An Inverse Function Theorem for Fréchet Spaces Admitting Generalized Smoothing Operators

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Introduction

An inverse function theorem of the Nash–Moser type is proved for Fréchet spaces admitting generalized smoothing operators; the proof is based on Newton's method. In particular, for Köthe sequence spaces, property (Ω) in the standard form and the topological condition (DN) in the sense of Vogt are shown to be sufficient for the Nash–Moser theorem to hold under classical assumptions on the mappings.

In the literature, inverse function theorems of so-called Nash–Moser type with "loss of derivatives" are proved for Fréchet spaces that admit smoothing operators as introduced by Nash [8]; a possible proof relies on Newton's method as suggested by Moser in [7] (see e.g. [2; 3; 5; 11; 12; 13] or [1; 6] for generalized results). For instance, Lojasiewicz and Zehnder [5] prove such a theorem showing that Newton's method still converges if the classical "tame" assumptions on the mappings (cf. [2]) are replaced by "linear-tame estimates with $1 \leq \lambda < 2$" while the theorem fails if $\lambda = 2$ (cf. [5]). This paper contains a generalization of [5]; the aim is to find out under which more general conditions on the Fréchet space Newton's method converges. The hypothesis of smoothing operators is replaced by the weaker assumption of the existence of generalized smoothing operators, and the (linear-) tame estimates supposed in [5] are replaced by more general estimates. It is then considered as a property of the Fréchet space under which assumptions on the mappings the inverse function theorem holds. This property of the Fréchet space is quantitatively measured by means of the existence of suitable generalized smoothing operators.

The first section contains preliminaries. Section 2 treats the standardized case, "loss of derivatives = 1"; for this situation, a generalization of the result in [5] is proved. It is carefully checked which property of the Fréchet space is needed to compensate this loss of derivatives in order to make Newton's method converge. In [5], the existence of classical smoothing operators and hence property (DN) in standard form are assumed; here only the

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weaker property ($\underline{\text{DN}}$) and certain generalized smoothing operators are required. In the case "loss of derivatives = 0" it is even sufficient to suppose only property ($\underline{\text{DN}}$). Section 3 shows that much more general problems—for instance, the full result in [5]—can be reduced to the above standardized situation by means of a formal transformation of the fundamental systems of seminorms. An inverse function theorem is obtained where the assumptions on the mappings are coupled with a property of the Fréchet space formulated by means of conditions of type ($S_{(a,d)}$) (cf. Definition 3.1) on the existence of generalized smoothing operators. In Section 4, the previous results are evaluated for Köthe sequence spaces. Sufficient conditions for the existence of the above generalized smoothing operators are given in terms of the quantitative variants ($DN_{(\phi,\psi)}$) and ($\Omega_{(\tau,\sigma)}$) (cf. [9]) of the topological properties (DN) of Vogt [14] and (Ω) of Vogt and Wagner [17]. In particular, it is shown that the Nash–Moser theorem holds under classical assumptions on the mappings for each Köthe sequence space that is an (Ω) space in standard form and satisfies the topological condition (DN). It seems to be remarkable that it is enough to suppose property (DN) in its topological form and that it is not necessary to assume a tamely invariant version of property (DN) in standard form, which might be suggested by the negative example in [5].

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1. Preliminaries

We use common notation on Fréchet spaces (cf. [4]). A Fréchet space E equipped with a fixed fundamental system $|\cdot|_0 \le |\cdot|_1 \le |\cdot|_2 \le \cdots$ of continuous seminorms defining the topology is called a graded Fréchet space (cf. [2]). The sequence of seminorms is called grading (cf. [2]); sometimes we shall also consider gradings $(|\cdot|_t)_{t\in J}$ for some set $J \subset \mathbb{R}$, with $|\cdot|_s \le |\cdot|_t$ if $s \le t$. A linear map $A: E \to F$ between graded Fréchet spaces is called (ϕ)-tame for a map $\phi: \mathbb{N}_0 \to \mathbb{N}_0$ if there exist k_0 and constants $c_k > 0$ such that $\|Ax\|_k \le c_k \|x\|_{\phi(k)}$ for all $k \ge k_0$ and $x \in E$; the map A is called linear-tame if $\phi(k) \le ak + b$ for all k and suitable fixed a, b. A is called tame if $a = 1$ and is called normwisely tame if $a = 1$ and $b = 0$ (cf. [2]).

For $\phi, \psi: \mathbb{N}_0 \to \mathbb{N}_0$, the space E is called a (ϕ, ψ)-tame direct summand of F if there exist a (ϕ)-tame linear map $A: E \to F$ and a (ψ)-tame linear map $B: F \to E$ such that $B \circ A = \mathrm{id}_E$; if, in addition, $A \circ B = \mathrm{id}_F$ then we say that $E \cong F$ is (ϕ, ψ)-tamely isomorphic, and A is called a (ϕ, ψ)-tame isomorphism. If $\phi(k) \le ak + b$ and $\psi(k) \le ak + b$ for all k and fixed a, b, then E is called a linear-tame direct summand of F (and, if $E \cong F$ is (ϕ, ψ)-tamely isomorphic, E and F are said to be linear-tamely isomorphic); the same notation is used with tame and normwisely tame in place of linear-tame.

Köthe sequence spaces are graded as follows: Let $a = (a_{j,k})_{j=1,k=0}^{\infty}$ be a matrix such that $0 \le a_{j,k} \le a_{j,k+1}$ for all j, k and $\sup_k a_{j,k} > 0$ for all j. For $1 \le q < \infty$ we put

$$\lambda^q(a) = \{x = (x_j)_{j=1}^{\infty} \subset \mathbb{K} : \|x\|_k = (\Sigma_j |x_j|^q a_{j,k}^q)^{1/q} < +\infty \text{ for all } k\}$$

(where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). $\lambda^{\infty}(a)$ is analogously defined with $\|x\|_k = \sup_j |x_j| a_{j,k}$. For $0 \le \alpha_1 \le \alpha_2 \le \cdots \nearrow +\infty$ and $r_0 < r_1 < r_2 < \cdots \nearrow R \in [0, \infty]$, we consider the power series space $\Lambda_R^q(\alpha) = \lambda^q(a)$ with $a_{j,k} = e^{r_k \alpha_j}$ (of finite type if $R < \infty$ or of infinite type if $R = \infty$, respectively).

We shall make use of the topological invariants (DN) (cf. Vogt [14]), (Ω) (cf. Vogt and Wagner [17]) and ($\underline{\text{DN}}$) (cf. [15]). For a seminorm $|\cdot|$ on E the extended real-valued dual norm is defined by $|\phi|^* := \sup\{|\phi(x)| : |x| \le 1\} \in [0, \infty]$, $\phi \in E'$.

Definition 1.1 (cf. [14; 15; 17]). Let E be a graded Fréchet space.

(i) E has property (DN) if there is a p such that for all k there are K and $c > 0$ with
$$|\cdot|_k^2 \le c|\cdot|_p|\cdot|_K.$$

(ii) E has property ($\underline{\text{DN}}$) if there is a p such that for all k there are K and $0 < \tau < 1$, $c > 0$, with
$$|\cdot|_k \le c|\cdot|_p^{1-\tau}|\cdot|^{\tau}_K.$$

(iii) E has property (Ω) if for all p there is a q such that for all Q there are $0 < \theta < 1$, $c > 0$, with
$$|\cdot|_q^* \le c|\cdot|_p^{*1-\theta}|\cdot|_Q^{*\theta}.$$

Properties (DN), ($\underline{\text{DN}}$), and (Ω) are topological invariants and independent of the chosen grading; (DN) and ($\underline{\text{DN}}$) are inherited by subspaces; and (Ω) is inherited by quotient spaces. The Fréchet space E has property (DN) (resp. (Ω)) if and only if there exist a grading $|\cdot|_k$ and constants $c_k > 0$ such that $|\cdot|_k^2 \le c_k|\cdot|_{k-1}|\cdot|_{k+1}$ for all k (resp. such that $|\cdot|_k^{*2} \le c_k|\cdot|_{k-1}^*|\cdot|_{k+1}^*$ for all k) (cf. [16]); $(E, |\cdot|_k)$ is then called a (DN)-space (resp. an (Ω)-space) in standard form. We shall further employ the following quantitative variants of properties (DN) and (Ω), which are defined in [9]. Let

$$\mathfrak{F} = \{\phi : \mathbb{N}_0 \to \mathbb{N}_0 : \phi(n+1) \ge \phi(n) + 1 \text{ for all } n\}.$$

Definition 1.2 (cf. [9]). Let E be a graded Fréchet space and $\phi, \psi \in \mathfrak{F}$.

(i) E has property $(\text{DN}_{(\phi,\psi)})$ if there exist b_0 and constants $c_m > 0$ such that
$$|\cdot|_n^{m-l} \le c_m|\cdot|_{\phi(l)}^{m-\psi(n)}|\cdot|_{\phi(m)}^{\psi(n)-l} \quad \text{for all } b_0 \le l < \psi(n) < m.$$

(ii) E has property $(\Omega_{(\phi,\psi)})$ if there exist b_0 and constants $c_m > 0$ such that
$$|\cdot|_{\phi(n)}^{*\psi(m)-\psi(l)} \le c_m|\cdot|_l^{*\psi(m)-n}|\cdot|_m^{*n-\psi(l)} \quad \text{for all } b_0 \le \psi(l) < n < \psi(m).$$

(iii) E has property (DN_{t}) (resp. (Ω_{t})) if there exist $\phi, \psi \in \mathfrak{F}$ such that $\phi(k), \psi(k) \le k + b$ for all k and some fixed b and $E \in (\text{DN}_{(\phi,\psi)})$ (resp. $E \in (\Omega_{(\phi,\psi)})$). If $\phi(k), \psi(k) \le ak + b$ for all k and fixed a, b then we analogously write $E \in (\text{DN}_{\text{l}})$ (resp. $E \in (\Omega_{\text{l}})$).

REMARKS 1.3 (cf. [9]). Let E, F be graded Fréchet spaces and $\phi, \psi, \sigma, \tau \in \mathfrak{F}$.

(i) Let E be a (σ,τ)-tame direct summand of F. If F is a (DN)-space in standard form then $E \in (\mathrm{DN}_{(\sigma,\tau)})$; if $F \in (\mathrm{DN}_{(\phi,\psi)})$ then $E \in (\mathrm{DN}_{(\sigma\circ\phi,\psi\circ\tau)})$. If F is an (Ω)-space in standard form then $E \in (\Omega_{(\sigma,\tau)})$; if $F \in (\Omega_{(\phi,\psi)})$ then $E \in (\Omega_{(\sigma\circ\phi,\psi\circ\tau)})$.

(ii) E has property (DN) (resp. (Ω)) if and only if there exist $\phi, \psi \in \mathfrak{F}$ such that $E \in (\mathrm{DN}_{(\phi,\psi)})$ (resp. $E \in (\Omega_{(\phi,\psi)})$).

(iii) Properties $(\mathrm{DN}_{\mathrm{t}})$ and (Ω_{t}) are invariant with respect to tame isomorphisms; properties $(\mathrm{DN}_{\mathrm{l}})$ and (Ω_{l}) are invariant with respect to linear-tame isomorphisms.

(iv) The Köthe space $E = \lambda^q(a)$ has property $(\mathrm{DN}_{(\phi,\psi)})$ (resp. $(\Omega_{(\phi,\psi)})$) if and only if there exists a Köthe space $\lambda^q(b)$ that is a (DN) (resp. (Ω)) space in standard form such that $\lambda^q(a) \cong \lambda^q(b)$ is (ϕ,ψ)-tamely isomorphic; this is proved in [9, 3.8].

2. A Standardized Inverse Function Theorem

In this section an inverse function theorem is proved under the standardized assumption "loss of derivatives = 1". This generalizes the result of Lojasiewicz and Zehnder [5]; the proof is based on Newton's method and on the technique of [5], and the full result proved in [5] for power series spaces follows from Theorem 2.2 by means of a simple formal reduction (cf. Corollary 4.9). In [5] the Fréchet space is presumed to admit smoothing operators and hence is a (DN) space in the standard form; in this section these assumptions are weakened and replaced by the existence of generalized smoothing operators in the form of condition $(\bar{\mathrm{S}}_{(1,1)})$ and property $(\underline{\mathrm{DN}})$.

DEFINITION 2.1. The graded Fréchet space E has property $(\bar{\mathrm{S}}_{(1,1)})$ if the following holds: There exist $\rho, \mu : \mathbb{N} \to]0,\infty[$, a number $\alpha > 1$, and constants $b \geq 1$ and $c_n > 0$ such that, for each $\theta \geq 1$, there is a (not necessarily linear) map $S_\theta : E \to E$ such that:

(a) $|S_\theta x|_n \leq c_n \theta^{\rho(n)} |x|_{n-1}$, $n \geq 1$, $x \in E$;

(b) $|x - S_\theta x|_1 \leq c_n \theta^{-\mu(n)} |x|_{n-1}$, $n \geq b$, $x \in E$; and

(c) $\sup_n \{\mu(n) - \alpha\rho(n)\} = +\infty$.

In this section we suppose that E and F are graded Fréchet spaces, where E satisfies properties $(\bar{\mathrm{S}}_{(1,1)})$ and $(\underline{\mathrm{DN}})$ with $p = 1$ in Definition 1.1(ii). Hence we assume that there exist numbers $0 < \epsilon_n < 1$, integers $\phi(n)$, and constants $c_n > 0$ such that

(d) $|x|_n \leq c_n |x|_1^{1-\epsilon_n} |x|_{\phi(n)}^{\epsilon_n}$, $n \geq 1$, $x \in E$.

Let $U = \{x \in E : |x|_1 < 1\}$ and let $\Phi : (U \subset E) \to F$ be a continuous (nonlinear) map so that $\Phi(0) = 0$. We assume that for each $x \in U$ the linear map

$$\Phi'(x) : E \to F, \quad \Phi'(x)v = \lim_{t\to 0} \frac{1}{t}(\Phi(x+tv) - \Phi(x))$$

exists, and we assume that for each $x \in U$ there is a map $L(x): F \to E$ such that $\Phi'(x) \circ L(x) = \mathrm{id}_F$. We further suppose that there are constants $c_n > 0$ such that:

(1) $|\Phi(x)|_n \le c_n |x|_n$, $x \in U$;

(2) $|\Phi'(x)v|_n \le c_n(|x|_n |v|_1 + |v|_n)$, $x \in U$, $v \in E$;

(3) $|\Phi(x+v) - \Phi(x) - \Phi'(x)v|_n \le c_n(|x|_n |v|_1^2 + |v|_1 |v|_n)$, $x, x+v \in U$; and

(4) $|L(x)y|_n \le c_n(|x|_{n+1}|y|_1 + |y|_{n+1})$, $x \in U$, $y \in F$.

THEOREM 2.2 (cf. [5]). *If $\Phi: (U \subset E) \to F$ is as above and E satisfies properties* ($\bar{\mathrm{S}}_{(1,1)}$) *and* ($\underline{\mathrm{DN}}$) *in the form of* (d), *then $\Phi(U)$ is a neighborhood of zero in F.*

REMARKS 2.3. (i) More precisely, the following holds in Theorem 2.2: If $\tau = 1 + 1/\alpha$ and $\mu_0 = (2/(2-\tau))\rho(1) + \tau/(2-\tau)$, and if $s_0 \ge b$ is chosen so that $\mu(s_0) \ge \alpha\rho(s_0) + \tau\mu_0 + 1$, then there exist $\delta > 0$ and a mapping ψ: $(V \subset F) \to E$ defined in $V = \{y \in F: |y|_{s_0} \le \delta\}$ such that $\psi(0) = 0$, $\Phi(\psi(y)) = y$, and $|\psi(y)|_1 \le c|y|_{s_0}$ for $y \in V$ and some $c > 0$. Furthermore, there are $\sigma(n)$, $\kappa(n)$ (which can be explicitly calculated from the given data) and constants $c_n > 0$ such that

$$|\psi(y)|_n \le c_n(|y|_{\sigma(n)} + |y|_{\sigma(n)}^{\kappa(n)}).$$

(ii) If also F has property ($\underline{\mathrm{DN}}$) then it is enough to assume that (2) and (3) hold for $n = 1$. If F has property (DN) then we can choose $\kappa(n) = 1$ in (i) (enlarging s_0, $\sigma(n)$).

(iii) If in the situation of Theorem 2.2 the map $\Phi'(x): E \to F$ is bijective, $x \in U$, then: Φ is injective in a (possibly smaller) neighborhood U of zero; the inverse map $\psi: V \to U$ is uniquely defined in a suitable neighborhood V of zero in F and continuous; and the Gâteaux derivative $\psi'(y)$ exists and $\psi'(y) = L(\psi(y))$ for $y \in V$. If $\Phi': U \times E \to F$ is continuous then $\psi': V \times F \to E$ is continuous as well, and Φ is a C^1-diffeomorphism near 0. The proof is standard (cf. [2]).

Proof of Theorem 2.2 (cf. [5]). We want to show that there is a neighborhood V of zero, $V \subset F$, such that for each $y \in V$ and $\theta_j = 2^{\tau^j}$, $\tau = 1 + 1/\alpha$, the iteration

$$x_0 = 0, \quad x_{j+1} = x_j + \Delta x_j, \quad \Delta x_j = S_{\theta_j} L(x_j) z_j, \quad z_j = y - \Phi(x_j)$$

is well-defined such that $x_j \in U$ for all j and $x_j \to x \in U$ with $\Phi(x) = y$. In order to show this we prove several lemmata.

LEMMA 2.4 (cf. [5, Lemma 1]). *Let $L(n) = \alpha\rho(n) + 1$. For every $n \in \mathbb{N}$ there is a constant $K_n > 0$ such that, for all $y \in F$ with $|y|_1 \le 1$, we have*

$$|x_j|_n \le K_n \theta_j^{L(n)} |y|_n \quad \textit{and} \quad |z_j|_n \le K_n \theta_j^{L(n)} |y|_n$$

for all j as long as $|x_j|_1 < 1$.

Proof. Let $|y|_1 \le 1$ and assume that $|x_i|_1 < 1$ for $i = 1, \ldots, j$. From (1) we have

$$|z_j|_n \le |y|_n + |\Phi(x_j)|_n \le |y|_n + c_n |x_j|_n.$$

In particular we have $|z_j|_1 \le 1 + c_1$. From (a) and (4) we obtain

$$|\Delta x_j|_n \le c_n \theta_j^{\rho(n)} |L(x_j) z_j|_{n-1} \le c_n' \theta_j^{\rho(n)} (|x_j|_n + |y|_n)$$

and thus $|x_{j+1}|_n + |y|_n \le (c_n' + 1)\theta_j^{\rho(n)}(|x_j|_n + |y|_n)$. The assertion follows because

$$|x_{j+1}|_n \le (c_n' + 1)^{j+1} 2^{\rho(n)((\tau^{j+1}-1)/(\tau-1))} |y|_n \le K_n \theta_{j+1}^{L(n)} |y|_n. \qquad \square$$

LEMMA 2.5 (cf. [5, Lemma 2]). *For each $\mu > 0$ there exist $s_0 = s_0(\mu)$, $\delta > 0$, and $M > 0$ such that, for all $y \in F$ with $|y|_{s_0} \le \delta$, we have the estimate*

$$|z_j|_1 \le M \theta_j^{-\mu} |y|_{s_0}$$

as long as $|x_j|_1 < 1$.

REMARK. If $\mu \ge (2/(2-\tau))\rho(1)$ then we can choose any $s_0 \ge b$ satisfying $\mu(s_0) \ge L(s_0) + \tau\mu$.

Proof. We choose μ, s_0 as in the remark. The proof is by induction on j. The case $j = 0$ is clear. We assume that the assertion holds for j and that $|x_i|_1 < 1$, $i = 0, \ldots, j+1$. We put $R(x; v) := \Phi(x+v) - \Phi(x) - \Phi'(x)v$ and see that $z_{j+1} = \Phi'(x_j)(I - S_{\theta_j})L(x_j)z_j - R(x_j; \Delta x_j)$. By means of (2), (b), (4), and Lemma 2.4, the first term is estimated by

$$\begin{aligned}|\Phi'(x_j)(I - S_{\theta_j})L(x_j)z_j|_1 &\le 2c_1 |(I - S_{\theta_j})L(x_j)z_j|_1 \le 2c_1 c_s \theta_j^{-\mu(s)} |L(x_j)z_j|_{s-1} \\ &\le c_s' \theta_j^{-\mu(s)} (|x_j|_s + |z_j|_s) \le c_s'' \theta_j^{L(s)-\mu(s)} |y|_s \le c_s'' \theta_{j+1}^{-\mu} |y|_s\end{aligned}$$

if $s \ge b$ and $L(s) - \mu(s) \le -\tau\mu$. From (a) and (4) we further obtain

$$|\Delta x_j|_1 \le c_1 \theta_j^{\rho(1)} |L(x_j) z_j|_0 \le c_1' \theta_j^{\rho(1)} |z_j|_1.$$

By means of (3) and the hypothesis of the induction, the second term is estimated by

$$\begin{aligned}|R(x_j; \Delta x_j)|_1 &\le 2c_1 |\Delta x_j|_1^2 \le c_1'' \theta_j^{2\rho(1)} |z_j|_1^2 \\ &\le c_1'' M^2 \theta_j^{2\rho(1) - 2\mu} |y|_{s_0}^2 \le c_1'' M^2 \theta_{j+1}^{-\mu} |y|_{s_0}^2.\end{aligned}$$

Altogether we get

$$|z_{j+1}|_1 \le C(1 + M^2 |y|_{s_0}) \theta_{j+1}^{-\mu} |y|_{s_0} \le M \theta_{j+1}^{-\mu} |y|_{s_0}$$

if we choose $M = 2C$ and $\delta \le M^{-2}$. This proves the assertion. $\square$

COROLLARY 2.6. *Let $\mu_0 = (2/(2-\tau))\rho(1) + \tau/(2-\tau)$, $\tau = 1 + 1/\alpha$, and choose $s_0 \ge b$ with $\mu(s_0) \ge \alpha\rho(s_0) + \tau\mu_0 + 1$. Then there is a $\delta > 0$ such that $|x_j|_1 \le \frac{1}{2}$ holds for all j if $|y|_{s_0} \le \delta$.*

Proof. If $|x_i|_1 \le \frac{1}{2}$ for $i = 0, \ldots, j$, then from Lemma 2.5 we conclude for $i = 0, \ldots, j$ that

$$|\Delta x_i|_1 \leq c_1' \theta_i^{\rho(1)} |z_i|_1 \leq c_1' M \theta_i^{\rho(1)-\mu_0} |y|_{s_0}$$

if δ and M are chosen as in the lemma and $|y|_{s_0} \leq \delta$. Since $\mu_0 > \rho(1)$, we have

$$|x_{j+1}|_1 \leq \sum_{i=0}^{j} |\Delta x_i|_1 \leq c_1' M \sum_{i=0}^{\infty} \theta_i^{\rho(1)-\mu_0} |y|_{s_0} \leq \frac{1}{2},$$

choosing a smaller δ if necessary. This gives the assertion. □

In the following we choose μ_0, s_0 as in Corollary 2.6; this is possible by means of (c). In contrast to [5] we do not suppose any (DN)-type condition in the following Lemma 2.7 and assume only property (<u>DN</u>) for E in Lemma 2.9 (in place of condition (DN) in the standard form in [5]); moreover, we have no assumptions on F at all.

LEMMA 2.7 (cf. [5, Lemma 3]). *Choose μ_0, s_0, δ as in Corollary 2.6. For each m there exist $c_m > 0$ and $\gamma(m), \nu(m) \geq 1$ such that, for every $y \in F$ with $|y|_{s_0} \leq \delta$ and all j, we have*

$$|z_j|_1 \leq c_m \theta_j^{-m} (|y|_{\gamma(m)} + |y|_{\gamma(m)}^{\nu(m)}).$$

Proof. By Lemma 2.5 the statement holds for $0 \leq m \leq \mu_0$ with $\nu(m) = 1$ and $\gamma(m) = s_0$; the case $j = 0$ is also clear. We assume the statement to hold for some $m \geq \mu_0$. Then

$$|\Phi'(x_j)(I - S_{\theta_j}) L(x_j) z_j|_1 \leq c_s'' \theta_{j+1}^{-(m+1)} |y|_s$$

follows from the proof of Lemma 2.5 if $s \geq b$ satisfies

$$\mu(s) \geq \alpha\rho(s) + (1 + 1/\alpha)(m+1) + 1.$$

Applying the hypothesis of the induction and observing the proof of Lemma 2.5, we further obtain

$$|R(x_j; \Delta x_j)|_1 \leq c_1'' \theta_j^{2\rho(1)} |z_j|_1^2 \leq 4 c_1'' c_m^2 \theta_{j+1}^{-(m+1)} (|y|_{\gamma(m)} + |y|_{\gamma(m)}^{2\nu(m)}),$$

since $2\rho(1) - 2m \leq -\tau(m+1)$ if $m \geq \mu_0$. This proves the statement for $m+1$. □

REMARKS 2.8. (i) In Lemma 2.7 we can choose $\nu(m) = 2^{m-\mu_0}$ if $m \geq \mu_0$ and $\gamma(\mu_0) = s_0$; $\gamma(m+1) = \max\{\gamma(m), s(m+1)\}$, where $s = s(m+1) \geq b$ is chosen so that $\mu(s) \geq \alpha\rho(s) + (1+1/\alpha)(m+1) + 1$.

(ii) Observing that $|z_j|_1^2 \leq c|z_j|_1^{1+\epsilon}$ if $1/\alpha < \epsilon \leq 1$ and enlarging μ_0, s_0, we also can choose $\nu(m) = (1+\epsilon)^{m-\mu_0}$ for $m \geq \mu_0$.

(iii) If F has property (DN) than a choice $\nu(m) = 1$ is possible. For instance, if $|\cdot|_n^2 \leq c_n |\cdot|_{s_0} |\cdot|_{\omega(n)}$ holds in F then we can choose $\gamma(\mu_0) = s_0$ and $\gamma(m+1) = \max\{\omega(\gamma(m)), s(m+1)\}$.

(iv) If $E = F = \Lambda_\infty^q(\alpha)$ and $r_k = k$ then $\rho(s) = 1$ and $\mu(s) = s - 2$. Hence, for $\alpha > 1$ in (i) it suffices to choose $s(m+1) \geq (1+1/\alpha)m + \alpha + 5$. Further, we have $\omega(n) = 2n$ in (iii). By (ii) and since $|\cdot|_n^{1+\epsilon} \leq |\cdot|_0^{\epsilon} |\cdot|_{(1+\epsilon)n}$ we can thus choose $\nu(m) = 1$ and $\gamma(m) = A(1+\epsilon)^m$ for $\epsilon > 1/\alpha$ and a suitable A.

LEMMA 2.9 (cf. [5, Lemma 4]). *Choose μ_0, s_0, δ as in Corollary 2.6. For all n, a there exist $\sigma = \sigma(n,a)$, $\kappa = \kappa(n,a)$, and $c_n > 0$ such that, for all $y \in F$ with $|y|_{s_0} \le \delta$, we have*

$$|\Delta x_j|_n \le c_n \theta_j^{-a}(|y|_\sigma + |y|_\sigma^\kappa),$$
$$|z_j|_n \le c_n \theta_j^{-a}(|y|_\sigma + |y|_\sigma^\kappa).$$

Proof. From Lemmas 2.5 and 2.7 we obtain the estimates

$$|\Delta x_j|_1 \le c_1' \theta_j^{\rho(1)} |z_j|_1 \le c_m' \theta_j^{\rho(1)-m}(|y|_{\gamma(m)} + |y|_{\gamma(m)}^{\nu(m)}).$$

Lemma 2.4 implies that $|\Delta x_j|_s \le |x_{j+1}|_s + |x_j|_s \le c_s \theta_{j+1}^{L(s)} |y|_s$. From (d) we obtain

$$\begin{aligned}
|\Delta x_j|_n &\le c_n |\Delta x_j|_1^{1-\epsilon_n} |\Delta x_j|_{\phi(n)}^{\epsilon_n} \\
&\le c_{n,m} \theta_j^{(1-\epsilon_n)(\rho(1)-m)+\epsilon_n(\alpha\tau\rho(\phi(n))+\tau)} (|y|_{\gamma(m)} + |y|_{\gamma(m)}^{\nu(m)})^{1-\epsilon_n} |y|_{\phi(n)}^{\epsilon_n} \\
&\le c_n' \theta_j^{-a}(|y|_\sigma + |y|_\sigma^\kappa)
\end{aligned}$$

if $m \ge \rho(1) + (\epsilon_n(\alpha\tau\rho(\phi(n)) + \tau) + a)/(1-\epsilon_n)$, $\sigma \ge \max\{\gamma(m), \phi(n)\}$, and $\kappa \ge \nu(m)(1-\epsilon_n) + \epsilon_n$. Next we examine the case $z_{j+1} = \Phi'(x_j)(I - S_{\theta_j})L(x_j)z_j - R(x_j; \Delta x_j)$. First we obtain

$$\begin{aligned}
&|\Phi'(x_j)(I-S_{\theta_j})L(x_j)z_j|_n \\
&\quad\le c_n(|x_j|_n |(I-S_{\theta_j})L(x_j)z_j|_1 + |(I-S_{\theta_j})L(x_j)z_j|_n) \\
&\quad\le c_n'(\theta_j^{L(n)} |y|_n |(I-S_{\theta_j})L(x_j)z_j|_1^{\epsilon_n} \\
&\qquad\quad + |(I-S_{\theta_j})L(x_j)z_j|_{\phi(n)}^{\epsilon_n}) |(I-S_{\theta_j})L(x_j)z_j|_1^{1-\epsilon_n} \\
&\quad\le c_{m,n}(\theta_j^{L(n)+\epsilon_n(L(b)-\mu(b))} |y|_n \\
&\qquad\quad + \theta_j^{\epsilon_n\rho(\phi(n))} |L(x_j)z_j|_{\phi(n)}^{\epsilon_n}) \theta_j^{-(1-\epsilon_n)\mu(m)} |L(x_j)z_j|_{m-1}^{1-\epsilon_n} \\
&\quad\le c_{m,n}'(\theta_j^{L(n)+\epsilon_n(L(b)-\mu(b))} |y|_n \\
&\qquad\quad + \theta_j^{\epsilon_n(\rho(\phi(n))+L(\phi(n)+1))} |y|_{\phi(n)+1}^{\epsilon_n}) \theta_j^{-(1-\epsilon_n)\mu(m)+(1-\epsilon_n)L(m)} |y|_m^{1-\epsilon_n} \\
&\quad\le c_n \theta_{j+1}^{-a}(|y|_\sigma + |y|_\sigma^\kappa)
\end{aligned}$$

if $\sigma \ge \max\{m, \phi(n)+1, n\}$, $\kappa \ge 2 - \epsilon_n$, and m is chosen so large that

$$\mu(m) - \alpha\rho(m) \ge 1 + \frac{\tau a}{1-\epsilon_n} + \frac{1}{1-\epsilon_n} \times \max\{L(n) + \epsilon_n(L(b) - \mu(b)), \epsilon_n(\rho(\phi(n)) + L(\phi(n)+1))\}.$$

Here we have used (2), (4), Lemma 2.4, (d), (a), and (b). Applying (3) together with Lemmas 2.4, 2.5, and 2.7, we conclude that

$$\begin{aligned}
|R(x_j; \Delta x_j)|_n &\le c_n'(\theta_j^{L(n)} |y|_n \theta_j^{2\rho(1)} |z_j|_1^2 + \theta_j^{\rho(1)} |z_j|_1 \theta_j^{\tau L(n)} |y|_n) \\
&\le c_{m,n} \theta_j^{\tau L(n)+2\rho(1)-m} |y|_n (|y|_{\gamma(m)} + |y|_{\gamma(m)}^{\nu(m)}) \le c_n \theta_{j+1}^{-a}(|y|_\sigma + |y|_\sigma^\kappa)
\end{aligned}$$

if $\sigma \ge \max\{n, \gamma(m)\}$, $\kappa \ge \nu(m) + 1$, and $m \ge \tau L(n) + 2\rho(1) + \tau a$. This proves Lemma 2.9. □

REMARKS 2.10. (i) If $F \in (\underline{\mathrm{DN}})$ as well (with $p=1$ in Definition 1.1) then the proof of Lemma 2.9 can be much simplified since z_{j+1} can be estimated directly by means of Lemmas 2.4 and 2.7 (much as $|\Delta x_j|_n$). In this case the proof of Lemma 2.9 and hence of Theorem 2.2 uses the estimates (2) and (3) only for $n=1$.

(ii) If $E, F \in (\mathrm{DN})$ then we can choose $\kappa = 1$ in Lemma 2.9.

Proof of Theorem 2.2. Lemma 2.9 implies that z_j is a null sequence in F and x_j is a Cauchy sequence in E. For the limit $x = \lim_j x_j \in U$ we have $\Phi(x) = \lim_j \Phi(x_j) = y$ since Φ is continuous. From the proof of Corollary 2.6 we obtain that $|x|_1 \le c|y|_{s_0}$, and the estimate in Remark 2.3(i) follows from Lemma 2.9. □

REMARK 2.11. Theorem 2.2 still holds if Definition 2.1(a) is only required for all $n \in I$ and some infinite set I such that $1 \in I$ and $\sup_{n \in I}\{\mu(n) - \alpha\rho(n)\} = +\infty$.

The same proof also gives the following inverse function theorem without "loss of derivatives". Here it is enough to assume property $(\underline{\mathrm{DN}})$, and condition $(\bar{\mathrm{S}}_{(1,1)})$ is not needed.

THEOREM 2.12. *Let E satisfy condition* $(\underline{\mathrm{DN}})$ *in the form of* (d). *Let* Φ: $(U \subset E) \to F$ *satisfy the assumptions of Theorem 2.2, where we assume in place of* (4) *that*

$$(4)_0 \quad |L(x)y|_n \le c_n(|x|_n |y|_1 + |y|_n), \; x \in U, \; y \in F.$$

Then $\Phi(U)$ is a neighborhood of zero in F.

Proof. In the proof of Theorem 2.2 we choose $S_\theta x = x$, $\rho(n) = 0$, and $L(n) = 1$ in Lemma 2.4. □

3. Generalized Smoothing Operators

In this section, Theorem 2.2 is applied in order to prove inverse function theorems for Fréchet spaces under more general assumptions. In place of the particular estimates (1), (2), (3), and (4) of the previous section, we now admit more general estimates and then give conditions on the Fréchet space E such that the inverse function theorem still holds; these conditions on E are formulated by means of the following variants of property $(\bar{\mathrm{S}}_{(1,1)})$ (cf. Definition 2.1). In particular, a tamely invariant condition $(\bar{\mathrm{S}}_t)$ is introduced; this property will be shown to be sufficient for the Nash–Moser theorem to hold under classical assumptions on the mappings.

DEFINITION 3.1. Let $(E, (|\cdot|_n)_{n=0}^\infty)$ be a graded Fréchet space.

(i) Let $a, d \in \mathbb{N}$ and $a \ge d$. E has property $(\mathrm{S}_{(a,d)})$ (we write $E \in (\mathrm{S}_{(a,d)})$) if there exist $\rho, \mu : \mathbb{N} \to]0, \infty[$, a set $I \subset \mathbb{N}$, a number $\alpha > 1$, and constants

$b \geq a+d$ and $c_n > 0$ such that for every $\theta \geq 1$ there exist (not necessarily linear) maps $S_\theta: E \to E$ such that:

(a) $|S_\theta x|_n \leq c_n \theta^{\rho(n)} |x|_{n-d}$, $n \in I \cup \{a\}$, $x \in E$;

(b) $|x - S_\theta x|_a \leq c_n \theta^{-\mu(n)} |x|_{n-d}$, $n \geq b$, $x \in E$; and

(c) $\sup_{n \in I}\{\mu(n) - \alpha\rho(n)\} = +\infty$.

(ii) $E \in (\bar{\mathrm{S}}_{(a,d)})$ means that (i) holds for $I = \{n : n \geq a\}$.

(iii) $E \in (\mathrm{S}_{(d)})$ means that for each $a_0 \geq d$ there is an $a \geq a_0$ such that $E \in (\mathrm{S}_{(a,d)})$.

(iv) $E \in (\bar{\mathrm{S}}_{(d)})$ means that there is an $a_0 \geq d$ such that $E \in (\bar{\mathrm{S}}_{(a,d)})$ for all $a \geq a_0$.

(v) $E \in (\mathrm{S}_t)$ means that for each d there is an $a \geq d$ such that $E \in (\mathrm{S}_{(a,d)})$.

(vi) $E \in (\bar{\mathrm{S}}_t)$ means that for each d there is an $a_0 \geq d$ such that $E \in (\bar{\mathrm{S}}_{(a,d)})$ for all $a \geq a_0$.

REMARKS 3.2.

(i) Property $(\bar{\mathrm{S}}_{(1,1)})$ coincides with the condition given in Definition 2.1.

(ii) $(\bar{\mathrm{S}}_{(a,d)})$ implies $(\mathrm{S}_{(a,d)})$, $(\bar{\mathrm{S}}_{(d)})$ implies $(\mathrm{S}_{(d)})$, and $(\bar{\mathrm{S}}_t)$ implies (S_t).

(iii) $(\mathrm{S}_{(a,d)})$ implies $(\mathrm{S}_{(a,d-1)})$ and $(\mathrm{S}_{(d)})$ implies $(\mathrm{S}_{(d-1)})$; the same holds for $(\bar{\mathrm{S}}_{(a,d)})$ and $(\bar{\mathrm{S}}_{(d)})$.

(iv) (S_t) is equivalent to $\bigcap_{d \geq 1}(\mathrm{S}_{(d)})$, and $(\bar{\mathrm{S}}_t)$ is equivalent to $\bigcap_{d \geq 1}(\bar{\mathrm{S}}_{(d)})$.

(v) $(\mathrm{S}_{(d)})$ and $(\bar{\mathrm{S}}_{(d)})$ are preserved when removing or adding a finite number of seminorms. For instance, $(E, |\cdot|_n) \in (\mathrm{S}_{(a,d)})$ implies that $(E, |\cdot|_{n+p}) \in (\mathrm{S}_{(a-p,d)})$.

(vi) $(E, |\cdot|_n) \in (\bar{\mathrm{S}}_{(d)})$ implies that $(E, |\cdot|_{dn}) \in (\bar{\mathrm{S}}_{(1)})$; more precisely, we notice that $(E, |\cdot|_n) \in (\bar{\mathrm{S}}_{(da,d)})$ implies that $(E, |\cdot|_{dn}) \in (\bar{\mathrm{S}}_{(a,1)})$.

(vii) $(E, |\cdot|_{dn}) \in (\mathrm{S}_{(1)})$ implies that $(E, |\cdot|_n) \in (\mathrm{S}_{(d)})$; more precisely, we notice that $(E, |\cdot|_{dn+b}) \in (\mathrm{S}_{(a,1)})$ if and only if $(E, |\cdot|_n) \in (\mathrm{S}_{(da+b,d)})$, $0 \leq b \leq d-1$.

(viii) Properties $(\mathrm{S}_{(a,d)})$ and $(\bar{\mathrm{S}}_{(a,d)})$ are inherited by normwisely tame direct summands.

(ix) (S_t) and $(\bar{\mathrm{S}}_t)$ are tame invariants and are inherited by tame direct summands.

Next we prove that the topological invariants (Ω) and (DN) are necessary for properties $(\bar{\mathrm{S}}_t)$ and $(\bar{\mathrm{S}}_{(d)})$. In view of Remarks 3.2(iii) and (iv), it is enough to show this for property $(\bar{\mathrm{S}}_{(1)})$.

LEMMA 3.3. *If E has property* $(\bar{\mathrm{S}}_{(1)})$ *then E has properties* (Ω) *and* (DN).

Proof. To show property (Ω), let $U_n = \{x \in E : |x|_n \leq 1\}$. Let p be fixed. We choose $a \geq p$ so that $E \in (\bar{\mathrm{S}}_{(a,1)})$. For each $n \geq \max\{a, b\}$ we then have

$$U_{n-1} \subset c_n(\theta^{\rho(n)} U_n + \theta^{-\mu(n)} U_a)$$

for all $\theta \geq 1$. Applying standard arguments (cf. [14; 17]), we conclude that

$$|\cdot|^*_{n-1} \leq c'_n |\cdot|^{*1-\sigma_n}_a |\cdot|^{*\sigma_n}_n, \quad \sigma_n = \frac{\mu(n)}{\rho(n)+\mu(n)}.$$

Inductively applying the above inequality, for $k = 0, 1, \ldots$ we obtain that

$$|\cdot|_{n-1}^* \leq c_{n,k} |\cdot|_a^{*1-\sigma_n \cdots \sigma_{n+k}} |\cdot|_{n+k}^{*\sigma_n \cdots \sigma_{n+k}}.$$

This proves (Ω). In order to show property (DN) we fix a_0 so that $E \in (\bar{S}_{(a,1)})$ for all $a \geq a_0$. For $k \geq a_0$ we must then show that there are $p \geq k$ and a constant $c > 0$ such that $|\cdot|_k^2 \leq c|\cdot|_{a_0} |\cdot|_p$. For a given $a \geq a_0$ we can choose $\rho_a(n)$, $\mu_a(n)$, and $b = b_a$ according to Definition 3.1 so that (i)(a) and (i)(b) hold and $\sup_n \mu_a(n) = +\infty$. Then we have

$$|x|_a \leq c_{n,a}(\theta^{\rho_a(a)} |x|_{a-1} + \theta^{-\mu_a(n)} |x|_{n-1}), \quad n \geq b_a, \ \theta \geq 1, \ x \in E$$

with suitable constants $c_{n,a} > 0$. From this we get

$$|x|_a \leq c'_{n,a} |x|_{a-1}^{1-\sigma} |x|_{n-1}^{\sigma}, \quad \sigma = \sigma(n,a) = \frac{\rho_a(a)}{\mu_a(n) + \rho_a(a)}.$$

Inductively applying these estimates, for $n_a \geq b_a$ (which will be chosen later) we obtain

$$|x|_a \leq c'_a |x|_{a_0}^{(1-\sigma(n_a,a))(1-\sigma(n_{a-1},a-1)) \cdots (1-\sigma(n_{a_0+1},a_0+1))} |x|_{n_a-1}^{\mu_a} \cdots |x|_{n_{a_0+1}-1}^{\mu_{a_0+1}},$$

where $0 < \mu_i < 1$ and $(1-\sigma(n_a,a)) \cdots (1-\sigma(n_{a_0+1}, a_0+1)) + \mu_a + \cdots + \mu_{a_0+1} = 1$. We choose $n_a, \ldots, n_{a_0+1}$ so that $(1-\sigma(n_a,a)) \cdots (1-\sigma(n_{a_0+1}, a_0+1)) \geq \frac{1}{2}$. This is possible since $\sup_n \mu_a(n) = +\infty$ for $a \geq a_0$. If $k = a$ is given then we obtain the assertion by choosing $p = \max\{a_0, n_a - 1, \ldots, n_{a_0+1} - 1\}$. □

As a result in the reverse direction, we show in the next section that a Köthe sequence space satisfying properties (Ω) in standard form and topological (DN) has property ($\bar{S}_t$) and hence ($\bar{S}_{(1)}$).

We now connect the conditions introduced in Definition 3.1 with the inverse function theorem. We consider the following situation. Let E, F be Fréchet spaces equipped with fundamental systems of seminorms $(|\cdot|_t)_{t \in J}$, $J \subset \mathbb{R}$, where $|\cdot|_s \leq |\cdot|_t$ for $s \leq t$ (e.g., we shall look at the cases $J = \mathbb{N}_0$ or $J = [0, \infty[$). Let $l \in J$ and $\eta > 0$ and put $U = \{x \in E : |x|_l < \eta\}$. Let Φ: $(U \subset E) \to F$ be a continuous (nonlinear) map with $\Phi(0) = 0$. Assume that the linear map $\Phi'(x): E \to F$ exists (where $\Phi'(x)v$ denotes the Gâteaux derivative) for all $x \in U$. It is useful to introduce the following notation.

DEFINITION 3.4. *Let* $\alpha, \beta, \gamma, \phi: J \to J$ be monotonically inceasing and let Φ be as above. We then call Φ an (α, β, γ)-map if the following holds. There is a map $L: (U \subset E) \times F \to E$ such that $\Phi'(x)L(x)y = y$, $x \in U$, and $y \in F$, and there exist $d, t_0 \in J$ and constants $c_t > 0$ such that, for all $t \in J$ with $t \geq t_0$, we have:

(1) $|\Phi(x)|_t \leq c_t |x|_{\alpha(t)}$, $x \in U$;
(2) $|\Phi'(x)v|_t \leq c_t(|x|_{\alpha(t)} |v|_l + |v|_{\alpha(t)})$, $x \in U$, $v \in E$;
(3) $|\Phi(x+v) - \Phi(x) - \Phi'(x)v|_t \leq c_t(|x|_{\alpha(t)} |v|_l^2 + |v|_l |v|_{\alpha(t)})$, $x, x+v \in U$; and
(4) $|L(x)y|_t \leq c_t(|x|_{\gamma(t)} |y|_d + |y|_{\beta(t)})$, $x \in U$, $y \in F$.

For the Fréchet space $(E, (|\cdot|_t)_{t \in J})$, we shall use the following notation:

(i) $E \in (\text{NM}: (\alpha, \beta, \gamma))$ means that for each (α, β, γ)-map $\Phi: (U \subset E) \to F$ the set $\Phi(U)$ is a neighborhood of zero in F; and
(ii) $E \in (\text{NM}_1: (\alpha, \beta, \gamma))$ means that (i) holds under the restriction $F = E$.

For $J=\mathbb{N}_0$ and $\alpha\colon \mathbb{N}_0\to[0,\infty[$, the term $\alpha(n)$ must be replaced by $[\alpha(n)]$ where $[x]:=\max\{z\in\mathbb{Z}: z\le x\}$. Using the notation just described, the result of Lojasiewicz and Zehnder [5] means that for a space E admitting linear smoothing operators (e.g. $E=\Lambda^q_\infty(\alpha)$, $r_k=k$) we have

$$E\in(\mathrm{NM_I}\colon(n+d,\lambda n+d,\lambda n+d)) \quad \text{for each } d\in\mathbb{N}_0 \text{ and } 1\le\lambda<2,$$

while $\Lambda^2_\infty(j)\notin(\mathrm{NM_I}\colon(n,2n,2n))$ (see Corollary 4.9 for generalizations).

In the following we shall investigate sufficient conditions for a graded Fréchet space $(E,(|\cdot|_t)_{t\in J})$ to satisfy $E\in(\mathrm{NM}\colon(\alpha,\beta,\gamma))$. In view of the applications it is useful to see that this property is inherited by direct summands.

LEMMA 3.5. *Let $J\subset\mathbb{R}$ and let $\alpha,\beta,\gamma,\phi,\psi\colon J\to J$ be monotically increasing so that $\sup f(J)=\sup J$ for $f=\alpha,\beta,\gamma,\phi,\psi$. Assume that $(F,|\cdot|_{t\in J})$ is a (ψ,ϕ)-direct summand of $(E,|\cdot|_{t\in J})$.*

(i) *If $E\in(\mathrm{NM}\colon(\phi\circ\alpha,\ \beta\circ\psi,\ \phi\circ\gamma\circ\psi))$, then $F\in(\mathrm{NM}\colon(\alpha,\beta,\gamma))$.*

(ii) *If $E\in(\mathrm{NM_I}\colon(\phi\circ\alpha\circ\psi,\ \phi\circ\beta\circ\psi,\ \phi\circ\gamma\circ\psi))$ and $\alpha(t),\beta(t),(\phi\circ\psi)(\tau)\ge t$, then $F\in(\mathrm{NM_I}\colon(\alpha,\beta,\gamma))$.*

Proof. Let $S\colon F\to E$ be ψ-tame and let $T\colon E\to F$ be ϕ-tame so that $T\circ S=\mathrm{id}_F$.

(i) Let $\Phi\colon(U\subset F)\to G$ be an (α,β,γ)-map. Then $V=T^{-1}(U)$ is a neighborhood of zero in E. We define $\Psi\colon(V\subset E)\to G$ where $\Psi(x)=\Phi(Tx)$ for $x\in V$. Then $\Psi'(x)v=\Phi'(Tx)Tv$ for $x\in V$ and $v\in E$, and $M(x)\colon G\to E$ defined by $M(x)y=S(L(Tx)y)$ satisfies $\Psi'(x)M(x)y=y$ for $x\in V$ and $y\in G$. Hence Ψ is a $(\phi\circ\alpha,\beta\circ\psi,\phi\circ\gamma\circ\psi)$-map and $\Psi(V)$ is a neighborhood of zero in G. Since $\Phi(U)\supset\Psi(V)$, $\Phi(U)$ is also a neighborhood of zero in G.

(ii) Let $\Phi\colon(U\subset F)\to F$ be an (α,β,γ)-map. Put $V=T^{-1}(U)$ and $\Psi\colon(V\subset E)\to E$ where $\Psi(x)=S\Phi(Tx)+x-STx$ for $x\in V$. Then $\Psi'(x)v=S\Phi'(Tx)Tv+v-STv$ for $x\in V$ and $v\in E$, and for $M(x)\colon E\to E$ defined by $M(x)y=S(L(Tx)Ty)+(I-ST)y$ we have $\Psi'(x)M(x)y=y$ for $x\in V$ and $y\in E$. Ψ is a $(\phi\circ\alpha\circ\psi,\phi\circ\beta\circ\psi,\phi\circ\gamma\circ\psi)$-map and $\Psi(V)$ is a neighborhood of zero in E. Hence, for $y\in E$ with $|y|_{s_0}\le\delta$ there exists $x\in V$ such that $T\Psi(x)=\Phi(Tx)=Ty$ and so $u=Tx\in U$ with $\Phi(u)=Ty$. Since T is surjective, the open mapping theorem implies that T is open; therefore the set $T\{y\in E\colon |y|_{s_0}\le\delta\}$ is a neighborhood of zero in F. Hence $\Phi(U)$ is a neighborhood of zero in F. □

Applying the standard inverse function theorems (2.2 and 2.12) yields the following.

THEOREM 3.6. *Let the graded Fréchet space $(E,(|\cdot|_t)_{t\in J})$ satisfy property $(\underline{\mathrm{DN}})$.*

(i) *If $J=\mathbb{N}_0$ then $E\in(\mathrm{NM}\colon(n,n,n))$.*

(ii) *If $J=\mathbb{N}_0$ and $E\in(\mathrm{S}_{(1)})$, then $E\in(\mathrm{NM}\colon(n,n+1,n+1))$.*

(iii) *Let $\alpha,\beta,\gamma\colon J\to J$ be monotonically increasing so that* $\sup\alpha(J)=\sup J$. *Let $(r_n)_{n=0}^\infty\subset J$ and $n_0\in\mathbb{N}$ be chosen so that $r_n\le r_{n+1}\nearrow\sup J$ and*

(5) $\gamma(\alpha(r_n)) \leq \alpha(r_{n+1})$ *and* $\beta(\alpha(r_n)) \leq r_{n+1}$ *for all* $n \geq n_0$.

If $(E, |\cdot|_{\alpha(r_n)}) \in (\mathrm{S}_{(1)})$ *then it follows that* $(E, |\cdot|_t) \in (\mathrm{NM}: (\alpha, \beta, \gamma))$.

Proof. (i) This follows from Theorem 2.12 after removing a finite number of seminorms.

(ii) By means of Remark 3.2(v), we may assume that $E \in (\underline{\mathrm{DN}})$ with $p = 1$ in Definition 1.1. Let Φ be an $(n, n+1, n+1)$-map with l, d, t_0 as in Definition 3.4. We choose $a \geq l + d + t_0 + 1$ so that $E \in (\mathrm{S}_{(a,1)})$. We then change the gradings in E and F by removing the first $a-1$ seminorms. With respect to the new gradings we obtain an $(n, n+1, n+1)$-map with $l = d = 1$ and $t_0 = 0$, and $E \in (\mathrm{S}_{(1,1)})$ holds by means of Remark 3.2(v). The assertion follows from Theorem 2.2 and Remark 2.11.

(iii) Let Φ be an (α, β, γ)-map as in Definition 3.4. We choose $a \in \mathbb{N}_0$ with $\alpha(r_a) \geq l$ and $r_a \geq d$. With respect to the new gradings $\|\cdot\|_n^E = |\cdot|_{\alpha(r_n)}^E$ and $\|\cdot\|_n^F = |\cdot|_{r_n}^F$, Φ is an $(n, n+1, n+1)$-map with $l = d = a$ in Definition 3.4. Now (ii) gives the result. □

For a given triplet (α, β, γ) one must check whether there exist r_n satisfying (5) so that $(E, |\cdot|_{s_n}) \in (\mathrm{S}_{(1)})$ for $s_n = \alpha(r_n)$. If α is strictly increasing then (5) is equivalent to

(5)′ $s_{n+1} \geq \phi(s_n)$, $n \geq n_0$, where $\phi(t) = \max\{(\alpha \circ \beta)(t), \gamma(t)\}$.

If $\sup \alpha(J) = \sup J$ and $\alpha^{-1}(t) := \sup\{s \in J: \alpha(s) \leq t\} \in J$ with $\alpha(\alpha^{-1}(t)) \leq t$ for all t (this holds e.g. if $J = \mathbb{N}_0$ or $J = [0, \infty[$ and α is continuous), then $E \in (\mathrm{NM}: (\mathrm{id}_J, \beta, \gamma))$ implies that $E \in (\mathrm{NM}: (\alpha, \alpha^{-1} \circ \beta, \gamma))$ (consider on F the grading $|\cdot|_{\alpha^{-1}(t)}^F$). In concrete cases it is obvious how r_n and $s_n = \alpha(r_n)$ should be chosen. Table 1 contains some examples.

Remark 3.7. Let $J = [0, \infty[$, b_1, b_2, $b_3 \geq 0$, and A, B, $C \geq 1$. Put $d = \max\{b_1 + b_2, b_3, 1\}$ and $D = \max\{AB, C\}$. Then the following choices of r_n and $s_n = \alpha(r_n)$ satisfy (5).

Table 1

$\alpha(t)$	$\beta(t)$	$\gamma(t)$	r_n	$s_n = \alpha(r_n)$
$t + b_1$	$t + b_2$	$t + b_3$	$dn - b_1$	dn
At	Bt	Ct	$(1/A)D^n$	D^n
t^A	t^B	t^C	$e^{(1/A)D^n}$	e^{D^n}

Corollary 3.8. *Let* $(E, |\cdot|_t)$ *be a graded Fréchet space with property* $(\underline{\mathrm{DN}})$. *Let* $b_1, b_2, b_3 \in \mathbb{N}_0$ *and* $A, B, C \geq 1$. *Put* $d = \max\{b_1 + b_2, b_3, 1\}$ *and* $D = \max\{AB, C\}$.

(i) *If* $J = \mathbb{N}_0$ *and* $(E, |\cdot|_{dn}) \in (\mathrm{S}_{(1)})$, *then* $(E, |\cdot|_n) \in (\mathrm{NM}: (n + b_1, n + b_2, n + b_3))$.

(ii) *If* $J=[0,\infty[$ *and* $(E,|\cdot|_{D^n})\in(\mathrm{S}_{(1)})$, *then* $(E,|\cdot|_t)\in(\mathrm{NM}\colon(At,Bt,Ct))$.
(iii) *If* $J=[0,\infty[$ *and* $(E,|\cdot|_{e^{D^n}})\in(\mathrm{S}_{(1)})$, *then* $(E,|\cdot|_t)\in(\mathrm{NM}\colon(t^A,t^B,t^C))$.
(iv) *If* $J=\mathbb{N}_0$ *and* $(E,|\cdot|_n)\in(\mathrm{S}_{(d)})$, *then* $(E,|\cdot|_n)\in(\mathrm{NM}\colon(n+b_1,n+b_2,n+b_3))$.
(v) *If* $J=\mathbb{N}_0$ *and* $(E,|\cdot|_n)\in(\mathrm{S}_{\mathrm{t}})$, *then* $(E,|\cdot|_n)\in(\mathrm{NM}\colon(n+d,n+d,n+d))$ *for all* d.

Proof. (i), (ii), and (iii) are clear by means of Theorem 3.6(iii) and Remark 3.7; (iv) and (v) follow from Remarks 3.2(iv) and (vii). □

If $E\in(\mathrm{S}_{\mathrm{t}})$ then the inverse function theorem holds for each (α,β,γ)-map $\Phi\colon(U\subset E)\to F$, where $\alpha(n),\beta(n),\gamma(n)\le n+b$ for some fixed b; this gives for $E\in(\mathrm{S}_{\mathrm{t}})$ the Nash–Moser theorem under classical assumptions on Φ (cf. [2; 3]). If $(E,|\cdot|_{D^n})\in(\mathrm{S}_{(1)})$ for every $D\in\mathbb{N}$ then the inverse function theorem can be applied to each (α,β,γ)-map $\Phi\colon(U\subset E)\to F$, where $\alpha(n),\beta(n)$, $\gamma(n)\le An+b$ for some fixed A,b. It is obvious how to obtain further corresponding results.

4. An Inverse Function Theorem for Köthe Spaces

In this section the conditions of type $(\mathrm{S}_{(a,d)})$ introduced in Definition 3.1 are evaluated for Köthe sequence spaces $\lambda^q(a)$. In view of Lemma 3.3 we assume that $\lambda^q(a)$ admits a continuous norm $|\cdot|_0$. Let $0<a_{j,k}\le a_{j,k+1}$ be a Köthe matrix, and let $1\le q\le\infty$.

THEOREM 4.1. *Let* $a,d\in\mathbb{N}$ *and* $a\ge d$. *Then* $\lambda^q(a)\in(\mathrm{S}_{(a,d)})$ *(resp.* $(\bar{\mathrm{S}}_{(a,d)})$*) holds if and only if the following is true. There exist* $\rho,\mu\colon\mathbb{N}\to]0,\infty[$, $\alpha>1$, *and* $b\ge a+d$, *as well as a set* $I\subset\mathbb{N}$ *(resp.* $I=\{n\colon n\ge a\}$*) and* $c_n>0$ *and* $\gamma_j\ge1$ *such that* $\sup_{n\in I}\{\mu(n)-\alpha\rho(n)\}=+\infty$ *and, for all* j, *the following condition holds*:

$$\sup_{n\in I\cup\{a\}} c_n^{-1}\left(\frac{a_{j,n}}{a_{j,n-d}}\right)^{1/\rho(n)}\le\gamma_j\le\inf_{n\ge b}c_n\left(\frac{a_{j,n-d}}{a_{j,a}}\right)^{1/\mu(n)}. \qquad \text{(a, b)*}$$

Proof. Let $\lambda^q(a)\in(\mathrm{S}_{(a,d)})$, where ρ,μ,α,b,I,c_n are chosen as in Definition 3.1. Putting in the unit vectors we see that, for all j and $\theta\ge1$, one of the following two alternatives holds: Either $a_{j,n}\le 2c_n\theta^{\rho(n)}a_{j,n-d}$ for all $n\in I\cup\{a\}$, or $a_{j,a}\le 2c_n\theta^{-\mu(n)}a_{j,n-d}$ for all $n\ge b$.

Put

$$\gamma_j=\sup_{n\in I\cup\{a\}}\left(\frac{a_{j,n}}{2c_na_{j,n-d}}\right)^{1/\rho(n)}\quad\text{and}\quad \nu_j=\inf_{n\ge b}\left(\frac{2c_na_{j,n-d}}{a_{j,a}}\right)^{1/\mu(n)}.$$

Then, for each j and $\theta\ge1$, either $\gamma_j\le\theta$ or $\nu_j\ge\theta$ holds. In particular this implies that $\gamma_j<+\infty$ and $\gamma_j\le\nu_j$. If, on the other hand, condition (a, b)* is fulfilled then we define $S_\theta(x_j)=(y_j)$ putting $y_j=x_j$ if $\gamma_j\le\theta$ and $y_j=0$ otherwise. Then (a) and (b) of Definition 3.1(i) are true. □

REMARK 4.2. For (a, b)* it is necessary that for $b\le n\in I$ and suitable $c_n'>0$ we have

$$c_n' a_{j,n-d}^{\rho(n)+\mu(n)}\ge a_{j,a}^{\rho(n)}a_{j,n}^{\mu(n)}.$$

Putting $\gamma_j=a_{j,a}/a_{j,a-d}$, we obtain the following sufficient condition for (a, b)*.

LEMMA 4.3. *Assume that for* $\rho,\mu:\mathbb{N}_0\to\,]0,\infty[$, $b\ge a+d$, *and* $c_n>0$ *we have*

$$\frac{a_{j,n}}{a_{j,n-d}}\le c_n\left(\frac{a_{j,a}}{a_{j,a-d}}\right)^{\rho(n)},\quad n\in I\cup\{a\};\tag{a)*}$$

$$a_{j,a}^{1+\mu(n)}\le c_n a_{j,a-d}^{\mu(n)}a_{j,n-d},\quad n\ge b.\tag{b)*}$$

Then condition (a, b)* *holds for* $\gamma_j=a_{j,a}/a_{j,a-d}$ (*the constants* c_n *may have changed*).

REMARKS 4.4. (i) If $\lambda^q(a)\in(\mathrm{DN})$ with dominant norm $|\cdot|_p$ (as in Definition 1.1) and if $p\le a-d$, then (b)* holds with $\sup_n\{\mu(n)\}=+\infty$.

(ii) Condition (a)* is not really a condition of (Ω)-type. However, if $\lambda^q(a)\in(\Omega)$ then for every p there is an a_0 such that, for all $a\ge a_0$ and d with $a-d=p$, condition (a)* holds for suitable $\rho(n)$ and $n\ge a_0+d$. This follows since, by means of (Ω), for each p there is a $q=:a_0$ such that for every $n\ge q$ there exist m and $c>0$ such that $ca_{j,q}^{m+1}\ge a_{j,p}^m a_{j,n}$; this implies

$$\frac{a_{j,n}}{a_{j,n-d}}\le\frac{a_{j,n}}{a_{j,q}}\le c\left(\frac{a_{j,q}}{a_{j,p}}\right)^m\quad\text{for } n\ge d+q.$$

COROLLARY 4.5. (i) *If* $\lambda^q(a)\in(\mathrm{DN})$ *with dominant norm* $|\cdot|_p$ *and, in addition,* $c_n a_{j,n}^2\ge a_{j,n-1}a_{j,n+1}$ *holds for suitable constants* $c_n>0$, *then it follows that* $\lambda^q(a)\in(\bar{\mathrm{S}}_{(a,d)})$ *for all* $a\ge p+d$. *In particular,* $\lambda^q(a)\in(\bar{\mathrm{S}}_{\mathrm{t}})$.

(ii) *If* $\lambda^q(a)\in(\mathrm{DN})\cap(\Omega_{\mathrm{t}})$ *then it follows that* $\lambda^q(a)\in(\bar{\mathrm{S}}_{\mathrm{t}})$.

Proof. Part (i) follows from Theorem 4.1, Lemma 4.3, and Remark 4.4(i); here we may choose $\rho(n)=1$. Part (ii) follows from (i) and Remark 3.2(ix) since, by means of [9], the space $\lambda^q(a)$ is tamely isomorphic to some Köthe sequence space, which is an (Ω) space in standard form. □

We notice that instead of assuming property (Ω) in standard form in Corollary 4.5(i) it is enough to assume that condition (a)* holds for some bounded ρ. For that is suffices to suppose that there are a and A with $a_{j,n}/a_{j,n-1}\le c_n(a_{j,a}/a_{j,0})^A$. For $\lambda^q(a)\in(\mathrm{DN})\cap(\Omega_{\mathrm{t}})$ and their tame direct summands, the inverse function theorem holds under classical assumptions on the mappings by means of Corollary 3.8(v) (cf. Remark 3.2(ix) and Lemma 3.5). It is remarkable that in this case condition (DN) is only needed in its topological form.

Next we evaluate the conditions of type $(\mathrm{S}_{(a,d)})$ for power series spaces. Let $0 \le \alpha_0 \le \alpha_1 \le \cdots \nearrow +\infty$ and $r_0 < r_1 < \cdots \nearrow R \in [0,\infty]$, and put $\Lambda_R^q(\alpha) = \lambda^q(a)$ with $a_{j,k} = e^{r_k \alpha_j}$. We need only consider power series spaces of infinite type, as follows.

THEOREM 4.6. *For $d \in \mathbb{N}$ and $\Lambda = \Lambda_R^q(\alpha)$, the following are equivalent:*

(i) $\Lambda \in (\mathrm{S}_{(a,d)})$ *for some* $a \ge d$;
(ii) $\Lambda \in (\bar{\mathrm{S}}_{(a,d)})$ *for all* $a \ge d$;
(iii) $\Lambda \in (\bar{\mathrm{S}}_{(d)})$ *or* $\Lambda \in (\mathrm{S}_{(d)})$ *or both*;
(iv) $R = \infty$, *and there is* $0 < \lambda < 2$ *such that* $r_n / r_{n-d} \le \lambda < 2$ *for infinitely many* n; *and*
(v) $R = \infty$, *and there is* $\mu > \frac{1}{2}$ *such that* $\sup_n \{r_{n-d} - \mu r_n\} = +\infty$.

Proof. The directions (ii) ⇒ (iii) ⇒ (i) and (iv) ⇔ (v) are clear. We prove the implications (i) ⇒ (iv) and (iv) ⇒ (ii). If (i) holds then Theorem 4.1 implies that

$$\sup_{n \in I} \exp\left\{\alpha_j (r_n - r_{n-d}) \frac{1}{\rho(n)} - c_n\right\} \le \gamma_j \le \inf_{n \ge b} \exp\left\{\alpha_j (r_{n-d} - r_a) \frac{1}{\mu(n)} + c_n\right\},$$

where $\sup_{n \in I}\{\mu(n) - \alpha \rho(n)\} = +\infty$ and $\alpha > 1$. For $b \le n \in I$ we conclude that

$$\frac{r_n - r_{n-d}}{\rho(n)} - \frac{c_n}{\alpha_j} \le \frac{\ln \gamma_j}{\alpha_j} \le \frac{r_{n-d} - r_a}{\mu(n)} + \frac{c_n}{\alpha_j}.$$

Since n can be chosen independently on the left- and on the right-hand side, respectively, we obtain for $j \to \infty$ that $\sup_n r_n = +\infty$. Further,

$$\frac{r_n - r_{n-d}}{r_{n-d} - r_a} \le \frac{\rho(n)}{\mu(n)} \le \frac{1}{\alpha}$$

holds for infinitely many n; hence

$$\frac{r_n}{r_{n-d} - r_a} \le 1 + \frac{1}{\alpha} + \frac{r_a}{r_{n-d} - r_a},$$

and $r_n \to +\infty$ gives (iv).

If (iv) holds and $a \ge d$ then we put $\rho(n) = r_n - r_{n-d}$ and $\mu(n) = r_{n-d} - r_a$, and choose $\alpha > 1$ so that $\lambda < 1 + 1/\alpha$. For infinitely many n we then obtain

$$\mu(n) - \alpha\rho(n) = r_{n-d}(1+\alpha) - \alpha r_n - r_a \ge r_{n-d}(1 + \alpha - \alpha\lambda) - r_a \to +\infty.$$

Then, putting $\gamma_j = e^{\alpha_j}$, (ii) follows from Theorem 4.1. □

COROLLARY 4.7. *$\Lambda_\infty^q(\alpha) \in (\bar{\mathrm{S}}_{\mathrm{t}})$ holds if and only if*

$$\liminf_n \frac{r_n}{r_{n-d}} < 2 \quad \textit{for all } d \in \mathbb{N}.$$

EXAMPLES 4.8. (i) For $r_n = A^n$, $A > 1$, we have $\Lambda_\infty^q(\alpha) \in (\bar{\mathrm{S}}_{(d)})$ if and only if $A^d < 2$.

(ii) If $\lim_n (r_{n+1}/r_n) = 1$ then $\Lambda^q_\infty(\alpha) \in (\bar{S}_t)$.

(iii) Let $\phi, \mu: \mathbb{N}_0 \to \mathbb{N}_0$ be monotonically increasing, where

$$\mu(n) < \mu(n+1) \le \phi(\mu(n)) \quad \text{and} \quad r_{\phi(n)}/r_n \le \lambda < 2$$

for all n. Then $(\Lambda^q_\infty(\alpha), |\cdot|_{r_{\mu(n)}}) \in (\bar{S}_{(1)})$.

Applying Corollary 3.8, we also obtain the full result of Lojasiewicz and Zehnder [5].

COROLLARY 4.9.

(i) *Let* $r_k = k$. *Then* $\Lambda^q_\infty(\alpha) \in (\text{NM}: (An+b, Bn+b, Cn+b))$ *holds for all* $b \ge 0$ *and* $A, B, C \ge 1$ *if* $D = \max\{AB, C\} < 2$. (In [5], the case $B = C$, $AB < 2$ is stated.)

(ii) *Let* $\alpha, \phi: \mathbb{N}_0 \to \mathbb{N}_0$ *be increasing, and let* $1 < r_{\phi(n)}/r_n \le \lambda < 2$ *for all* n. *Then* $(\Lambda^q_\infty(\alpha), |\cdot|_{r_n}) \in (\text{NM}: (\text{id}, \phi, \phi))$ *and hence also* $\in (\text{NM}: (\alpha, \alpha^{-1} \circ \phi, \phi))$.

Proof. Part (i) follows from Example 4.8(i) with Corollary 3.8(ii). To show part (ii), we put $s_0 = 0$ and $s_{n+1} = \phi(s_n)$. By means of Theorem 4.6 we have $(\Lambda^q_\infty(\alpha), |\cdot|_{r_{s_n}}) \in (\bar{S}_{(1)})$, and Theorem 3.6 gives the assertion. □

In Lemma 3.3 we proved that conditions (Ω) and (DN) are necessary for property $(\bar{S}_{(1)})$. We now consider Köthe spaces satisfying both properties (Ω) and (DN) and look for sufficient conditions for properties of type $(S_{(a,d)})$. For that, the quantitative variants $(\text{DN}_{(\phi,\psi)})$ and $(\Omega_{(\tau,\sigma)})$ introduced in [9] are useful (cf. Section 1).

Let $\mathfrak{F}$ be defined as in Section 1, and let $\phi^{-1}(k) := \max\{l: \phi(l) \le k\}$, $\phi \in \mathfrak{F}$. Then

$$\phi^{-1}(k) \le \phi(\phi^{-1}(k)) \le k \le \phi^{-1}(\phi(k)) \le \phi(k), \quad \phi \in \mathfrak{F}.$$

We assume that the space $\lambda^q(a)$ has properties $(\text{DN}_{(\phi,\psi)})$ and $(\Omega_{(\tau,\sigma)})$ for $\phi, \psi, \sigma, \tau \in \mathfrak{F}$ (motivated by Remark 1.3(ii) and Lemma 3.3). This means that there exist $b_0 \ge 0$ and $c_m > 0$ such that:

$$a_{j,n}^{m-l} \le c_m a_{j,\phi(l)}^{m-\psi(n)} a_{j,\phi(m)}^{\psi(n)-l} \text{ for all } j \text{ and } b_0 \le l < \psi(n) < m; \text{ and} \qquad (*)$$

$$a_{j,\tau(n)}^{\sigma(m)-\sigma(l)} \ge c_m^{-1} a_{j,l}^{\sigma(m)-n} a_{j,m}^{n-\sigma(l)} \text{ for all } j \text{ and } b_0 \le \sigma(l) < n < \sigma(m). \qquad (**)$$

Our goal is to establish the conditions (a)* and (b)* of Lemma 4.3 for suitable $\rho(n)$, $\mu(n)$ and then to state assumptions on ϕ, ψ, σ, τ so that also condition (c) of Definition 3.1(i) holds.

For that we fix $a \ge d \ge 1$. With $l = \phi^{-1}(a-d)$ and $m = \phi^{-1}(n-d)$, from (*) for $b_0 \le \phi^{-1}(a-d) < \psi(a) < \phi^{-1}(n-d)$ we obtain the estimate

$$a_{j,a}^{\phi^{-1}(n-d)-\phi^{-1}(a-d)} \le c_n a_{j,a-d}^{\phi^{-1}(n-d)-\psi(a)} a_{j,n-d}^{\psi(a)-\phi^{-1}(a-d)}.$$

Hence for $\mu(n) = (\phi^{-1}(n-d) - \psi(a))/(\psi(a) - \phi^{-1}(a-d))$ we have the inequalities

$$a_{j,a}^{1+\mu(n)} \le c_n a_{j,a-d}^{\mu(n)} a_{j,n-d}, \qquad b_0 \le \phi^{-1}(a-d) < \psi(a) < \phi^{-1}(n-d). \qquad \text{(b)*}$$

In order to derive condition (a)* of Lemma 4.3, we discuss two different possibilities; the first one is simpler while the second one yields better results if for instance (DN_t) holds.

We first use the *decomposition* $a_{j,n}/a_{j,n-d} = (a_{j,n}/a_{j,a})(a_{j,a}/a_{j,n-d})$ and the estimates

$$\text{(i)}\quad \frac{a_{j,n}}{a_{j,a}} \overset{(**)}{\leq} c_n \left(\frac{a_{j,a}}{a_{j,a-d}}\right)^{\gamma(n)} \quad \text{and} \quad \text{(ii)}\quad \frac{a_{j,a}}{a_{j,n-d}} \overset{(*)}{\leq} c_n \left(\frac{a_{j,a-d}}{a_{j,a}}\right)^{\mu(n)}.$$

Condition (a)* follows for $\rho(n) = \gamma(n) - \mu(n)$. For $l = a-d$ and $m = n$, from (**) we derive

$$a_{j,a}^{\sigma(n)-\sigma(a-d)} \geq c_n^{-1} a_{j,a-d}^{\sigma(n)-\tau^{-1}(a)} a_{j,n}^{\tau^{-1}(a)-\sigma(a-d)},$$
$$b_0 \leq \sigma(a-d) < \tau^{-1}(a) < \sigma(n).$$

For $b_0 \leq \sigma(a-d) < \tau^{-1}(a)$ and $n > a$ we obtain (i) with

$$\gamma(n) = \frac{\sigma(n) - \tau^{-1}(a)}{\tau^{-1}(a) - \sigma(a-d)}.$$

In a second attempt we use the *estimates*

$$\text{(i)}\quad \frac{a_{j,n}}{a_{j,n-d}} \overset{(*)}{\leq} c_{n,k}\left(\frac{a_{j,n+k}}{a_{j,n}}\right)^{\alpha}, \quad \text{(ii)}\quad \frac{a_{j,n+k}}{a_{j,n}} \overset{(**)}{\leq} c_{n,k}\left(\frac{a_{j,n}}{a_{j,a}}\right)^{\beta},$$
$$\text{(iii)}\quad \frac{a_{j,n}}{a_{j,a}} \overset{(**)}{\leq} c_n\left(\frac{a_{j,a}}{a_{j,a-d}}\right)^{\gamma}.$$

This condition implies (a)* for $\rho(n) = \alpha\beta\gamma$, where k must be chosen later. (iii) holds for $\gamma = \gamma(n)$.

(i) For $l = \phi^{-1}(n-d)$ and $m = \phi^{-1}(n+k)$, (*) implies that

$$a_{j,n}^{\phi^{-1}(n+k)-\phi^{-1}(n-d)} \leq c_{n,k} a_{j,n-d}^{\phi^{-1}(n+k)-\psi(n)} a_{n,n+k}^{\psi(n)-\phi^{-1}(n-d)}$$

for all $b_0 \leq \phi^{-1}(n-d) < \psi(n) < \phi^{-1}(n+k)$. This gives (i) for $\alpha = (\psi(n) - \phi^{-1}(n-d))/(\phi^{-1}(n+k) - \psi(n))$.

(ii) By means of (**), for $l = a$ and $m = n+k$ we have the estimate

$$a_{j,n}^{\sigma(n+k)-\sigma(a)} \geq c_{n,k}^{-1} a_{j,a}^{\sigma(n+k)-\tau^{-1}(n)} a_{j,n+k}^{\tau^{-1}(n)-\sigma(a)}$$

for $b_0 \leq \sigma(a) < \tau^{-1}(n)$, $k \geq 1$. This gives (ii) with

$$\beta = \frac{\sigma(n+k) - \tau^{-1}(n)}{\tau^{-1}(n) - \sigma(a)}.$$

LEMMA 4.10. *Let* $\lambda^q(a) \in (DN_{(\phi,\psi)}) \cap (\Omega_{(\tau,\sigma)})$ *with* $\phi, \psi, \sigma, \tau \in \mathfrak{F}$. *Let* $a \geq d \geq 1$ *with* $\phi^{-1}(a-d) \geq b_0$ *and* $\tau^{-1}(a) > \sigma(a-d)$. *Then, for every* $n > a$ *with* $\phi^{-1}(n-d) > \max\{b_0, \psi(a)\}$ *and* $\tau^{-1}(n) > \sigma(a)$, *conditions* (a)* *and* (b)* *of Lemma 4.3 hold where*

$$\mu(n) = \frac{\phi^{-1}(n-d) - \psi(a)}{\psi(a) - \phi^{-1}(a-d)} \quad \textit{and} \quad \rho(n) = \frac{\sigma(n) - \tau^{-1}(a)}{\tau^{-1}(a) - \sigma(a-d)} - \mu(n).$$

This holds also for $k \geq 1$ *and* $\phi^{-1}(n+k) > \psi(n)$ *if*

$$\rho(n) = \frac{\sigma(n+k)-\tau^{-1}(n)}{\phi^{-1}(n+k)-\psi(n)} \frac{\sigma(n)-\tau^{-1}(a)}{\tau^{-1}(n)-\sigma(a)} \frac{\psi(n)-\phi^{-1}(n-d)}{\tau^{-1}(a)-\sigma(a-d)}.$$

In Lemma 4.10 we have $\lim_n \mu(n) = +\infty$ for each a and d. Hence, for condition (c) of Definition 3.1(i) it is sufficient that $\rho(n)/\mu(n) \leq \delta < 1$ for infinitely many n.

Corollary 4.11. *Let* $\lambda^q(a) \in (\mathrm{DN}_{(\phi,\psi)}) \cap (\Omega_{(\tau,\sigma)})$ *with* $\phi, \psi, \sigma, \tau \in \mathfrak{F}$. *Let* $a \geq d \geq 1$ *with* $\phi^{-1}(a-d) \geq b_0$ *and* $\tau^{-1}(a) > \sigma(a-d)$. *Assume that either* (i) *or* (ii) *holds*:

(i) $$\frac{\sigma(n)-\tau^{-1}(a)}{\phi^{-1}(n-d)-\psi(a)} \frac{\psi(a)-\phi^{-1}(a-d)}{\tau^{-1}(a)-\sigma(a-d)} \leq \lambda < 2 \text{ for infinitely many } n.$$

(ii) *There is* $0 < \delta < 1$ *such that for all* n_0 *there exist* $n \geq n_0$ *and* $k \geq 1$ *with*

$$\frac{\sigma(n+k)-\tau^{-1}(n)}{\phi^{-1}(n+k)-\psi(n)} \frac{\sigma(n)-\tau^{-1}(a)}{\tau^{-1}(n)-\sigma(a)}$$
$$\times \frac{\psi(n)-\phi^{-1}(n-d)}{\phi^{-1}(n-d)-\psi(a)} \frac{\psi(a)-\phi^{-1}(a-d)}{\tau^{-1}(a)-\sigma(a-d)} \leq \delta < 1, \quad \phi^{-1}(n+k) > \psi(n).$$

In both cases it follows that $\lambda^q(a) \in (\mathrm{S}_{(a,d)})$.

From Corollary 4.11 we can obtain conditions behaving in a stable fashion with respect to certain isomorphisms; this is an advantage when compared to the easier but more unstable conditions stated in Corollary 4.5.

In order to evaluate Corollary 4.11, we assume that $\lambda^q(a) \in (\mathrm{DN}_{(\phi,\psi)}) \cap (\Omega_{(\tau,\sigma)})$ with $\phi, \psi, \sigma, \tau \in \mathfrak{F}$. We fix $a \geq d \geq 1$ with $\phi^{-1}(a-d) \geq b_0$ and $\tau^{-1}(a) > \sigma(a-d)$. We are looking for sufficient conditions for $\lambda^q(a) \in (\mathrm{S}_{(a,d)})$, and consider several cases.

Case I: Let $\phi(n) \leq n+b_1$ and $\psi(n) \leq n+b_2$. In view of Corollary 4.11(ii) it is enough to have

$$\liminf_n \frac{\sigma(n+k)-\tau^{-1}(n)}{k-b_1-b_2} \frac{\sigma(n)}{\tau^{-1}(n)} \frac{d+b_1+b_2}{n-d-b_1-\psi(a)} = 0,$$

where $k = k(n)$ and $\phi^{-1}(n+k) > \psi(n)$.

(a) If $\sigma(n) \leq An$ then $\limsup_k((\sigma(n+k)-\tau^{-1}(n))/(k-b_1-b_2)) \leq A$ holds for fixed n, and moreover $\limsup_n(\sigma(n)/(n-d-b_1-\psi(a))) \leq A$. Hence the assertion follows for arbitrary τ.

(b) In the general case we choose $k = n > b_1+b_2$ and obtain the sufficient condition

$$\liminf_n \frac{\sigma(2n)\sigma(n)}{\tau^{-1}(n)n^2} = 0.$$

If $\sigma(n) \leq An^\alpha$ and $\tau(n) \leq Bn^\beta$ with $\alpha, \beta \geq 1$, then this holds if $\alpha < 1+1/2\beta$.

Case II: Let $\phi(n) \le An+b$, $\psi(n) \le Bn+b$, $\sigma(n) \le Cn+b$, and $\tau(n) \le Dn+b$ with $A, B, C, D \ge 1$. We want to apply Corollary 4.11(ii). For a fixed n we have

$$\limsup_k \frac{\sigma(n+k)-\tau^{-1}(n)}{\phi^{-1}(n+k)-\psi(n)} \le AC,$$

and moreover

$$\limsup_n \frac{\sigma(n)-\tau^{-1}(a)}{\tau^{-1}(n)-\sigma(a)} \frac{\psi(n)-\phi^{-1}(n-d)}{\phi^{-1}(n-d)-\psi(a)} \le (AB-1)CD.$$

From this we obtain the sufficient condition

$$(AB-1)AC^2D\frac{Ba+b-(1/A)(a-b-d-A)}{(1/D)(a-b-D)-C(a-d)-b} < 1.$$

In all of the above situations we have established that $\lambda^q(a) \in (\mathrm{S}_{(a,d)})$ for $a \ge d \ge 1$ if $\phi^{-1}(a-d) \ge b_0$ and $\tau^{-1}(a) > \sigma(a-d)$. We next vary a and d as well and look for sufficient conditions for $\lambda(a) \in (\mathrm{S}_\mathrm{t})$. First we notice that for all a_0, b_0 there exist $a \ge d \ge a_0$ such that $\phi^{-1}(a-d) \ge b_0$ and $\tau^{-1}(a) > \sigma(a-d)$. This follows since there are $p \ge a_0$ with $\phi^{-1}(p) \ge b_0$ and $q \ge p+a_0$ with $\tau^{-1}(q) > \sigma(p)$, and we can put $a=q$ and $d=q-p$. We further notice the following: If for any a_0 there are $a \ge d \ge a_0$ such that $E \in (\mathrm{S}_{(a,d)})$, then $E \in (\mathrm{S}_\mathrm{t})$ (cf. Remark 3.2(iii)).

In Case II we can choose for a_0 numbers $a \ge d \ge a_0$ so that $\phi^{-1}(a-d) \ge b_0$ and $\tau^{-1}(a) > \sigma(a-d)$. For fixed $p = a-d$ we have

$$\limsup_a \frac{Ba+b-(1/A)(p-b-A)}{(1/D)(a-b-D)-Cp-b} \le BD.$$

In Case II we hence obtain the sufficient condition $(AB-1)ABC^2D^2 < 1$.

We next consider condition (i) of Corollary 4.11. For $d \in \mathbb{N}$ we put

$$X_d = \liminf_n \frac{\sigma(n)}{\phi^{-1}(n-d)} \quad \text{and} \quad Y = \liminf_n \frac{\psi(n)}{\tau^{-1}(n)}.$$

If $X_dY < 2$ for all $d \in \mathbb{N}$ then $\lambda^q(a) \in (\mathrm{S}_\mathrm{t})$. If ϕ, ψ, σ, τ are chosen as in Case II then we obtain the sufficient condition $ABCD < 2$.

Altogether we have proved that $\lambda^q(a) \in (\mathrm{S}_\mathrm{t})$ holds in all the cases listed in Theorem 4.12. This implies that the Nash-Moser theorem holds for $E = \lambda^q(a)$ under classical assumptions on the map Φ (cf. Corollary 3.8(v)). It seems remarkable that the conditions below are in general not tamely invariant. In particular, the conditions $(\mathrm{DN}) \cap (\Omega_\mathrm{t})$ or $(\mathrm{DN}_\mathrm{t}) \cap (\Omega_\mathrm{l})$ are sufficient for $\lambda^q(a) \in (\mathrm{S}_\mathrm{t})$. For properties $(\mathrm{DN}_{(\phi,\psi)})$ and $(\Omega_{(\tau,\sigma)})$ see also Remarks 1.3(i) and (iv).

THEOREM 4.12. *Let* $\phi, \psi, \sigma, \tau \in \mathfrak{F}$ *and* $\lambda^q(a) \in (\mathrm{DN}_{(\phi,\psi)}) \cap (\Omega_{(\tau,\sigma)})$, *and let* $A, B, C, D, \alpha, \beta \ge 1$ *and* $b \in \mathbb{N}_0$. *Assume there exists an* n_0 *such that* $\phi(n)$, $\psi(n)$, $\sigma(n)$, $\tau(n)$ *are for* $n \ge n_0$ *less than or equal to the terms listed in Table 2,*

Table 2

	$\phi(n)\le$	$\psi(n)\le$	$\sigma(n)\le$	$\tau(n)\le$	Condition
(i)	$\phi(n)$	$\psi(n)$	$n+b$	$n+b$	—
(ii)	$n+b$	$n+b$	$An+b$	$\tau(n)$	—
(iii)	$n+b$	$n+b$	$\sigma(n)$	$\tau(n)$	$\liminf_n \frac{\sigma(2n)\sigma(n)}{\tau^{-1}(n)n^2}=0$
(iv)	$n+b$	$n+b$	An^α	Bn^β	$1\le\alpha<1+1/2\beta$
(v)	$An+b$	$Bn+b$	$Cn+b$	$Dn+b$	$ABCD<2$
(vi)	$An+b$	$Bn+b$	$Cn+b$	$Dn+b$	$(AB-1)ABC^2D^2<1$
(vii)	$\phi(n)$	$\psi(n)$	$\sigma(n)$	$\tau(n)$	$\forall d\colon \liminf_n \frac{\sigma(n)}{\phi^{-1}(n-d)} \liminf_n \frac{\psi(n)}{\tau^{-1}(n)}<2$

and that the stated condition holds. Then it follows in each of the seven cases listed that $\lambda^q(a)\in(\mathrm{S}_t)$.

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