Minkowski Dimension and the Poincaré Exponent

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1. Statement of Results

If $K \subset \mathbb{R}^2$ is compact, let $N(K, \epsilon)$ be the minimal number of ϵ-balls needed to cover K. We define the *upper* and *lower Minkowski dimension* as

$$\overline{\mathrm{Mdim}}(K) = \limsup_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon},$$

$$\underline{\mathrm{Mdim}}(K) = \liminf_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon}.$$

If the two values agree, the common value is simply called the Minkowski dimension of K and is denoted $\mathrm{Mdim}(K)$.

Consider a group G of Möbius transformations acting on the 2-sphere S^2. Such transformations are identified with elements of $\mathrm{PSL}(2, \mathbb{C})$ in a natural way, and G is called *Kleinian* if it is discrete in this topology (i.e., the identity is isolated in G). G is called *elementary* if it contains a finite-index Abelian subgroup. In this paper we will consider only non-elementary groups. For a non-elementary group, the limit set $\Lambda(G)$ is the accumulation set (on S^2) of the orbit of any point $z_0 \in S^2$ (and is independent of the point). The complement $\Omega(G) = S^2 \setminus \Lambda$ is called the ordinary set. In this paper we will always assume that Ω is non-empty and that the group is conjugated in $\mathrm{PSL}(2, \mathbb{C})$ so $\infty \in \Omega$.

For any Kleinian group, the quotient $R = \Omega/G$ is a union of Riemann surfaces. We say that G is *analytically finite* if $R = R_1 \cup \cdots \cup R_s$ is a union of finite-type surfaces (i.e., each R_j is compact or compact with a finite number of punctures). The Ahlfors finiteness theorem says that if G is finitely generated then G is analytically finite.

If $z_0 \in \Omega(G)$ then the critical exponent (or Poincaré exponent) is defined as

$$\delta(G) = \inf\left\{s \colon \sum_{g \in G} \mathrm{dist}(g(z_0), \Lambda)^s < \infty\right\},$$

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where distance is in the spherical metric. It is easy to show that this exponent does not depend on the choice of z_0. Usually $\delta(G)$ is defined by extending the action of G on S^2 to a group of isometries of the hyperbolic 3-ball $\mathbb{B}^3 \subset \mathbb{R}^3$, and then considering the series

$$\sum_G \exp(1-|g(0)|)^s.$$

However, it is easy to see that this definition gives the same number.

THEOREM 1.1. *Suppose G is an analytically finite, non-elementary Kleinian group. If* $\text{area}(\Lambda(G))=0$ *then* $\delta(G)=\overline{\text{Mdim}}(\Lambda(G))$.

The assumption that G is non-elementary is needed in Theorem 1.1, for if G is a rank-1, cyclic, parabolic group then $\delta(G)=1/2$, but Λ is a single point. Define the Hausdorff content

$$H_\alpha^\infty(K)=\inf\{\sum r_j^\alpha : K\subset \bigcup_j D(x_j,r_j)\};$$

the infimum is over all coverings of K by disks, and

$$\dim(K)=\inf\{\alpha: H_\alpha^\infty(K)=0\}.$$

This is the Hausdorff dimension of K, and it is easy to see that $\dim(K)\le \underline{\text{Mdim}}(K)$. It is proved in Bishop and Jones [4, Thm. 1.1] that $\delta(G)\le\dim(\Lambda)$ for any non-elementary Kleinian group. Combining this result and Theorem 1.1 we easily deduce the following.

COROLLARY 1.2. *If G is an analytically finite Kleinian group then the Minkowski dimension of Λ exists and equals the Hausdorff dimension.*

COROLLARY 1.3. *If G is an analytically finite, non-elementary Kleinian group and $\Lambda(G)$ has zero area, then $\delta(G)=\dim(\Lambda)$.*

Different proofs of these results are given in [3] and [4] using estimates for the heat kernel on the hyperbolic 3-manifold associated to the Kleinian group G. The proof given here does not require these techniques—it is a purely "two-dimensional" argument. As such, it may be easier to adapt to other settings, for example, Julia sets of rational mappings.

G is called *geometrically finite* if it is finitely generated and there is a finite-sided fundamental polyhedron for the action of G on $\mathbb{B}$. The limit sets of such groups must have zero area [2], so our results apply to them. For geometrically finite groups, Corollary 1.2 was independently established by Stratmann and Urbański in [11]. Corollary 1.3 is also well known in this case (see e.g. [12]).

The paper is organized as follows. In Section 2 we define a related critical exponent δ_{Whit} and show that $\delta_{\text{Whit}}(K)\le\overline{\text{Mdim}}(K)$ for any compact K, with equality if $\text{area}(K)=0$. In Section 3 we show that $\delta\le\delta_{\text{Whit}}$ for analytically finite groups, with equality if $\Omega(G)/G$ is compact, and in Section 4 we define good and bad horoballs and prove a lemma giving some of their properties.

Section 5 contains our proof of the main theorem when most horoballs of G are good, and Section 6 our proof of the theorem in the case $\dim(\Lambda) = 2$. In Section 7 we state a lemma, and finish the proof assuming the lemma and $\dim(\Lambda) < 2$; the lemma is proved in Section 8.

NOTATION. In this paper $A \simeq B$ means that A/B is bounded and bounded away from 0. Given a square S in the plane and $\lambda > 0$, λS denotes the concentric square with $\operatorname{diam}(\lambda S) = \lambda \operatorname{diam}(S)$.

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2. Whitney Squares and Minkowski Dimension

A Whitney decomposition of a domain $\Omega \subset \mathbb{R}^2$ is a collection of disjoint (except for boundaries) squares $\{Q_j\}$ such that $\Omega = \bigcup_j Q_j$ and

$$\operatorname{diam}(Q) \le \operatorname{dist}(Q_j, \partial\Omega) \le 4 \operatorname{diam}(Q_j).$$

The existence of a Whitney decomposition for any open set is a standard fact in real analysis (see e.g. [10, Thm. VI.1]). One can simply take a maximal collection of dyadic squares in Ω such that $\operatorname{dist}(Q, \partial\Omega) \le \operatorname{diam}(Q)$.

For any compact set $K \subset \mathbb{R}^2$ we can define an exponent of convergence

$$\delta_{\text{Whit}} = \delta_{\text{Whit}}(K) = \inf\left\{s: \sum_{Q_j:\, \operatorname{dist}(Q_j, K) \le 1} \operatorname{diam}(Q_j)^s < \infty\right\}.$$

The sum is taken over all squares in a Whitney decomposition of $\Omega = K^c$ that are within distance 1 of K (we must drop the "far away" squares or the series will not converge). It is easy to check that this does not depend on the particular choice of Whitney decomposition.

LEMMA 2.1. *For any compact set K, $\delta_{\text{Whit}} \le \overline{\text{Mdim}}(K)$. If, in addition,* $\operatorname{area}(K) = 0$, *then $\delta_{\text{Whit}} = \overline{\text{Mdim}}(K)$.*

Proof. Suppose $\{Q_j\}$ is a Whitney decomposition of $\Omega = \mathbb{R}^n \setminus K$. For each Q_j with $\operatorname{diam}(Q_j) \le \dim(K)$, there is a dyadic cube Q_j' of the same size that hits K and satisfies $\operatorname{dist}(Q_j, Q_j') \le C \operatorname{diam}(Q_j)$. Clearly each Q_j' is associated to only a bounded number of Whitney cubes. Therefore the number of dyadic cubes of size 2^{-n} that hit K is at least $C2^{n(\delta_{\text{Whit}} - \epsilon)}$ (for n large enough, depending on ϵ). Thus $\delta_{\text{Whit}}(K) \le \overline{\text{Mdim}}(K)$.

Conversely, if K has zero area, Q is a dyadic square hitting K, and $\{Q_k\}$ is the collection of Whitney squares for Ω contained in Q, then

$$\sum_k \operatorname{diam}(Q_k)^2 = \operatorname{diam}(Q)^2.$$

Hence, for any $s \le 2$ (since $\operatorname{diam}(Q) \le 1$),

$$\sum_k \operatorname{diam}(Q_k)^s \ge \operatorname{diam}(Q)^s.$$

Since there are more than $C2^{n(\overline{\text{Mdim}}(K)-\epsilon)}$ such squares Q, the sum over the whole Whitney collection is greater than

$$C2^{-ns}2^{n(\overline{\text{Mdim}}(K)-\epsilon)},$$

which diverges if $s < \overline{\text{Mdim}}(K)-\epsilon$. Thus $\delta_{\text{Whit}}(K) \geq \overline{\text{Mdim}}(K)$, as desired. □

We can have strict inequality if K has positive area. For example, one can choose a set of disjoint disks $D(x_j, r_j) \subset D(0,1)$ so that $K = \overline{D(0,1)} \setminus \bigcup_j D_j$, K is nowhere dense and has positive area, and $r_j \to 0$ as fast as we wish. Summing the Whitney decomposition of a single disk yields

$$\sum_{Q_k \subset D_j} \text{diam}(Q_j)^s \simeq r_j^s$$

if $s > 1$ and ∞ if $s \leq 1$. By taking $r_j \to 0$ very fast, we can get

$$\delta_{\text{Whit}}(K) = 1 < 2 = \overline{\text{Mdim}}(K).$$

3. Whitney Squares and the Poincaré Series

In this section we explain the elementary relations between δ and δ_{Whit}.

Suppose Ω is a domain in S^2 with more than two boundary points. Then Ω has a hyperbolic metric ρ defined by the covering map from the disk to Ω. Let $d(z) = \text{dist}(z, \partial\Omega)$. For a general domain (e.g., [6, Thm. 4.3]),

$$\frac{(1+o(1))|dz|}{d(z)\log 1/d(z)} \leq |d\rho(z)| \leq 2\frac{|dz|}{d(z)}.$$

A set $K \subset \mathbb{R}^2$ is called *uniformly perfect* if there is a constant $C < \infty$ such that

$$\frac{1}{C}\frac{|dz|}{d(z)} \leq |d\rho(z)| \leq 2\frac{|dz|}{d(z)}$$

on each component Ω of $S^2 \setminus K$. (This is one of many equivalent definitions; see [7] and [8].)

The limit set of any finitely generated group is uniformly perfect [5; 9]. In fact, the proof in Canary's paper [5] shows this to be true under the weaker assumption that there is an $\epsilon_0 > 0$ such that any closed geodesic on Ω/G has length $\geq \epsilon_0$. This is certainly true if Ω/G is a finite union of finite-type surfaces, so the result is still true for analytically finite groups.

LEMMA 3.1. *If G is any non-elementary Kleinian group with $\Lambda \neq S^2$ then $\delta \leq \delta_{\text{Whit}}$. If $\Omega(G)/G$ is compact then $\delta = \delta_{\text{Whit}}$.*

Proof. Fix a point $z_0 \in \Omega(G)$ (not an elliptic fixed point). There is a small hyperbolic disk around z_0 (with radius r_0 depending on z_0) that projects injectively to $R = \Omega/G$ under the quotient map. Hence points in $G(z_0)$, the orbit of z_0 under G, are separated by at least r_0 in the hyperbolic metric. By (3.1) each Whitney square has a uniformly bounded hyperbolic diameter and area. Thus each Whitney square for $\Omega = S^2 \setminus \Lambda$ contains at most a bounded number M (depending on z_0 and G) of points in $G(z_0)$. Therefore,

$$\sum_{g\in G} \operatorname{dist}(g(z_0), \Lambda)^s \leq M \sum_j \operatorname{diam}(Q_j)^s$$

and hence $\delta(G) \leq \delta_{\text{Whit}}(\Lambda(G))$.

Now suppose $R = \Omega(G)/G = R_1 \cup \cdots \cup R_s$ is a finite union of compact Riemann surfaces. We can choose points $E = \{z_1, \ldots, z_s\} \subset \Omega$ so that z_j projects into R_j, $j = 1, \ldots, s$, under the quotient map. By compactness, any point $z \in \Omega$ is a bounded hyperbolic distance from $G(E)$, the orbit of E under G. For each square Q with $\operatorname{dist}(Q, \Lambda) \leq 1$, choose a closest point $z_Q \in G(E)$. Then z_Q is only a bounded hyperbolic distance from Q, so the uniform perfectness of Λ implies

$$\operatorname{diam}(Q) \leq C \operatorname{dist}(z_Q, \Lambda).$$

Furthermore, only a bounded number (say M) of the Q_js are associated to any given point of $G(E)$. Thus

$$\sum_j \operatorname{diam}(Q_j)^s \leq MC^s \sum_{z_j \in E} \sum_{g\in G} \operatorname{dist}(g(z_j), \Lambda)^s,$$

and therefore $\delta_{\text{Whit}}(\Lambda(G)) \leq \delta(G)$. □

One of the main results of [3] is that $\delta_{\text{Whit}} = \delta$ for any non-elementary analytically finite group. This fact and Lemma 2.1 imply Theorem 1.1, but the fact seems harder than the theorem. The purpose of this note is to give a proof of the theorem that does not require proving $\delta_{\text{Whit}} = \delta$.

4. Good and Bad Horoballs

A horoball in $\Omega(G)$ is a Euclidean ball $B \subset \Omega \subset S^2$ that is invariant under a rank-1 parabolic subgroup of G. The fixed point p of the parabolic element is on the boundary of the horoball and corresponds to a cusp on the surface $\Omega(G)/G$. We say that B is doubly cusped if there is another (disjoint) ball B_1 fixed by the same subgroup.

Suppose $R = \Omega(G)/G$ is a finite union of finite-type Riemann surfaces $R_1, \ldots, R_N$; that is, each is a compact surface with at most a finite number of punctures. Let $\{p_1, \ldots, p_m\}$ be the punctures in $R = \bigcup_{i=1}^N R_i$, and for each p_i let B_i^* be a neighborhood of p_i which lifts to a Euclidean ball B_i in Ω that is invariant under some parabolic element of G (see [1, Lemma 1]). Then $X = R \setminus \bigcup_j B_j^*$ is compact, so we can choose a finite set of points $E = \{z_1, \ldots, z_P\} \subset \Omega(G)$ that project to a 1-dense subset of X (i.e., every point of X is within hyperbolic distance 1 of a point of E). For the remainder of the paper we fix a finite collection of such horoballs $\mathcal{B} = \{B_j\}$, one from each equivalence class of horoballs in Ω. It will also be convenient to assume that B_i is contained inside a larger horoball $\hat{B}_i \subset \Omega$ of twice the Euclidean diameter (we can always do this by just taking smaller balls if necessary). If $\infty \in \Omega$

then we may also assume that ∞ is not contained in any of the B_i. Finally, if a parabolic point is doubly cusped, we require that $\mathcal{B}$ contain horoballs of equal size for both "sides" of the parabolic point.

For $z \in \Omega(G)$ we define $d(z) = \text{dist}(z, \Lambda)$ in the Euclidean metric. Normalize the group so that $\infty \in \Omega$ and $\Lambda(G)$ has diameter 1. Now suppose that p is a parabolic fixed point of G and that $B \subset \Omega$ is a horoball at p. By our preceding remarks, B must be the image of one of our finite collection of horoballs $\{B_j\}$ under some element g of G; that is, $B = g(B_j)$.

We would like to call B good or bad depending on whether the map g: $B_j \to B$ is "close" to being linear. Using a linear Möbius transformation, we can map p to 0 and the horoball B to the disk $D(i/2, 1/2)$. The parabolic element of G fixing p is conjugated by the linear map to a transformation of the form

$$\tau(z) = \frac{z}{1+\eta z}.$$

We define $\eta(B) = |\eta|$. Given $\eta > 0$, we say a horoball $B = g(B_i)$ is an "η-bad" horoball if $\eta(B) \leq \eta$, and is "η-good" otherwise.

A helpful way to think about good and bad horoballs is as follows. Suppose h is a generator of the parabolic subgroup fixing B. Define

$$\eta'(B) = \sup_{z \in \partial B} \frac{|h(z) - z|}{\text{diam}(B)}.$$

Then it is easy to see that $\eta' \simeq \eta$. In other words, B is η-bad if the parabolic subgroup fixing B has a generator that is close to the identity on B. Alternatively, the group G looks less and less discrete on horoballs with smaller and smaller η.

LEMMA 4.1. *Suppose G is analytically finite and normalized so that $\infty \in \Omega(G)$ and* $\text{diam}(\Lambda) = 1$. *Fix a finite collection of horoballs $\mathcal{B}$ as above, and assume ∞ is not in any of these horoballs. Then*:

(1) *There is a C_1 (depending only on η) such that, for any η-good horoball B and any $w \in \partial B$, there is a point $z \in G(E)$ such that*

$$C_1^{-1} d(w) \leq d(z) \leq C_1 d(w),$$
$$C_1^{-1} d(w) \leq |z - w| \leq C_1 d(w).$$

(2) *There is an η_2 such that if B is η_2-bad then it is singly cusped; that is, there is not a disjoint horoball also tangent to p.*

(3) *There is an $\eta_3 > 0$ such that, if $\eta \leq \eta_3$ and B is η-bad and $D(x, r) \subset \Omega$ with* $\frac{3}{2}\text{diam}(B) \leq \text{dist}(x, B) \leq \text{diam}(B)/(2\eta)$, *then* $r \leq C_2 \eta\, \text{dist}(x, B)^2$, *where the constant C_2 depends only on G.*

(4) *For any $\delta > 0$ there is an $\eta_4 > 0$ (depending only on δ) such that if B is an η_4-bad horoball then there is a disk $D \subset 3B$ such that* $\text{diam}(D) \geq \frac{1}{3}\text{diam}(B)$, *and $D \setminus \Lambda$ contains no balls of radius* $\geq \delta\, \text{diam}(D)$.

(5) *There is an $\eta_5 > 0$ such that, if B_1 and B_2 are η_5-bad horoballs with* $\text{diam}(B_1) \leq \text{diam}(B_2)$, *then* $\text{dist}(B_1, B_2) \geq 100\, \text{diam}(B_1)$.

(6) *If B_1, B_2 are horoballs with*

$$\operatorname{diam}(B_1) \le \operatorname{diam}(B_2) \le 2\operatorname{diam}(B_1),$$

and if $\operatorname{dist}(B_1, B_2) \le A \operatorname{diam}(B_1)$, *then both B_1 and B_2 are A^{-2}-good.*

Proof. To prove (1), note that it is true for all the balls in $\mathcal{B}$ by finiteness. If B is η-good then B is the image of some B_j in our fixed, finite collection under a map that is the composition of a conformal linear map and a map of bounded distortion (depending on η).

To prove (2), suppose $B_1, B_2 \in \mathcal{B}$ are paired horoballs at a doubly cusped parabolic point p with parabolic generator h. For $i = 1, 2$, let z_i be the point on ∂B_i farthest from p and consider the cross-ratio of $z_1, h(z_1), z_2, h(z_2)$. Now suppose $B = g(B_1)$ is some η-bad image of B_1, and suppose we have chosen g so that $|g(z_1) - g(p)|$ is maximized among elements mapping B_1 to B. Since cross-ratio is preserved by Möbius transformations, we can deduce that

$$|g(z_1) - g(h(z_1))| \simeq |g(z_1) - g(z_2)| \simeq |g(z_2) - g(h(z_2))| \simeq \eta \operatorname{diam}(B).$$

Thus, all four points are mapped within $C\eta(B)\operatorname{diam}(B)$ of $g(z_1)$. If η is small enough, this is only possible if $\infty \in g(B_2)$, contary to hypothesis.

To prove (3) note that it is sufficient to demonstrate the case where $p = 0$, $B = D(i/2, 1/2)$, and the subgroup fixing p is generated by

$$h(z) = \frac{z}{1 \pm \eta z}.$$

Then

$$|h(x) - x| = \left| x - \frac{x}{1 \pm \eta x} \right| \le 2\eta |x|^2 \le C_0 \eta \operatorname{dist}(x, B)^2.$$

Suppose $C_2 > 100 C_0$ and that $r \ge C_2 \eta \operatorname{dist}(x, B)^2$. Then both x and $h(x)$ are in the disk $D(x, r) \subset \Omega$, so the line segment connecting them projects to a loop on Ω/G of hyperbolic length $\le 10 C_0 / C_2$. If C_2 is large enough (depending only on G), this implies the loop is contained in a neighborhood of a cusp on Ω/G; hence the lift is in a horoball of Ω that is tangent to 0. This horoball is obviously not B since B does not contain x, so there is a second horoball B' at 0. This contradicts part (2), so we are done.

The final three statements are all easy consequences of part (3). □

We noted in the previous section that if $\Omega(G)/G$ is compact then there is a close connection between the Poincaré and Whitney sums. When there are punctures in $\Omega(G)/G$, we must take account of the fact that horoballs contain many Whitney squares but no orbit points. The next observation is very easy and is left to the reader.

LEMMA 4.2. *Suppose Ω is an open set and $B \subset \Omega$ is a Euclidean ball such that $\partial B \cap \partial \Omega \ne \emptyset$. Let $\{Q_j\}$ be a Whitney decomposition for Ω. Then if $s > 1$,*

$$\sum_{j: Q_j \cap B \neq \emptyset} \operatorname{diam}(Q_j)^s \simeq \operatorname{diam}(B)^s,$$

where the constants depend only on s.

The following lemma will be useful later.

LEMMA 4.3. *Suppose G is analytically finite and $E \subset \Omega$ is a finite set with one point in each equivalence class of components. Assume the group has been normalized so that $\infty \in \Omega$. Suppose $\{Q_j\}$ is a Whitney decomposition for $\Omega(G)$, and let $\mathcal{B}$ be a choice of horoballs for G as before. Then if $s > 1$,*

$$\sum_{j: \operatorname{diam}(Q_j) \leq 1} \operatorname{diam}(Q_j)^s \simeq \sum_{z \in E} \sum_{g \in G} \operatorname{dist}(g(z), \Lambda)^s + \sum_{B \in \mathcal{B}} \operatorname{diam}(B)^s,$$

where the constants depend on G, s, E, $\mathcal{B}$. If $\eta > 0$ then

$$\sum_{j: \operatorname{diam}(Q_j) \leq 1} \operatorname{diam}(Q_j)^s \simeq \sum_{z \in E} \sum_{g \in G} \operatorname{dist}(g(z), \Lambda)^s + \sum_{B \in \mathcal{B}, \eta(B) \leq \eta} \operatorname{diam}(B)^s,$$

where the constants depend on G, s, E, $\mathcal{B}$, η.

Proof. The first equation is simply the observation that Whitney squares that do not hit any horoballs can be associated (as in the previous section) to orbits of E, whereas the Whitney squares that do hit a horoball are controlled by the previous lemma. The second equation is proved using part (1) of Lemma 4.1 to associate to each η-good horoball a nearby orbit point. Thus, the part of the horoball sum we are omitting is controlled by the orbit sum. □

5. Theorem 1.1 When G Has Many Good Horoballs

We now start the proof of Theorem 1.1. It is enough to prove the following.

THEOREM 5.1. *If G is an analytically finite group and* $\operatorname{area}(\Lambda(G)) = 0$, *then* $\delta(G) = \delta_{\text{Whit}}(\Lambda(G))$.

Let $D = \overline{\operatorname{Mdim}}(\Lambda) = \delta_{\text{Whit}}(\Lambda(G))$ and $d = \dim(\Lambda)$. If G has no parabolic elements then $\Omega(G)/G$ is compact, and we have already proven this case in Lemma 3.1. We may therefore assume that G has parabolics, which implies $\delta(G) \geq 1/2$. If $D = 1/2$, we have nothing to do, so we may assume that $D > 1/2$.

Suppose $\epsilon > 0$ is so small that $D - \epsilon > 1/2$. Let $E = \{z_1, \dots, z_s\}$ be a finite collection of points in Ω, one projecting to each component of Ω/G. We will show that

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} = \infty$$

and thus $\delta(g) \geq D$.

Choose an integer n_0 so that

$$N(\Lambda, 2^{-n_0}) \geq 1000 \cdot 2^{n_0(D-\epsilon/2)}.$$

By passing to a subcollection with at least $2^{n_0(D-\epsilon/2)}$ elements, we claim that we may assume that for any two squares S_j, S_k we have $9S_j \cap 9S_k = \emptyset$. This is easy—just enumerate the list of squares and inductively remove any square S_k for which there is a $j<k$ with $9S_j \cap 9S_k \neq \emptyset$. Since $9S_j \cap 9S_k \neq \emptyset$ implies $S_k \subset 15S_j$, each S_j can cause at most $30^2 = 900$ later squares to be removed. Thus the final list has at least $2^{n_0(D-\epsilon/2)}$ elements.

Let $r = 3\cdot 2^{-n_0}$. *Let* $\mathcal{S} = \{S_k\}$ be a collection of $2^{n_0(D-\epsilon/2)}$ squares of size r, so that the triples $3S_k$ are pairwise disjoint and $\frac{1}{3}S_k \cap \Lambda \neq \emptyset$ for each k.

First we deal with the case when most of the horoballs of size r are good. Let $\eta > 0$ (to be fixed later). For each η-good horoball with $\operatorname{diam}(B) \geq r/3$, let $\mathcal{G}_B$ be the collection of squares in $\mathcal{S}$ which are such that $\frac{1}{3}S$ hits B. Let $\mathcal{G}$ be the union of all the $\mathcal{G}_B$.

The proof breaks into three cases:

(1) $\#(\mathcal{S} \cap \mathcal{G}) \geq \frac{1}{2}\#(\mathcal{S})$ for all large enough n;
(2) $\dim(\Lambda) = 2$;
(3) $\#(\mathcal{S} \cap \mathcal{G}) < \frac{1}{2}\#(\mathcal{S})$ for infinitely many n and $\dim(\Lambda) < 2$.

If η is chosen small enough (depending on G), then one can show that the last case is impossible (see e.g. [4]). However, this is a hard result using heat kernel estimates on hyperbolic 3-manifolds, and one of our purposes here is to give a self-contained proof that uses only two-dimensional techniques.

Proof of Case 1. By part (1) of Lemma 4.1, there is an orbit point $z \in G(E) \cap B$ such that $d(z) \simeq \operatorname{diam}(B)$. For this point,

$$d(z)^{D-\epsilon} \geq C \sum_{S \in \mathcal{G}_B} \operatorname{diam}(S)^{D-\epsilon}.$$

If more than half the squares in $\mathcal{S}$ belong to $\mathcal{G}$ then this argument shows

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} \geq \frac{1}{2} C 2^{n_0 \epsilon/2}.$$

If this happens for arbitrarily large n_0 then

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} = \infty,$$

so we have shown that $\delta(G) \geq D - \epsilon$, as desired. □

6. Proof of Case 2 of Theorem 1.1

Case 2 follows easily from the following lemma.

LEMMA 6.1. *Suppose G is an analytically finite Kleinian group, normalized so that $\infty \in \Omega(G)$ and* $\operatorname{diam}(\Lambda(G)) = 1$. *If $\delta_{\text{Whit}} = 2$ then $\delta = 2$.*

Proof. We know the lemma if $\Omega(G)/G$ is compact (see Section 2), so we may assume that $\Omega(G)$ contains horoballs. Consider the sum over all Whitney squares for Ω,

$$\sum_j \operatorname{diam}(Q_j)^{2-2\epsilon}.$$

By the definition of δ_{Whit}, this diverges. Using Lemma 4.3, we can split the sum into two pieces; one corresponding to all squares Q_j that hit some horoball and the other corresponding to all Whitney squares that miss every horoball:

$$\sum_{j:\,\operatorname{diam}(Q_j)\le 1} \operatorname{diam}(Q_j)^{2-2\epsilon} \simeq \sum_{z\in E}\sum_{g\in G} \operatorname{dist}(g(z),\Lambda)^{2-2\epsilon} + \sum_{B\in\mathcal{B}} \operatorname{diam}(B)^{2-2\epsilon}.$$

If the first sum on the right diverges then $\delta > 2-2\epsilon$ and we are done. Thus we may assume the second sum on the right diverges for all $\epsilon > 0$. Hence if

$$\mathcal{B}_n = \{B_j : 2^{-n-1} \le \operatorname{diam}(B_j) < 2^{-n}\}$$

and $N_n = \#\mathcal{B}_n$, we must have $N_n \ge 2^{n(2-2\epsilon)}$ for infinitely many values of n. Fix a value of n_0 where this holds and note that, for at least half the balls B in $\mathcal{B}_n$, there is a second ball $B' \in \mathcal{B}_n$ such that

$$\operatorname{dist}(B, B') \le 2^{-n(1-\epsilon)} \le \operatorname{diam}(B)2^{n\epsilon};$$

otherwise we would have so many disjoint balls of radius $2^{-n(1-\epsilon)}$ that they could not all be contained in a bounded neighborhood of Λ (recall that $\operatorname{diam}(\Lambda) = 1$).

By part (4) of Lemma 4.1, this implies that for such a pair both B and B' are $2^{-2\epsilon n}$-good horoballs. Let $\mathcal{G}_{n_0} \subset \mathcal{B}_{n_0}$ be the subcollection of $2^{-2\epsilon n}$-good horoballs. For any η-good horoball B let z be the point given in part (1) of Lemma 4.1 such that $d(z) \ge C\eta \operatorname{diam}(B)$. Let H be the parabolic subgroup fixing B_j. Then an easy calculation shows that there are at least $C\eta^{-1}$ orbits of z under H with distance $\ge C\eta \operatorname{diam}(B)$ from Λ. Thus

$$\sum_{h\in H} d(h(z))^\alpha \ge C \operatorname{diam}(B)^\alpha \eta^{\alpha-1}$$

for any $1 < \alpha \le 2$ and some C depending on G and α.

Thus, if z_j is the good point in B_j given by (1) of Lemma 4.1,

$$\begin{aligned}\sum_{z\in G(E)} d(z)^\alpha &\ge C \sum_{B_j\in\mathcal{G}_{n_0}} \sum_{k\in\mathbb{Z}} d(h^k(z_j))^\alpha \\ &\ge C \sum_{B_j\in\mathcal{G}_{n_0}} \operatorname{diam}(B)^\alpha 2^{-2\epsilon n_0(\alpha-1)} \\ &\ge C 2^{n_0(2-\epsilon)} 2^{-n_0\alpha} 2^{-2\epsilon n_0(\alpha-1)}.\end{aligned}$$

Taking $\epsilon = 0$ and solving for α, we see this diverges for small enough ϵ if $\alpha < 2$. Thus $\delta \ge 2$, as desired. □

7. Theorem 1.1 When G Has Few Good Horoballs

Case 3 assumes that fewer than half the elements of $\mathcal{S}$ are in $\mathcal{G}$ for all large enough n_0 (i.e., we assume most horoballs are bad) and that $\dim(\Lambda) < 2$.

We use the "stopping time" construction described by the following lemma. Recall that $d = \dim(\Lambda)$ and $D = \overline{\text{Mdim}}(\Lambda)$.

LEMMA 7.1. *Suppose $\epsilon > 0$ and $r > 0$. There is a constant C_0 (depending only on G and ϵ) and constants $\eta_0 > 0$ and ν_0 (depending on G, ϵ, and r) such that the following holds: Suppose we have a square S such that $\frac{1}{3}S \cap \Lambda \neq \emptyset$ and S does not intersect any η_0-good horoball with diameter* $\geq \text{diam}(S)/3$. *Then either* (a)

$$\sum_{z \in S \cap G(E)} d(z)^{D-\epsilon} \geq \nu_0 \, \text{diam}(S)^{D-\epsilon}$$

or (b) *there is a collection of subsquares $\mathcal{C}(S) = \{S_j\} \subset S$ such that*

(1) $\text{diam}(S_j) \leq r \, \text{diam}(S)$ *for all j,*
(2) $\{3S_j\} \subset S$ *and are pairwise disjoint,*
(3) $S_j \backslash \Lambda$ *does not contain a ball of radius* $\text{diam}(S_j)/10$, *and*
(4) $\sum_j \text{diam}(S_j)^2 \geq C_0 \, \text{diam}(S)^2$.

Let us assume that the lemma holds and see how to finish the proof. We will prove the lemma in the next section.

Suppose $\epsilon > 0$. Let $C_0 = C_0(\epsilon, G)$ be as given by the lemma. Let $r = C_0^{1/\epsilon}$ and let η_0, ν_0 be as given by the lemma. We define generations of nested squares $\mathcal{S}_0, \mathcal{S}_1, \ldots$ as follows. Let $\mathcal{S}_0 = \mathcal{S}$ be the collection considered in the previous section; that is, $\mathcal{S}_0$ is a collection of $\frac{1}{2} \cdot 2^{n_0(D-\epsilon/2)}$ squares of size $r = 2^{-n_0}$ with disjoint triples, so that $\frac{1}{3}S \cap \Lambda \neq \emptyset$ and S does not hit any η-good horoball of size $\geq \text{diam}(S)/3$.

Suppose $\mathcal{S}_n$ has been defined. Let $\mathcal{A}_1^n \subset \mathcal{S}_n$ be the collection of squares for which alternative (a) holds and $\mathcal{A}_2^n$ the subcollection for which alternative (b) holds. Then we define

$$\mathcal{S}_{n+1} = \bigcup_{S \in \mathcal{A}_2^n} \mathcal{C}(S).$$

In other words: Given $\mathcal{S}_n$, we defined $\mathcal{S}_{n+1}$ by throwing away all the squares where alternative (a) of Lemma 7.1 holds; for each square $S \in \mathcal{S}_n$ where alternative (b) holds, we replace it by the collection $\mathcal{C}(S)$ of subsquares satisfying (1)-(4) in Lemma 7.1.

To each square $S \in \mathcal{S}_\infty = \bigcup_n \mathcal{S}_n$ we associate a positive number $\mu(S)$ as follows. For $S \in \mathcal{S}_0$ let

$$\mu(S) = \text{diam}(S)^{D-\epsilon}. \tag{7.1}$$

For $S \in \mathcal{S}_n$, $n \geq 1$, there is a unique $S_0 \in \mathcal{S}_{n-1}$ containing S (i.e., S_0 is its "parent") and by definition alternative (b) holds for S_0. Set

$$\mu(S) = \frac{\text{diam}(S)^2}{\sum_{S' \in \mathcal{C}(S_0)} \text{diam}(S')^2} \mu(S_0). \tag{7.2}$$

Note that $\sum_{S' \in \mathcal{C}(S_0)} \mu(S') = \mu(S_0)$. Let $\mu(n) = \sum_{S \in \mathcal{S}_n} \mu(S)$. It is clear that $\{\mu(n)\}_{n=0}^{\infty}$ is non-increasing.

LEMMA 7.2. *If* $\mu(n) \not\to 0$ *then* $\dim(\Lambda) \geq 2-\epsilon$. (*Recall that* μ *depends on* ϵ.)

Proof. Suppose $\mu(n) \not\to 0$. Then the numbers $\{\mu(S)\}$ define a positive measure on the set $Y = \bigcap_n \bigcup_{S \in \mathcal{S}_n} S \subset \Lambda$. We claim that

$$\mu(S) \leq C \operatorname{diam}(S)^{2-\epsilon}$$

for every square with $\operatorname{diam}(S) \leq r$, where C is a constant that depends on the choice of n_0 but not on S. We first verify this by induction for squares in $\mathcal{S}_\infty$. If $S \in \mathcal{S}_0$, the claim is true by definition with $C = A_0 = 2^{n_0(2-D)}$.

If $S \in \mathcal{S}_n$ is contained in $S_0 \in \mathcal{S}_{n-1}$, then part (4) of Lemma 7.1 implies

$$\mu(S) = \frac{\operatorname{diam}(S)^2}{\sum_{S' \in \mathcal{C}(S_0)} \operatorname{diam}(S')^2}\mu(S_0) \leq A_0 \frac{\operatorname{diam}(S)^2}{C_0 \operatorname{diam}(S_0)^2} \operatorname{diam}(S_0)^{2-\epsilon}$$

and hence

$$\begin{aligned}\mu(S) &\leq A_0 C_0^{-1} \operatorname{diam}(S)^2 \operatorname{diam}(S_0)^{-\epsilon} \\ &\leq A_0 C_0^{-1} r^{-\epsilon} \operatorname{diam}(S)^{2-\epsilon} \\ &\leq A_0 \operatorname{diam}(S)^{2-\epsilon}\end{aligned}$$

as desired. Also note (for future use) that this proves the weaker estimate

$$\mu(S) \leq \operatorname{diam}(S)^{D-\epsilon}. \tag{7.3}$$

Now consider a general square $S \subset S_0 \in \mathcal{S}_0$. Let S_1 be the smallest square in $\mathcal{S}_\infty$ containing S. Suppose $S_1 \in \mathcal{S}_{n-1}$. Then, by (7.2),

$$\begin{aligned}\mu(S) &= \frac{\sum_{S' \in \mathcal{S}_n, S' \subset S} \operatorname{diam}(S')^2}{\sum_{S' \in \mathcal{S}_n, S' \subset S_1} \operatorname{diam}(S')^2}\mu(S_1) \\ &\leq A_0 \frac{\operatorname{diam}(S)^2}{C_0 \operatorname{diam}(S_1)^2} \operatorname{diam}(S_1)^{2-\epsilon} \\ &\leq A_0 C_0^{-1} \operatorname{diam}(S)^2 \operatorname{diam}(S_1)^{-\epsilon} \\ &\leq A_0 C_0^{-1} \operatorname{diam}(S)^{2-\epsilon} \left(\frac{\operatorname{diam}(S_1)}{\operatorname{diam}(S)}\right)^{-\epsilon} \\ &\leq A_0 C_0^{-1} \operatorname{diam}(S)^{2-\epsilon}.\end{aligned}$$

Thus the inequality holds for general squares with the constant $C = A_0 C_0^{-1}$.

If $\{S_j\}$ were any covering of Λ then we would have

$$0 < \mu(\Lambda) \leq \sum_j \mu(S_j) \leq C \sum_j \operatorname{diam}(S_j)^{2-\epsilon}.$$

Therefore, $\dim(\Lambda) \geq 2-\epsilon$ as desired. □

By Lemma 7.2, if $\epsilon < 2 - \dim(\Lambda)$, then we must have $\mu(n) \to 0$. Assume this is the case.

Recall that $\mathcal{A}_1^n$ is the collection of all squares in $\mathcal{S}_n$ where alternative (a) held (i.e., the construction above stopped) and that $\mathcal{A}_2^n$ are the remaining squares where alternative (b) held. Note that $\bigcup_n \mathcal{A}_1^n$ is a union of disjoint squares. For $S \in \mathcal{A}_2^{n-1}$, let $\mathcal{A}_1(S)$ be the collection of all squares in $\mathcal{C}(S) \subset \mathcal{S}_n$ where alternative (a) holds and let $\mathcal{A}_2(S)$ be the squares where alternative (b) holds. For $S' \in \mathcal{A}_1(S) \in \mathcal{A}_1^n$, define

$$\nu(S') = \sum_{z \in S' \cap \mathcal{G}(E)} d(z)^{D-\epsilon}.$$

Since alternative (a) of Lemma 7.1 holds for S', $\nu(S') \geq \nu_0 \operatorname{diam}(S')^{D-\epsilon}$ and so $\nu(S') \geq \nu_0 \mu(S')$ by (7.3) ($\mu(S')$ must be defined because alternative (b) holds for its parent). Thus,

$$\sum_{S' \in \mathcal{A}_1(S)} \nu(S') \geq \nu_0 \sum_{S' \in \mathcal{A}_1(S)} \mu(S') = \nu_0 \left[\mu(S) - \sum_{S' \in \mathcal{A}_2(S)} \mu(S') \right].$$

Therefore,

$$\begin{aligned} \sum_{S' \in \mathcal{A}_1^n} \nu(S') &\geq \nu_0 \left[\sum_{S' \in \mathcal{A}_2^{n-1}} \mu(S') - \sum_{S'' \in \mathcal{A}_2^n} \mu(S'') \right] \\ &= \nu_0 [\mu(n-1) - \mu(n)]. \end{aligned}$$

Hence, since $\mu(n) \to 0$, a telescoping series argument gives

$$\sum_{n=0}^{\infty} \sum_{S' \in \mathcal{A}_1^n} \nu(S') \geq \nu_0 \sum_{n=0}^{\infty} (\mu(n-1) - \mu(n)) = \nu_0 \mu(0).$$

Thus by (7.1),

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{S' \in \mathcal{A}_1^n} \nu(S') &\geq \nu_0 \sum_{S \in \mathcal{S}_0} \operatorname{diam}(S)^{D-\epsilon} \\ &\geq \nu_0 \#(\mathcal{S}_0) 2^{-n_0(D-\epsilon)} \\ &= \nu_0 2^{n_0(D-\epsilon/2)} 2^{-n_0(D-\epsilon)} \\ &= \nu_0 2^{n_0 \epsilon/2}. \end{aligned}$$

Thus, since the squares in $\bigcup_n \mathcal{A}_1^n$ are all disjoint,

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} \geq \sum_n \sum_{S \in \mathcal{A}_1^n} \nu(S) \geq \nu_0 2^{n_0 \epsilon/2}.$$

Taking $n_0 \to \infty$ proves $\delta(G) \geq D - \epsilon$. Taking $\epsilon \to 0$ shows that $\delta(G) \geq D = \overline{\text{Mdim}}(\Lambda)$, as desired.

8. Proof of Lemma 7.1

The idea of the proof is that the two alternatives for S in the lemma simply depend on whether S contains many bad horoballs or not. If it does not then alternative (a) holds, but if it does then (b) is true.

Let $\eta_0 > 0$ (we will show the lemma is correct if η_0 is small enough, depending only on G, ϵ, and r). Let H_g be the union of the η_0-good horoballs and let H_b be the union of all the η_0-bad horoballs. Let $U = \Omega \setminus (H_b \cup H_g)$ (this is part of $\Omega(G)$ that projects to the "compact part" of $\Omega(G)/G$. The proof divides into three cases depending on the relative sizes of $S \cap H_b$,

$S \cap H_g$, and $S \cap U$. Fix $\epsilon > 0$ and let $E \subset \Omega$ be a finite set so that points in $G(E)$ are at least hyperbolic distance ϵ apart, but so that every point of U is within 10ϵ of some point of $G(E)$.

Case A: First suppose that "a lot" of S corresponds to the compact part of $\Omega(G)/G$. More precisely, let $U_S = \frac{1}{3}S \cap U$ and assume

$$\text{area}(U_S) \geq \frac{1}{2000}\,\text{area}(S).$$

We will show that alternative (a) holds for S. If ϵ in the definition of E is small enough (depending only on the uniformly perfect constant of Λ), then for each point $w \in U$ there is a point $z \in G(E)$ such that $|z-w| \leq d(z)$. Moreover, each $z \in G(E)$ is the center of a disk $D_z = D(z, (\epsilon/100)d(z))$. These disks are all disjoint, but they cover a fixed fraction of the area of U. This is because, given any point of U, we can move hyperbolic distance $\leq 10\epsilon$ and reach one of the ϵ-disks. Moreover, since $\Lambda \cap \frac{1}{3}S \neq \emptyset$, moving hyperbolic distance 10ϵ does not allow leaving S from starting inside $\frac{1}{3}S$ (if ϵ is small enough).

Hence,

$$\sum_{z \in G(E) \cap U_S} \text{area}(D_z) \geq A\,\text{area}(U_S) \geq \frac{A}{2000}\,\text{area}(S).$$

Thus,

$$\sum_{z \in G(E) \cap \frac{1}{3}S} d(z)^2 \geq C\,\text{diam}(S)^2.$$

Then, since $d(z) \leq \text{diam}(S)$ for all $z \in G(E) \cap S$,

$$\sum_{z \in G(E) \cap \frac{1}{3}S} d(z)^{D-\epsilon} \geq C\,\text{diam}(S)^{D-\epsilon},$$

and we are done.

Case B: Now we assume that "a lot" of S lies in H_g. Let $V_S = \frac{1}{3}S \cap H_g$, and suppose

$$\text{area}(V_S) \geq \frac{1}{2000}\,\text{area}(S)$$

(i.e., the part of S in good horoballs has large area). By hypothesis, the only η_0-good horoballs hitting $\frac{1}{3}S$ have diameter $\leq \frac{1}{3}S$, and so we can associate to each such horoball B a point $z \in G(E) \cap CS$ such that $d(z) \simeq \epsilon\,\text{diam}(B)$. Hence, as before,

$$\sum_{z \in G(E) \cap CS} d(z)^2 \geq C \sum_{\text{good horoballs in } S} \text{diam}(B)^2 \geq C\,\text{diam}(S)^{D-\epsilon} \simeq \text{area}(S).$$

Since $D-\epsilon < 2$, this and bounded overlaps of the squares CS imply

$$\sum_{z \in G(E) \cap CS} d(z)^{D-\epsilon} \geq C\,\text{diam}(S)^{D-\epsilon}.$$

Thus alternative (a) holds in this case also.

Case C: Now assume

$$\text{area}(U_S)+\text{area}(V_S) \le \frac{1}{1000}\,\text{area}(S).$$

Since Λ has zero area, the "bad" part of S must have large area; that is,

$$\text{area}\left(\frac{1}{3}S \cap H_b\right) \ge \frac{99}{100}\,\text{area}(S).$$

Next we want to show that we may assume that $\frac{1}{3}S$ does not hit any bad horoball of comparable size.

Suppose that $\frac{1}{3}S$ hits an η_0-bad horoball with $\text{diam}(B) \ge (1/10)\,\text{diam}(S)$. If η_0 is small enough then part (3) of Lemma 4.1 implies that B can be the only η_0-bad horoball hitting $\frac{1}{3}S$ with diameter $\ge (1/50)\,\text{diam}(S)$. Thus we can find another square $S' \subset S$ with $\text{diam}(S') = \frac{1}{3}(S)$ that hits only small horoballs. More precisely, we can choose $S' \subset S$ such that

$$\text{area}(S' \cap H_b) \ge \frac{1}{100}\,\text{area}(S')$$

and such that S' does not hit any η_0-horoballs with diameter greater than or equal to $(1/10)\,\text{diam}(S')$.

So, by replacing S by S' if necessary, we may now assume that S is a square such that

$$\text{area}(S \cap W_g) \ge \frac{1}{100}\,\text{area}(S)$$

and such that S does not hit any η_0-bad horoballs with diameter greater than or equal to $(1/10)\,\text{diam}(S')$. Let $\{B_j\}$ be the collection of bad horoballs that hit $\frac{1}{3}S$. Since they all have diameter $\le \text{diam}(S)/10$, they are contained in S and their areas sum to be at least $C\,\text{area}(\frac{1}{3}S)$.

The horoballs are all disjoint, so by the simple Vitali covering lemma (see e.g. [13, Lemma 7.4]) there is a subcollection of these balls $\{\hat{B}_j\}$ which have disjoint triples and whose areas sum to be more than $C'\text{area}(S)$.

By part (2) of Lemma 4.1, to each of these balls we can associate a square $S_j \subset 3\hat{B}_j$ such that

$$\text{diam}(S_j) = \tfrac{1}{2}\,\text{diam}(\hat{B}_j),$$

and such that $S_j \setminus \Lambda$ contains no balls of radius $\delta\,\text{diam}(S_j)$. Here δ may be as small as we wish, assuming we take η_0 small enough. Choose η_0 so small that $\delta < r/1000$ (where r is the number given in the statement of Lemma 7.1). Inside S_j choose a collection $\{S_{jk}\}$ of r^{-2} squares of diameter $r/100$ that have disjoint triples. Then $\bigcup_j \bigcup_k S_{jk}$ obviously has all the desired properties. □

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