_{K_2}|h|^2 d^n(t) \sup_{s\in[t-h,t+h]}\ \sup_{v\in s+\rho(s)Q_0}\left|\nabla_s\left[\phi\left(\frac{v-s}{\rho(s)}\right)\rho^{-n}(s)\right]\right|.$$

An easy computation shows the inner supremum does not exceed $C\rho^{-(n+2)}(s)$. Hence, by (2.1), (2.2), and (5.12), we obtain, upon recalling that the function $\omega(r)/r^2$ is nonincreasing, the following estimate:

$$|\Delta_h^2\tilde{g}_2(t)| \leq CA\omega(d(t))\frac{|h|^2}{d^2(t)} \leq CA\omega(|h|).$$

Thus, $|\tilde{g}_2|_{\Lambda^\omega} \le CA$, which together with (5.14) yields (5.10). The proof of Theorem 1 is now complete. □

5.3. Proof of Lemma 6

Denote by λ a C^∞-function on $\mathbb{R}_+$ such that $\lambda(t) = 1$ for $0 \le t \le 1$ and $\lambda(t) = 0$ for $t \ge 2$. Let $\rho(z) := \rho(z, E)$ be the regularized distance from a point $z \in \mathbb{R}^{n+1}$ to the set E, and let $\tilde{E}_\alpha := \{ z \in \mathbb{R}^{n+1} : \rho(z) \le \alpha \}$, $\alpha > 0$. Given $\delta \in (0, \delta_0/20)$, for $\beta \in (4\delta, \delta_0/5)$ we set

$$v_\beta(z) := u(z)\lambda\left(\frac{\rho(z)}{\beta}\right), \quad w_\beta := u - v_\beta, \quad z \in \mathbb{R}^{n+1}.$$

Observe that functions $v_\beta, w_\beta \in C^1(\mathbb{R}^{n+1}) \cap C^\infty(\mathbb{R}^{n+1} \setminus \mathbb{R}^n)$ and have compact support. Moreover, $v_\beta = u$ on $\tilde{E}_\beta$, and v_β vanishes outside $\tilde{E}_{2\beta}$.

We claim that

$$\int_{\tilde{Q}\setminus E_\delta} F(z)\Delta w_\beta(z)\,dz = 0, \quad F \in \mathcal{F}_\delta. \tag{5.15}$$

(The left-hand side of (5.15) is thought of as an improper integral, but—as we shall see later—this integral converges absolutely.) In fact, let $\tilde{Q}_\sigma := \{ z = (x, y) \in \tilde{Q} : |y| \ge \sigma \}$. Integrating by parts and using the harmonicity of F, for every $\sigma > 0$ we have

$$\int_{\tilde{Q}_\sigma} F(z)\Delta w_\beta(z)\,dz = \int_Q [\theta(x, \sigma) - \theta(x, -\sigma)]\,dx, \tag{5.16}$$

where

$$\theta(x, \sigma) := F(x, \sigma)\frac{\partial w_\beta}{\partial y}(x, \sigma) - \frac{\partial F}{\partial y}(x, \sigma)w_\beta(x, \sigma).$$

Since w_β vanishes in $\tilde{E}_{2\delta}$, θ is a continuous function on $Q \times \mathbb{R}_+$ with compact support. Hence, passing in (5.16) to the limit as $\sigma \to 0$, we see that the improper integral in (5.15) exists and equals zero.

Equality (5.15) implies

$$\Phi(F) = \int_{\tilde{Q}\setminus E_\delta} F(z)\Delta u(z)\,dz = \int_{\tilde{E}_{2\beta}\setminus E_\delta} F(z)\Delta v_\beta(z)\,dz, \quad F \in \mathcal{F}_\delta. \tag{5.17}$$

We will now estimate the last integral. Toward this end, write

$$\begin{aligned}\Delta v = \Delta(u\lambda(\rho/\beta)) &= \Delta u\lambda(\rho/\beta) + 2\beta^{-1}\lambda'(\rho/\beta)\nabla u\nabla\rho \\ &\quad + u\,[\beta^{-2}\lambda''(\rho/\beta)(\nabla\rho)^2 + \beta^{-1}\lambda'(\rho/\beta)\Delta\rho].\end{aligned}$$

Note that if $\rho(z) \le 2\beta$ then, by virtue of (2.2), $d(z) := d(z, E) \le 5\beta \le \delta_0$. Using Lemma 2, relation (5.4), and estimate (2.3), for $\rho(z) \le 2\beta$ we have

$$\begin{aligned}|\Delta v_\beta| &\le C(M + \varepsilon)\frac{\omega(y)}{|y|} \\ &\quad + C\|f\|_{\Lambda^\omega}\left[\beta^{-1}d(z)\int_{d(z)}^1 \frac{\omega(t)}{t^2}\,dt + \beta^{-2}d^2(z)\int_{d(z)}^1 \frac{\omega(t)}{t^2}\,dt\right]\chi_{\tilde{E}_{2\beta}\setminus\tilde{E}_\beta}(z).\end{aligned}$$

Invoking Lemma 3 and property (2.5) of the majorant ω, we use (5.17) to obtain

$$|\Phi(F)| \le C(M+\varepsilon)\|F\|_{\mathcal{F}_\delta} + C\|f\|_{\Lambda^\omega} \int_\beta^1 \frac{\omega(t)}{t^2}\,dt \cdot \int_{\tilde{E}_{2\beta}\setminus\tilde{E}_\beta} |F(z)|\,dz. \tag{5.18}$$

Now our claim is as follows.

CLAIM. *There exists a $\delta \in (0, \delta_0/20)$ such that, for every function $F \in \mathcal{F}_\delta$, one can find a $\beta \in (4\delta, \delta_0/5)$ for which*

$$C\|f\|_{\Lambda^\omega} \int_\beta^1 \frac{\omega(t)}{t^2}\,dt \cdot \int_{\tilde{E}_{2\beta}\setminus\tilde{E}_\beta} |F(z)|\,dz \le \varepsilon\|F\|_{\mathcal{F}_\delta}. \tag{5.19}$$

To show this, suppose the claim is false. Then, for every $\delta \in (0, \delta_0/20)$, there is a function $F_\delta \in \mathcal{F}_\delta$ with $\|F_\delta\|_{\mathcal{F}_\delta} = 1$ such that

$$\int_{\tilde{E}_{2\beta}\setminus\tilde{E}_\beta} |F(z)|\,dz \ge \gamma\left[\int_\beta^1 \frac{\omega(t)}{t^2}\,dt\right]^{-1}, \quad 4\delta \le \beta \le \delta_0/5,$$

where $\gamma := \varepsilon/(C\|f\|_{\Lambda^\omega})$. Multiplying both sides of this inequality by $\omega(\beta)/\beta^2$, integrating, and recalling that $\int_0^1 (\omega(t)/t^2)\,dt = +\infty$, we obtain

$$\begin{aligned} &\int_{4\delta}^{\delta_0/5} \frac{\omega(\beta)}{\beta^2} \int_{\tilde{E}_{2\beta}\setminus\tilde{E}_\beta} |F_\delta(z)|\,dz\,d\beta \\ &\quad\ge \gamma \int_{4\delta}^{\delta_0/5} \frac{\omega(\beta)}{[\beta^2 \int_\beta^1 (\omega(t)/t^2)\,dt]}\,d\beta \\ &\quad= \gamma \log\left[\int_{4\delta}^1 \frac{\omega(t)}{t^2}\,dt \Big/ \int_{\delta_0/5}^1 \frac{\omega(t)}{t^2}\,dt\right] \to +\infty \quad \text{as } \delta \to 0. \end{aligned} \tag{5.20}$$

On the other hand,

$$\begin{aligned} \int_{4\delta}^{\delta_0/5} \frac{\omega(\beta)}{\beta^2} \int_{\tilde{E}_{2\beta}\setminus\tilde{E}_\beta} |F_\delta(z)|\,dz\,d\beta &= \int_{\tilde{E}_{2\delta_0/5}\setminus\tilde{E}_{4\delta}} |F_\delta(z)| \int_{\max\{\rho(z)/2,4\delta\}}^{\min\{\rho(z),\delta_0/5\}} \frac{\omega(\beta)}{\beta^2}\,d\beta\,dz \\ &\le \int_{\tilde{Q}\setminus\tilde{E}_{4\delta}} |F_\delta(z)| \int_{\rho(z)/2}^{\rho(z)} \frac{\omega(\beta)}{\beta^2}\,d\beta\,dz. \end{aligned}$$

Observe that if $\rho(z) \ge 4\delta$ then, in view of (2.2), $d(z) \ge 2\delta$, which implies $d(z, E_\delta) \ge \delta$. Therefore, in the last integral, $\rho(z) \le 2d(z) \le 2[d(z, E_\delta) + \delta] \le 4d(z, E_\delta)$. Also, again by (2.2), $\rho(z) \ge \frac{2}{5}d(z) \ge \frac{2}{5}d(z, E_\delta)$. Now we are in a position to continue the above estimates:

$$\begin{aligned} &\int_{4\delta}^{\delta_0/5} \frac{\omega(\beta)}{\beta^2} \int_{\tilde{E}_{2\beta}\setminus\tilde{E}_\beta} |F_\delta(z)|\,dz\,d\beta \\ &\quad\le \int_{\tilde{Q}\setminus\tilde{E}_{4\delta}} |F_\delta(z)| \int_{d(z,E_\delta)/5}^{4d(z,E_\delta)} \frac{\omega(\beta)}{\beta^2}\,d\beta\,dz \\ &\quad\le C\int_{\tilde{Q}\setminus E_\delta} |F_\delta(z)| \frac{\omega(d(z, E_\delta))}{d(z, E_\delta)}\,dz = C\|F_\delta\|_{\mathcal{F}_\delta} = C < +\infty. \end{aligned} \tag{5.21}$$

Comparison of the relations (5.20) and (5.21) leads to contradiction, and claim (5.19) follows.

We finish our argument as follows. Fix $\delta \in (0, \delta_0/20)$ satisfying the above claim. For any function $F \in \mathcal{F}_\delta$, we pick an appropriate $\beta \in (4\delta, \delta_0/5)$. The above construction carried out with this particular β yields (5.19), which in combination with (5.18) leads us finally to the estimate

$$|\Phi(F)| \leq C(M + \varepsilon)\|F\|_{\mathcal{F}_\delta}, \quad F \in \mathcal{F}_\delta.$$

Lemma 6 is proved. □

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