Equivariant Elliptic Homology and Finite Groups

Jorge A. Devoto

1. Introduction

The *elliptic homology* $X \to \mathrm{Ell}_*(X)$ is the generalized homology theory defined by the equality

$$\mathrm{Ell}_*(X) = MSO_*(X) \otimes_{MSO_*} \mathbb{Z}[\tfrac{1}{2}][\delta, \epsilon, ((\delta^2-\epsilon)^2\epsilon)^{-1}], \tag{1.1}$$

where $MSO_*(X)$ denotes the oriented bordism of the space X,

$$MSO_* = MSO_*(pt),$$

and δ and ϵ are two indeterminates of degree 4 and 8, respectively. The ring

$$\mathrm{Ell}_* = \mathbb{Z}[\tfrac{1}{2}][\delta, \epsilon, ((\delta^2-\epsilon)^2\epsilon)^{-1}] \tag{1.2}$$

in (1.1) is considered as a graded algebra over MSO_* via a ring homomorphism $\Phi_{\mathrm{Ell}}: MSO_* \to \mathrm{Ell}_*$ called the *universal elliptic genus.* The dual cohomology theory, *elliptic cohomology,* is defined using the Spanier–Whitehead duality operator. If X is a finite CW complex, then the elliptic cohomology $\mathrm{Ell}^*(X)$ of X is naturally isomorphic [11, Thm. 1] to

$$\mathrm{Ell}^*(X) = MSO^*(X) \otimes_{MSO^*} \mathrm{Ell}^*, \tag{1.3}$$

where $MSO^*(X)$ denotes the oriented cobordism ring of the space X, $MSO^k = MSO_{-k}$, and $\mathrm{Ell}^k = \mathrm{Ell}_{-k}$.

When X is a compact and closed spin manifold, $\Phi_{\mathrm{Ell}}([X])$ has a natural interpretation in terms of nonlinear sigma models. Let $\mathcal{L}X = \{f: S^1 \to X \mid f$ is smooth$\}$ be the space of smooth free loops on X. The group S^1 acts on $\mathcal{L}X$ by rotation of loops. The theory of nonlinear sigma models predicts the existence of a S^1-equivariant differential operator D on $\mathcal{L}X$ called the *Dirac-Ramond* operator. One can then consider, at least in a formal way, the S^1-equivariant index of D. This can be done in two ways. The first uses the index formula of Atiyah and Singer and the second uses path integrals. In each case one obtains a formal power series $\Phi_D(X) = \sum_{n\geq 0} a_n q^n$, $a_n \in \mathbb{Z}$. This power series is [29] the Laurent expansion of a modular form for the group

$$\Gamma_0(2) = \left\{\begin{pmatrix} a & b \\ e & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),\ e \text{ is even}\right\} \tag{1.4}$$

Received April 27, 1993. Final revision received July 10, 1995.
Michigan Math. J. 43 (1996).

at the cusp $i\infty$. The ring of modular forms for the group $\Gamma_0(2)$, whose power series expansions at $i\infty$ have coefficients in $\mathbb{Z}[\frac{1}{2}]$, is isomorphic to $\mathbb{Z}[\frac{1}{2}][\delta,\epsilon]$. The isomorphism ev is given [19] by

$$\delta \to -\frac{1}{8} - 3\sum_{n\geq 1}\Bigg[\sum_{\substack{d\mid n\\ d\text{ odd}}} d\Bigg]q^n, \qquad \epsilon \to \sum_{n\geq 1}\Bigg[\sum_{\substack{d\mid n\\ n/d\text{ odd}}} d^3\Bigg]q^n.$$

Witten showed in [29] that the formal series $\Phi_D(X)$ is the Laurent expansion at $i\infty$ of the modular form $ev(\Phi_{\text{Ell}}([X]))$.

The relation between sigma models and elliptic homology provides some reason to believe that elliptic cohomology has a geometric model related either to conformal field theories or to Virasoro equivariant vector bundles on loop spaces [5; 24].

In this paper we will define, for each finite group G of order $|G| = 2s+1$, a G-equivariant generalization $\text{Ell}_G^*(\cdot)$ of $\text{Ell}^*(\cdot)\otimes_{\mathbb{Z}}\mathbb{Z}[1/|G|]$. The first step will be to define a ring homomorphism $\Phi_G\colon M\text{SO}_G^* \to \text{Ell}_G^*$, called the *homotopy theoretic twisted elliptic genus,* from $M\text{SO}_G^*$, the equivariant oriented cobordism ring of [8], to a ring Ell_G^* that is related to the moduli of G-coverings over *Jacobi quartics*

$$y^2 = 1 - 2\epsilon x^2 + \delta x^4. \tag{1.5}$$

DEFINITION 1.6. The *equivariant elliptic cohomology* $\text{Ell}_G^*(X)$ of a finite G-CW complex X is the graded tensor product

$$\text{Ell}_G^*(X) = M\text{SO}_G^*(X) \otimes_{M\text{SO}_G^*} \text{Ell}_G^*, \tag{1.7}$$

where Ell_G^* is considered as a graded algebra over $M\text{SO}_G^*$ via the twisted elliptic genus.

We can now state our main result.

THEOREM 1.8. *The functor $X \to \text{Ell}_G^*(X)$ from finite G-CW complexes to graded rings is a stable G-equivariant cohomology theory.*

The study of the image of the twisted elliptic genus leads naturally to the consideration of modular problems of higher level. The formalism of Jacobi quartics is inadequate to deal with moduli problems of even order because, among other reasons, the functions that parameterize (1.5) have double points at the points of order 2 of the curve (the restriction to moduli problems of odd order also appears in the fundamental work of Igusa on the relation between Jacobi quartics and the transformation theory of elliptic curves [14]). This is the main reason why we impose the condition that $|G|$ be odd. The second reason concerns the orientability of fixed point sets. It is possible that this hypothesis can be removed by a careful analysis of orbifold sigma models. We could also use another version of elliptic cohomology—related, for example, to Weierstrass cubics—but then we would lose the insight given by the use of loop spaces. The second condition, that $|G|$ be invertible in the coefficient ring, is due to the fact that it is not known if there is an equivariant version of Landweber's exact functor theorem. The exact functor theorem is used to prove the exactness of functors like (1.1). Our hypothesis about $|G|$ allows us to show that the functor (1.7) is exact by using localization along fixed points.

The idea behind the definition of the twisted elliptic genus comes from the geometry of loop spaces. Let X be a closed, compact, Riemannian spin manifold of dimension $2k$. Suppose that G acts on X by isometries in a way compatible with the spin structure. Then, for each $g_1 \in G$, we define the *twisted loop space*

$$\mathcal{L}_{g_1}X = \{f\colon \mathbb{R} \to X \mid f(t+1) = g_1 f(t) \text{ and } f \text{ is smooth}\}. \tag{1.9}$$

Two groups act on $\mathcal{L}_{g_1}X$ in a compatible way. The first group is $S^1 = \mathbb{R}/|g_1|\mathbb{Z}$ which, since g_1 has finite order, acts on $\mathcal{L}_{g_1}X$ by rotation of twisted loops. The action is induced by the action of $\mathbb{R}$ on itself by translation. The second group, $C_{g_1}(G)$, is the centralizer of g_1 in G. The action is in this case induced by the action of G on X. We can formally consider the $S^1 \times C_{g_1}(G)$-equivariant index of the Dirac–Ramond operator D on $\mathcal{L}_{g_1}X$. In this way we obtain a formal power series $\Phi_G(X) = \sum_{n\geq 0} R_n q^{n/|g_1|}$ with R_n a character of $C_{g_1}(G)$. For $g_2 \in C_{g_1}(G)$ and $\tau \in \mathfrak{h} = \{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$, we define $\Phi_G(g_1, g_2, \tau)$ as the evaluation of $\Phi_G(G)$ at g_2 and $q = \exp(2\pi i \tau)$. The formal power series $\Phi_G(X)(g_1, g_2, \tau)$ satisfies

$$\Phi_G(X)\left((g_1^d g_2^e, g_1^b g_2^a), \frac{a\tau+b}{e\tau+d}\right) = (e\tau+d)^k \Phi_G(X)(g_1, g_2, \tau), \tag{1.10}$$

where $\left(\begin{smallmatrix} a & b \\ e & d \end{smallmatrix}\right) \in \Gamma_0(2)$. Let TG be $\{(g_1, g_2) \in G \times G \mid g_1 g_2 = g_2 g_1\}$. The ring Ell_G^* is, roughly, the ring of functions $\vartheta\colon TG \times \mathfrak{h} \to \mathbb{C}$ that satisfy (1.10).

As an application of the theory Ell_G^* we shall study a conjecture of Atiyah and Segal about equivariant Euler numbers. In [2] they proved that if G is a finite group that acts smoothly on the compact manifold X, then the Euler characteristic of the equivariant K-theory of X is given by

$$\chi_K(X) = \operatorname{rank}_{\mathbb{Z}} K_G^0(x) - \operatorname{rank}_{\mathbb{Z}} K_G^{-1}(X) = \frac{1}{|G|} \sum_{(g_1, g_2) \in TG} \chi(X^{g_1, g_2}),$$

where X^{g_1, g_2} is the set of points fixed by g_1 and g_2 and $\chi(Z)$, for a space Z, denotes the usual Euler characteristic of Z. In [2] it was conjectured that the number

$$\chi_{\mathrm{Ell}}(X) = \frac{1}{|G|} \sum_{(g_1, g_2, g_3) \in TTG} \chi(X^{g_1, g_2, g_3}), \tag{1.11}$$

where the sum is now being taken over

$$TTG = \{(g_1, g_2, g_3) \in G \times G \times G \mid g_i g_j = g_j g_i \ \forall\, 1 \leq i, j \leq 3\}$$

and X^{g_1, g_2, g_3} is now the space of common fixed points of g_1, g_2, g_3, might be related to elliptic cohomology.

Let $\mathbb{F}^*$ denote the graded field of fractions of Ell^* (see Section 6 for an explanation of the use of graded fields and the modular interpretation of $\mathbb{F}^*$). If X is a compact G-manifold, then the initial term in Segal's spectral sequence—which is deduced from [22, Prop. 5.1] applies to the case of a G-covering (see also [23, Prop. 5.3] for the case of K-theory), $H^*(X/G, \mathcal{E}_G^*) \Rightarrow \mathrm{Ell}_G^*(X)$, where $\mathcal{E}_G^*$ is the sheaf over X/G associated to the presheaf $U \to \mathrm{Ell}_G^*(\pi^{-1}(U))$—is a finitely generated module over Ell_G^* and therefore the same is true for $\mathrm{Ell}_G^{\mathrm{odd}}(X)$ and $\mathrm{Ell}_G^{\mathrm{even}}(X)$. Thus the graded $\mathbb{F}^*$ modules $\mathrm{Ell}_G^{\mathrm{odd}}(X) \otimes_{\mathrm{Ell}^*} \mathbb{F}^*$ and $\mathrm{Ell}_G^{\mathrm{even}}(X) \otimes_{\mathrm{Ell}^*} \mathbb{F}^*$ have a well-defined rank over $\mathbb{F}^*$.

THEOREM 1.12. *If X is a compact G-manifold, then*

$$\chi_{\mathrm{Ell}}(X) = \mathrm{rank}_{\mathbb{F}^*}[\mathrm{Ell}_G^{\mathrm{even}}(X) \otimes \mathbb{F}^*] - \mathrm{rank}_{\mathbb{F}^*}[\mathrm{Ell}_G^{\mathrm{odd}}(X) \otimes \mathbb{F}^*]. \quad (1.13)$$

REMARK 1.14. We shall denote also by $\chi(\mathrm{Ell}_G^*(X))$ the right-hand side of (1.13).

We wish to express our gratitude to Peter Braam, Eduard Loojienga, Dave Benson, Domingo Toledo, and Herb Clemens for their suggestions. Part of this paper was written at the Max Planck Institute for Mathematics, a warm and friendly working environment. We wish also to express our gratitude to the referee, whose comments were particularly helpful in improving the exposition of the original manuscript.

2. The Twisted Elliptic Genus

Witten's Twisted Class

The K-theoretical version of *Witten's characteristic class* θ [19] is a stable exponential characteristic class $\theta: KO^*(X) \to K^*(X)[[q]]$, where $KO^*(X)$ denotes the real K-theory of the space X and $K^*(X)[[q]]$ denotes the ring of formal power series with coefficients in the complex K-theory of X. Witten's characteristic class of a real vector bundle $E \to X$ is explicitly given by

$$\theta(E) = \bigotimes_{n=1}^{\infty} [\Lambda_{-q^{2n-1}}(E \otimes \mathbb{C})] \otimes [S_{q^{2n}}(E \otimes \mathbb{C})],$$

where $\Lambda_a(E \otimes \mathbb{C}) = \sum_{n \geq 0} a^n \Lambda^n(E \otimes \mathbb{C})$ and $S_a(E \otimes \mathbb{C}) = \sum_{n \geq 0} a^n S^n(E \otimes \mathbb{C})$. Let G be a finite group of odd order. Then a G-equivariant analog of the functor $K^*(X)[[q]]$ appropriate for the theory of orbifold sigma models is the functor $\mathcal{K}_G$: G-spaces $\to$ rings, given by

$$X \to \mathcal{K}_G^*(X) = \bigoplus_{(g_1, g_2) \in TG} \{K^*(X^{g_1, g_2}) \otimes_{\mathbb{Z}} R(\langle g_2 \rangle)\}[[q^{1/|g_1|}]], \quad (2.1)$$

where $K^*(X^{g_1, g_2})$ denotes the complex K-theory of the fixed point set X^{g_1, g_2} and $R(\langle g_2 \rangle)$ is the complex representation ring of the group $\langle g_2 \rangle$ generated by g_2.

PROPOSITION 2.2. *The functor $X \to \mathcal{K}_G^*(X)$ defines a G-equivariant cohomology theory.*

Proof. Since a direct sum of exact sequences is exact, it suffices to prove (2.2) for each one of the summands in (2.1). In this case the result follows from two basic facts. The first fact is that, since $R(\langle g_2\rangle)$ is flat over $\mathbb{Z}$, the functor $Y\to\{K^*(Y)\otimes R(\langle g_2\rangle)\}[[q^{1/|g_1|}]]$ is a cohomology theory. The second fact is that if H is any subgroup of G and E^* is any generalized cohomology theory, then the functor $X\to E^*(X^H)$, where $X^H=\{x\in X\mid hx=x\ \forall h\in H\}$, is a G-equivariant cohomology theory. This result can be proved by checking the axioms in [28, Def. 6.7]. □

Let X be a compact G space and let $E\to X$ be a G-equivariant complex vector bundle. Then, for each pair $(g_1,g_2)\in TG$, the restriction $E|_{X^{g_1,g_2}}\to X^{g_1,g_2}$ admits a decomposition

$$E|_{X^{g_1,g_2}}=\bigoplus_{-|g_1|/2<j<|g_1|/2}\left\{\bigoplus_{-|g_2|/2<k<|g_2|/2}F_{ij}\right\}. \tag{2.3}$$

where the $\langle g_1,g_2\rangle$-equivariant complex vector bundles F_{jk} are characterized by the fact that g_1 acts fiberwise as $\exp\{2\pi ij/|g_1|\}$ and g_2 as $\exp\{2\pi ik/|g_2|\}$. We define

$$\bar{\theta}_G(E|_{X^{g_1,g_2}})=\bigotimes_{\substack{-|g_1|/2<j<|g_1|/2\\-|g_2|/2<k<|g_2|/2}}\left[\bigotimes_{s\geq 1}(\Lambda_{[w^{2k/c'}][-q^{2s-1}]}[F_{jk}])\otimes\bigotimes_{s\geq 0}(S_{[w^{2k/c'}][q^{2s}]}[F_{jk}])\right]. \tag{2.4}$$

In (2.4) we are taking $c=|g_1|$, $s=(nc+j)/c$ with $n\in\mathbb{Z}$, and $R\langle g_2\rangle=\mathbb{Z}[w]$. If E is a real G-equivariant vector bundle, then we define

$$\theta_G(E|_{X^{g_1,g_2}})=\bar{\theta}_G((E\otimes\mathbb{C})|_{X^{g_1,g_2}}). \tag{2.5}$$

The conventions used in the decomposition of $(E\otimes\mathbb{C})|_{X^{g_1,g_2}}$ are the usual ones in index theory (see [3]).

DEFINITION 2.6. *Witten's twisted class* $\theta_G\colon KO_G^*\to\mathcal{K}_G$, *evaluated in the G-equivariant real vector bundle $E\to X$, is the class whose (g_1,g_2) component is* $\theta_G(E|_{X^{g_1,g_2}})$.

The class $\theta_G(E)$ depends only on the G-equivariant isomorphism class $[E]$ of E and satisfies $\theta_G(E\oplus F)=\theta_G(E)\theta_G(F)$. We shall see later (see Remark 4.9) that if E is of the form $M\times X$, where M is a real G-module, then $\theta_G([E])$ is invertible. These two properties imply that θ_G has a unique extension as an exponential characteristic class, $\theta_G\colon KO_G^*\to\mathcal{K}_G$.

The Twisted Elliptic Genus

Let X be a closed, oriented, compact, Riemannian manifold of dimension $2k$, where G acts by orientation-preserving isometries. We shall always suppose, just to simplify the formulas, that each X^{g_1,g_2} is connected. This is a minor assumption that can easily be removed. As the order $|G|$ of G is odd, the orientation on X induces an orientation on each of the submanifolds

X^{g_1,g_2} [3, p. 584]. Recall that, since BSpin and BSO are homotopically equivalent at odd primes [27, p. 336], orientable manifolds are orientable for $(K^*(\cdot)\otimes\mathbb{Z}[\frac{1}{2}])$-theory. Therefore, for each pair $(g_1,g_2)\in TG$, there exists a Gysin map $\pi_!^{g_1,g_2}: K^*(X^{g_1,g_2})\otimes\mathbb{Z}[\frac{1}{2}]\to K^*(pt)\otimes\mathbb{Z}[\frac{1}{2}]$ induced by the projection $\pi: X^{g_1,g_2}\to pt$. The family of maps $\pi_!^{g_1,g_2}$ induces a Gysin map $\pi_!: \mathcal{K}_G^*(X)\to\mathcal{K}_G^*$.

DEFINITION 2.7. The *geometric twisted elliptic genus* Φ_G is defined by the equality

$$\Phi_G(X)=\pi_!\left(\frac{\theta_G([TX])}{\theta_G([\dim(TX)])}\right)=\sum_{(g_1,g_2)\in TG}\Phi(X^{g_1,g_2})\in\mathcal{K}_G(pt),$$

where $[\dim(TX)]$ is the element of $\mathcal{K}_G(X)$ obtained by replacing all the bundles F_{jk} in formula (2.3) by topologically trivial bundles T_{jk}, where $\dim_{\mathbb{C}} T_{jk}=\dim_{\mathbb{C}} L_{jk}$ and where g_1 and g_2 at in the same way as in F_{jk}.

It is easy to show that Φ_G induces a ring homomorphism $\Omega_*^G\to\mathcal{K}_G(pt)$, where Ω_*^G is the geometric equivariant bordism of [6]. However, since we are interested in cohomology, we shall consider that it is defined on $\Omega_G^*=\Omega_{-*}^G$.

Modular Properties of the Twisted Elliptic Genus

Pick $\tau\in\mathfrak{h}$ and let $q=\exp\{2\pi i\tau\}$. We define $\Phi_G(X)(g_1,g_2,\tau)\in\mathbb{C}\cup\infty$ as the evaluation of $\Phi_G(x^{g_1,g_2})$ at g_2 and τ. The evaluation at g_2 is performed via the identification between representations and characters. We shall show that, for each $(g_1,g_2)\in TG$, the function $\tau\to\Phi_G(g_1,g_2,\tau)$ is an entire function and therefore $\Phi_G(g_1,g_2,\tau)\in\mathbb{C}$ for all $\tau\in\mathfrak{h}$.

It is easier to study the properties of the twisted genus using cohomology. The Chern character $\mathrm{ch}: K^*(\cdot)\to H^*(\cdot,\mathbb{Q})$ induces, for any $(g_1,g_2)\in TG$, a homomorphism

$$[K^*(X^{g_1,g_2})\otimes R(\langle g_1\rangle)][[q^{1/|g_1|}]] \xrightarrow{\mathrm{ch}_q} [H^*(X^{g_1,g_2},\mathbb{Q})\otimes_{\mathbb{Q}}(\mathbb{Q}\otimes R(\langle g_2\rangle))][[q^{1/|g_1|}]]. \quad (2.8)$$

Using this homomorphism together with the standard techniques of index theory, we obtain the cohomological form of the elliptic genus:

$$\Phi_G(X)(g_1,g_2,\tau)=\epsilon\langle\hat{A}(X^{g_1,g_2})\,\mathrm{ch}_q\{\Phi(X^{g_1,g_2})\},[X^{g_1,g_2}]\rangle, \quad (2.9)$$

where $[X^{g_1,g_2}]$ denotes the orientation class in cohomology of X^{g_1,g_2} and the factor $\epsilon=\pm1$ depends on the choice of orientation of the bundles involved (see [3] for an explanation of the cohomological form of the index theorem and of the terms involved). Let

$$[TX\otimes\mathbb{C}]\,|_{X^{g_1,g_2}}=[TX^{g_1,g_2}\otimes\mathbb{C}]\oplus\left[\bigoplus_{ij}K_{jk}\otimes\bar{K}_{ij}\right] \quad (2.10)$$

be the decomposition of the restriction of the complexified tangent bundle of X under the action of g_1 and g_2 (see (2.3)), and let x_r and z_{jks} be formal variables such that $p(TX^{g_1,g_2})=\prod_r(1-(2\pi ix_r)^2)$ and $c(K_{jk})=\prod_s(1+(2\pi iz_{jks}))$, where $p(TX^{g_1,g_2})$ (resp. $c(K_{jk})$) denotes the total Pontrjagin (resp. Chern) class of the bundle TX^{g_1,g_2} (resp. K_{jk}). Then the twisted elliptic genus $\Phi_G(X)$ can be written in terms of these variables as

$$\Phi_G(X)(g_1,g_2,\tau)$$

$$=\epsilon\left\langle \prod_{r,j,k,s}\left(\frac{\pi i x_r \vartheta_{(0,1/2)}(x_r,\tau)}{\vartheta_{(1/2,1/2)}(x_r,\tau)}\right)\left(\frac{\vartheta_{(j/c,1/2+k/c')}(z_{jks},\tau)}{\vartheta_{(1/2+j/c,1/2+k/c')}(z_{jks},\tau)}\right),[X^{g_1,g_2}]\right\rangle, \tag{2.11}$$

where $c=|g_1|$, $c'=|g_2|$, and, for $m,n\in\mathbb{Q}$, the theta function $\vartheta_{(m,n)}:\mathbb{C}\times\mathfrak{h}\to\mathbb{C}$ is defined by

$$\vartheta_{(m,n)}(z,\tau)=\sum_{k\in\mathbb{N}}\exp\{\pi i[(k+m)^2\tau+2(k+m)(z+n)]\}.$$

[21, p. 10]. The transformation rule (1.10) of the twisted elliptic genus can be derived from the equality

$$\vartheta_{(m',n')}\left(\frac{z}{e\tau+d},\frac{a\tau+b}{e\tau+d}\right)=\mu\exp\left\{\frac{2\pi i e z^2}{e\tau+d}\right\}(e\tau+d)^{1/2}\vartheta_{(m,n)}(z,\tau),$$

where $(m',n')=(dm-en,an-bm)+\frac{1}{2}(ed,ab)$ and μ is a phase factor that depends only on $\left(\begin{smallmatrix}a & b\\ e & d\end{smallmatrix}\right)$.

We can rewrite the twisted elliptic genus using the function $s(u,\tau)$ that parameterizes the Jacobi quartic curve (1.5). This function is related to the theta functions with characteristics via

$$(s(u,\tau))^{-1}=\psi(u,\tau)=\frac{\vartheta_{(0,1/2)}(u,\tau)}{\vartheta_{(1/2,1/2)}(u,\tau)}. \tag{2.12}$$

The twisted elliptic genus does not change if we take $j'=jc'$, $k'=kc$, and $c''=cc'$, so we do not lose any generality if we suppose $|g_1|=|g_2|=c$. Let us call

$$s_{jk}(\tau)=s\left(4\pi i\left(\frac{j\tau}{c}+\frac{k}{c},\tau\right),\tau\right). \tag{2.13}$$

The functions $s_{jk}(\tau)$ are the x-coordinate functions of the torsion points of order c of the Jacobi quartic parameterized by s.

LEMMA 2.14. *Let $(g_1,g_2)\in TG$ with $|g_1|=|g_2|=c$. Then the twisted genus $\Phi_G(X)(g_1,g_2,\tau)$ belongs to the ring $\mathbb{Z}[\frac{1}{2}][\delta,\epsilon,\Delta^{-1},s_{jk}]$, $j,k=1,\dots,c-1$.*

Proof. Let $s'_{jk}(\tau)$ be the y-coordinate functions of the points of order c of the Jacobi quartic parameterized by the function s defined in (2.12). The group structure of the Jacobi quartic (1.5) induces the identity [18]

$$s(u+v)=\frac{s(u)s'(v)+s(v)s'(u)}{1-\epsilon s^2(u)s^2(v)}. \tag{2.15}$$

This identity can also be written in terms of $\psi(u)$. In this case it is given by

$$\psi(u+v)=\frac{\psi^2(u)-\epsilon s^2(v)}{\psi(u)s'(v)-\psi'(u)s(v)}.$$

The denominator of the last expression can be formally expanded as

$$(\psi(u)s'(v)-\psi'(u)s(v))^{-1}=\sum_{n\geq 0}[1-\psi(u)s'(v)+\psi'(u)s(v)]^n.$$

If we substitute $u=2\pi i z_{jks}$ and $v=4\pi i(j\tau/c+k/c)$ in the last expression, then we can see that the contribution of z_{jk} to the twisted elliptic genus is

$$\begin{aligned}2\psi&\left(2\pi i z_{jks}+4\pi i\left(\frac{j\tau}{c}+\frac{k}{c'}\right)\right)\\&=\left[\psi^2(2\pi i z_{jks})-\epsilon s^2\left(4\pi i\left(\frac{j\tau}{c}+\frac{k}{c}\right)\right)\right]\sum_{n\geq 0}[1-\psi(2\pi i z_{jks})s'_{jk}(\tau)\\&\qquad+\psi'(2\pi i z_{jks})s_{jk}(\tau)]^n.\end{aligned}$$

From this formula we can conclude that $\Phi_G(X)(g_1,g_2,\tau)$ has a formal expansion $\Phi_G(X)(g_1,g_2,\tau)=\sum_{I,S_{jk}}P_{IS}\langle x^I z_{jk}^{S_{jk}},[X^{g_1,g_2}]\rangle$, where $I=\{i_1,\ldots,i_r\}$, and $S_{jk}=\{s(jk)_1,\ldots,s(jk)_l\}$ are multi-indices, each P_{IS} is a polynomial in the functions $s_{jk}(\tau)$ and $s'_{jk}(\tau)$, and $x^I=x_1^{i_1}\ldots x_r^{i_r}$, $z_{jk}^{S_{jk}}=z_{jk1}^{s(jk)_1}\ldots z_{jkl}^{s(jk)_l}$. It follows that $\Phi_G(X)(g_1,g_2,\tau)\in\mathbb{Z}[\frac{1}{2}][\delta,\epsilon,s_{jk}(\tau),s'_{jk}(\tau)]$. Using (2.12) and the results about the localization of the zeroes of theta functions of [21, Lemma 4.1], we can see that the functions $\psi(\tau)$ are holomorphic on $\mathfrak{h}$. From the power series expansion

$$\begin{aligned}s_{jk}(\tau)=&\frac{1}{2\sinh\{2\pi i(j\tau/c)+2\pi i(k/c)\}}-\sum_{\substack{m,d\geq 1\\ d\text{ odd}}}\exp\left\{2\pi i\left(md\tau+\frac{dj}{2c}\tau+\frac{dk}{2c}\right)\right\}\\&-\sum_{\substack{m,d\geq 1\\ d\text{ odd}}}\exp\left\{2\pi i\left(md\tau+\frac{-dj}{2c}\tau+\frac{-dk}{2c}\right)\right\},\end{aligned}$$

of the functions s_{jk} at $i\infty$, we can conclude that the power series expansions at $i\infty$ of the functions $1/s_{jk}(\tau)$ have coefficients in $\mathbb{Z}[1/2,1/c,\exp(2\pi i/|g_2|)]$. Using these facts and the identity $(1-\epsilon s^4(u))s(2u)=2s(u)s'(u)$, which can be derived from (2.15), we see that the functions $s'_{jk}(\tau)$ can be written in terms of the functions s_{jk}. □

3. The Coefficient Ring Ell_G

Basic Definitions

The group $\Gamma_0(2)\times G$ acts on the left on $TG\times\mathfrak{h}$ by

$$\left(\begin{pmatrix}a&b\\e&d\end{pmatrix},g\right)\times(g_1,g_2,\tau)\to\left(g(g_1^d g_2^e,g_1^b g_2^a)g^{-1},\frac{a\tau+b}{e\tau+d}\right).\tag{3.1}$$

This action induces, for each $k\in\{0\}\cup\mathbb{N}$, an action ρ_k of $\Gamma_0(2)\times G$ on the holomorphic functions $\vartheta\colon TG\times\mathfrak{h}\to\mathbb{C}$. The action ρ_k is defined by

$$\rho_k\left(\left(\begin{smallmatrix} a & b \\ e & d\end{smallmatrix}\right), g\right)\vartheta(g_1, g_2, \tau) = (e\tau+d)^{-k}\vartheta\left(g(g_1^d g_2^e, g_1^b g_2^a)g^{-1}, \frac{a\tau+b}{e\tau+d}\right).$$

It is easy to prove, using the cocycle condition $(eA'\tau+d)(e'\tau+d')=(e''\tau+d'')$ with $A=\left(\begin{smallmatrix} a & b \\ e & d\end{smallmatrix}\right)$, $A'=\left(\begin{smallmatrix} a' & b' \\ e' & d'\end{smallmatrix}\right)$, and $A''=AA'=\left(\begin{smallmatrix} a'' & b'' \\ e'' & d''\end{smallmatrix}\right)$, that ρ_k is well-defined.

DEFINITION 3.2. We define Ell_G^{-2k} as the abelian group of holomorphic functions $\vartheta\colon TG\times\mathfrak{h}\to\mathbb{C}$ satisfying the following conditions:

(1) $\rho_k\left(\left(\begin{smallmatrix} a & b \\ e & d\end{smallmatrix}\right), g\right)\vartheta=\vartheta$ for all $\left(\left(\begin{smallmatrix} a & b \\ e & d\end{smallmatrix}\right), g\right)\in\Gamma_0(2)\times G$;

(2) for each $(g_1, g_2)\in TG$, the function $\vartheta(g_1, g_2, \cdot)\colon\mathfrak{h}\to\mathbb{C}$ has a power series expansion at $i\infty$ of the form $\vartheta(g_1, g_2, \tau)=\sum_{n\geq K} a_n q^{n/|g_1|}$, where $K\in\mathbb{Z}$, $q=\exp\{2\pi i\tau\}$, and $a_n\in\mathbb{Z}[1/2, 1/|G|, \exp\{2\pi i/|g_1 g_2|\}]$;

(3) if n and $|C_{g_1}(G)|$ are coprime, and if σ_n is the automorphism of $\mathbb{Z}[1/|G|, \psi]$, where $\psi=\exp\{2\pi i/|C_{g_1}|\}$, defined by $\sigma_n(\psi)=\psi^n$, then

$$\sigma_n(a_m(g_1, g_2)) = a_m(g_1, g_2^n).$$

REMARK 3.3. The conditions (1) and (2) in definition 3.2 are slightly redundant. For example, using the argument of [20, Chap. 6, Sec. 2], we can see that a function $\vartheta\colon TG\times\mathfrak{h}\to\mathbb{C}$ that satisfies condition (1) has a power series expansion of the form $\vartheta(g_1, g_2, \tau)=\sum_{n\in\mathbb{Z}} a_n q^{1/|g_1|}$ at $i\infty$. From the theory of level structures and the theory of the curve of Tate, it follows that the minimal ring (see [15, p. 80, (Ka-12)]) containing the coefficients of these expansions is $\mathbb{Z}[1/2, 1/|G|, \exp\{2\pi i/|g_2|\}]$.

REMARK 3.4. The third condition in Definition 3.2 corresponds to the usual Galois action of the group $(\mathbb{Z}/|C_{g_1}(G)|)^*$ on modular forms of higher level [20, Chap. 6, Sec. 3]. It implies that $a_n(g_1, \cdot)\in R(C_{g_1}(G))\otimes\mathbb{Q}$ [12, Prop. 1.5]. Using the second condition and the usual scalar product of class functions on G [25, Part 1, Sec. 2.3], we can see that $a_n(g_1, \cdot)\in R(C_{g_1}(G))[1/|G|]$.

REMARK 3.5. If $\theta\in\mathrm{Ell}_G^{-2k}$ and $\theta'\in\mathrm{Ell}_G^{-2k'}$, then $\theta\theta'\in\mathrm{Ell}_G^{-2(k+k')}$. Hence the direct sum $\mathrm{Ell}_G^*=\bigoplus_{k\geq 0}\mathrm{Ell}_G^{-2k}$ has a natural structure of a graded ring.

The Mackey Functor Structure of $G\to\mathrm{Ell}_G^*$

Mackey functors are intimately related to equivariant cohomology theories (see [28], which will be our main reference). Let $\mathcal{G}$ be the category whose objects are subgroups of G, and whose morphisms are generated by inclusions $H\subset K$ of subgroups and conjugations $c_g\colon H\to g^{-1}Hg=H^g$ by elements g of G.

DEFINITION 3.6. Let R be a unital ring. A *Mackey functor* $M\colon\mathcal{G}\to R$-modules is a family of R-modules $M(H)$ for $H\in$ objects $\mathcal{G}$, together with R-linear homomorphisms; for $K\subset H$ and $g\in G$,

M1: $\mathrm{rest}_K^H\colon M(H)\to M(K)$ (restriction),
M2: $\mathrm{ind}_K^H\colon M(K)\to M(H)$ (transfer or induction),
M3: $c_g\colon M(H)\to M(H^g)$ (conjugation).

These morphisms must satisfy the following conditions:

M4: if $L \subset K \subset H$, then $\mathrm{rest}_L^K \mathrm{rest}_K^H = \mathrm{rest}_L^H$ and $\mathrm{ind}_K^H \mathrm{ind}_L^K = \mathrm{ind}_L^H$;
M5: $\mathrm{rest}_H^H = \mathrm{ind}_H^H = \mathrm{id}_{M(H)}$ and $c_{gh} = c_h c_g$;
M6: if $h \in H$, then $c_h \colon M(H) \to M(H)$ is the identity;
M7: if $K \subset H$, then $c_g \,\mathrm{rest}_K^H = \mathrm{rest}_{K^g}^{H^g} c_g$ and $c_g \,\mathrm{ind}_K^H = \mathrm{ind}_{K^g}^{H^g}$, where $H^g = c_g(H)$ and $K^g = c_g(K)$;
M8: (Mackey's axiom) If $L, K \subset H$ then

$$\mathrm{rest}_L^H \mathrm{ind}_K^H = \sum_{H \backslash G/L} \mathrm{ind}_{L \cap K^h}^L \mathrm{rest}_{L \cap K^h}^{K^h} c_h.$$

If the functor M takes values in the category of R-algebras, and if the following conditions are satisfied:

G1: the restriction and conjugation morphisms are R-algebra homomorphisms mapping 1 to 1;
G2: if $K \subset H$, $a \in M(K)$, and $b \in M(H)$, then

$$\mathrm{ind}_K^H(a.r_K^H(b)) = \mathrm{ind}_K^H(a).b, \quad \text{and} \quad \mathrm{ind}_K^H(r_K^H(b).a) = b.\mathrm{ind}_K^H(a),$$

then we shall say that M is a *Green functor.*

THEOREM 3.7. *Let G be a finite group. Then the functor $H \to \mathrm{Ell}_H^*$ admits the structure of a Green functor.*

Proof. If K is a subgroup of H, then the restriction map $\mathrm{rest}_K^H \colon \mathrm{Ell}_H^* \to \mathrm{Ell}_K^*$ is induced by the restriction of functions to the subset $TK \times \mathfrak{h} \subset TH \times \mathfrak{h}$. If $g \in G$ and K is a subgroup of G, then the conjugation homomorphism $c_{g^{-1}}\colon K^g \to K$ induces a map $TK^g \times \mathfrak{h} \to TK \times \mathfrak{h}$. The homomorphism $c_g \colon \mathrm{Ell}_K^* \to \mathrm{Ell}_{K^g}^*$ is defined as the composition $\vartheta \to \vartheta c_{g^{-1}}$. We still need to define an induction map $\mathrm{ind}_K^H \colon \mathrm{Ell}_K^* \to \mathrm{Ell}_H^*$. The group H acts on the right on $K \backslash H$ by multiplication and on the left on TH by $g \times (g_1, g_2) \to (g g_1 g^{-1}, g g_2 g^{-1})$. To each element $(g_1, g_2) \in TH$ we can associate $\alpha \colon \mathbb{Z}^2 \to H$ defined by $\alpha(m, n) = g_1^n g_2^m$. We define $(K \backslash H)^{\mathrm{im}\,\alpha} = \{Kg \in K \backslash H \mid \forall k \in \mathrm{im}\,\alpha, Kgk = Kg\}$. If $\theta \in \mathrm{Ell}_K^*$ and $(g_1, g_2) \in TH$, then we define

$$(\mathrm{ind}_K^H \theta)(g_1, g_2, \tau) = \sum_{Kg \in (K \backslash H)^{\mathrm{im}\,\alpha}} \theta(g g_1 g^{-1}, g g_2 g^{-1}, \tau). \tag{3.8}$$

Of course, one must check that this is well-defined and that it satisfies the axioms of Green functor. This can be done copying the techniques used in representation theory (see [25, Part 2, Sec. 7), which we shall leave to the reader. □

The Ring Ell_G^* *as a Module over the Burnside Ring*

Among the Green functors there is a universal object $G \to A(G)$ called the *Burnside ring.* The ring $A(G)$ is the Grothendieck ring of the semiring of isomorphism classes of finite G-sets. Owing to the universal properties of $A(G)$, any Green functor has a structure of a module over the Burnside ring.

Let us describe the prime ideals of $A(G)$. A subgroup H of G induces a homomorphism $\psi_H: A(G) \to \mathbb{Z}$, defined on a G-set S by $S \to \#\{S^H\}$, where S^H is the set of H fixed elements of S. Let p be a prime or zero. Then we can associate to H and p the ideal $q(H,p) = \psi_H^{-1}(p\mathbb{Z})$. All the prime ideals of $A(G)$ are of this form. The product $\psi = \prod_{(H)} \psi_H$ defines a homomorphism $\psi: A(G) \to \prod_{(H)} \mathbb{Z}$, where the product is over the set of conjugacy classes of subgroups of G. Proposition 1.2.3 of [28] implies that, after tensoring everything with $\mathbb{Z}[1/|G|]$, the homomorphism ψ becomes an isomorphism. This result implies that the unit 1 of $A(G)$ can be written as an orthogonal sum of idempotents e_H, one for each conjugacy class in G. Using these idempotents, we obtain a direct decomposition

$$M = \bigoplus_{(H)} e_H M \tag{3.9}$$

on each $A(G)$-module.

LEMMA 3.10. *Let $e_H \in A(G)$ be an idempotent corresponding to a subgroup $H \subset G$. Then $e_H \mathrm{Ell}_G = 0$ unless $H = \langle g_1, g_2 \rangle$ for some pair of commuting elements $g_1, g_2 \in TG$.*

Proof. The idempotents e_H of the Burnside ring have been explicitly computed by Yoshida [30] and Araki [1]. They are given by the formula

$$e_H = \frac{1}{|N_G(H)|} \sum_{D \subset H} |D| \mu(D,H) \left[\frac{G}{D}\right],$$

where the sum is over the set of subgroups D of H, $N_G(H)$ is the normalizer of H in G, μ is the Möbius function of the lattice of subgroups of $N_G(H)$, and $[G/D]$ is the class of G/D as a G-space. Recall that, for a partially ordered set $(L, \leq)$, one defines inductively the Möbius function μ by: $\mu(s,s) = 1$; $\mu(s,t) = -\sum_{s \leq u < t} \mu(s,u)$ if $s < t$; and $\mu(s,t) = 0$ if $s \not\leq t$. The product $[G/H]\vartheta$ is, as a consequence of [28, Prop. 6.2.3], given by

$$\left[\frac{G}{H}\right] \vartheta(g_1, g_2, \tau) = \sum_{Hg \in (H \backslash G)^{g_1 g_2}} \vartheta(g g_1 g^{-1}, g g_2 g^{-1}, \tau) = \kappa(H, g_1, g_2) \vartheta(g_1, g_2, \tau),$$

where $\kappa(D, g_1, g_2) = \#\{Hg \in (H \backslash G)^{(g_1, g_2)}\}$. Then

$$e_H \vartheta(g_1, g_2, \tau) = \frac{1}{|N_G(H)|} \sum_{D \subset N_G(H)} |D| \mu(D,H) \kappa(D, g_1, g_2) \vartheta(g_1, g_2, \tau).$$

Suppose that H is not conjugate to $\langle g_1, g_2 \rangle$ for any pair $(g_1, g_2) \in TG$. Then there are two possibilities. First, if $\kappa(D, g_1, g_2) = 0$ for all $D \subset H$, then $e_H \mathrm{Ell}_G^* = 0$. Therefore let us suppose that there exists $D \subsetneq H$ such that $n = \kappa(D, g_1, g_2) \neq 0$. In this case, $\kappa(U, g_1, g_2) = n' \geq n$ for any subgroup $D \subseteq U \subsetneq H$. Then

$$|D| \mu(D,H) \kappa(D, g_1, g_2) + \sum_{D \subsetneq U \subsetneq H} |U| \mu(U,H) \kappa(U, g_1, g_2)$$

$$= |D| \mu(D,H) \kappa(D, g_1, g_2) + \sum_{D \subsetneq U \subsetneq H} |D| \mu(U,H) \kappa(D, g_1, g_2) + K, \tag{3.11}$$

where K has no contribution from D. The inductive definition of the Möbius function μ implies that

$$|D|\mu(D,H)\kappa(D,g_1,g_2)+\sum_{D\subsetneq U\subsetneq H}|D|\mu(U,H)\kappa(D,g_1,g_2)=0.$$

Hence the contribution of D to $e_H\vartheta(g_1,g_2,\tau)$ disappears. Working inductively, in this way we can show that $e_H\vartheta(g_1,g_2,\tau)=0$. Finally, if the only subgroup D of H for which $\kappa(D,g_1,g_2)\neq 0$ is H itself, then H is conjugate to $\langle g_1,g_2\rangle$. □

The next proposition is a reformulation of Lemma 3.10.

PROPOSITION 3.12. *If $\mathfrak{Z}G$ denotes the category whose objects are the subgroups $\langle g_1,g_2\rangle$ generated by elements of TG and whose morphisms are generated by inclusions and conjugation by elements of G, then:*

(1) *the restrictions* $\mathrm{rest}^G_{\langle g_1,g_2\rangle}\colon \mathrm{Ell}^G_*\to \mathrm{Ell}^*_{\langle g_1,g_2\rangle}$ *induce an isomorphism*

$$\mathrm{Ell}^*_G \xrightarrow{\text{rest}} \varprojlim_{\langle g_1,g_2\rangle}\mathrm{Ell}^*_{\langle g_1,g_2\rangle}; \tag{3.13}$$

(2) *the family of induction maps induce an epimorphism*

$$\varinjlim_{\langle g_1,g_2\rangle}\mathrm{Ell}^*_{\langle g_1,g_2\rangle}\to \mathrm{Ell}^*_G; \tag{3.14}$$

and

(3) *if $C(TG)$ is a set of representatives of conjugacy classes in $\mathfrak{Z}G$ then*

$$\mathrm{Ell}^*_G\sim \bigoplus_{H\in C(TG)}\{\mathrm{Ell}^*_H\}^{W(H)}, \tag{3.15}$$

where $W(H)$ is the Weyl group of H.

REMARK 3.16. Formula (3.13) follows directly from Lemma 3.10 and, by [28, Thm. 6.3.3], implies (3.14). Finally, the last formula follows from our Lemma 3.10 and the exact sequence (6.1.4), Proposition 6.1.6, and (6.1.8) of [28] (see [28, Cor. 7.7.10] for a similar formula for the representation ring).

REMARK 3.17. One should in principle take the localization of Ell_H at the set $S(H)=q(H,0)$, but the elements $S_{(H)}$ are units (they correspond to the functions $s_{jk}(\tau)$) in Ell^*_H so we do not need to localize.

The Structure and Geometry of Ell^*_G

Our aim in this section is to find generators for Ell^*_G and to establish a flatness condition for Ell^*_G.

PROPOSITION 3.18. *The ring* Ell_G *is a flat* Ell *module.*

Using (3.15) and the fact that, since $|G|$ is invertible, we can obtain the $W(H)$-invariant elements using a projector, we see that it suffices to consider the case $G=\langle g_1,g_2\rangle$. We can also suppose, by the structure theorem of finite Abelian groups, that $|g_1| \mid |g_2|$. In this case $C_{g_1}(G)=G$ and the actions of $(\mathbb{Z}/|G|\mathbb{Z})^*$ and $\Gamma_0(2)\times G$ on TG combine in the usual way, sending $n \in \mathbb{Z}/|G|\mathbb{Z}$ to the matrix $\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$, into an action of $\mathrm{GL}(2,(\mathbb{Z}/|G|))\times G$ (by the exact sequence of [20, p. 62]).

Choosing a fixed representative $[g_1,g_2]$ in each orbit $\overline{(g_1,g_2)}$ of the action of $\mathrm{GL}(2,(\mathbb{Z}/|G|\mathbb{Z}))\times G$ on TG, we obtain a ring homomorphism

$$\Lambda : \mathrm{Ell}^*_G[\psi] \to \bigoplus_{[g_1,g_2]} \mathrm{Ell}^*(\Gamma([g_1,g_2])), \qquad \Lambda(\vartheta) = \sum_{[g_1,g_2]} \vartheta([g_1,g_2],\cdot),$$

where $\Gamma([g_1,g_2])$ is the isotropy group of $[g_1,g_2]$ in $\Gamma_0(2)$, and where $\mathrm{Ell}^{-2k}(\Gamma([g_1,g_2]))$ is the group of holomorphic functions $\vartheta : \mathfrak{h}\to\mathbb{C}$ such that:

(1) $\vartheta\left(\dfrac{a\tau+b}{e\tau+d}\right) = (e\tau+d)^k\vartheta(\tau)$ for all $\left(\begin{smallmatrix} a & b \\ e & d \end{smallmatrix}\right)\in\Gamma([g_1,g_2])$;

(2) ϑ has a power series expansion at $i\infty$ of the form

$$\vartheta(\tau) = \sum_{n\ge m} a_n q^{2\pi i/|g_1|} \quad \text{with } a_n \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{|G|}, \exp\frac{2\pi i}{|g_2|}\right].$$

We define $\mathrm{Ell}^*(\Gamma([g_1,g_2]) = \bigoplus_{k\ge 0}\mathrm{Ell}^{-2k}(\Gamma([g_1,g_2])$.

PROPOSITION 3.19. *The morphism* Λ *is an isomorphism.*

Proof. It is obvious that Λ is injective. It is also an epimorphism. This follows from the minimality conditions on the ring on coefficients and the results of [9, Chap. VII]. The morphism Λ gives the values of a function ϑ at the elements $[g_1,g_2]$. Then, using the power series expansions of the modular form $\vartheta([g_1,g_2],\cdot)$ at the other cusps of $\mathfrak{h}/\Gamma([g_1,g_2])$ and the transformation formula (1) of Definition 3.2, we can compute the values of the function ϑ at the other elements of the orbit, under the action of $\Gamma_0(2)\times G$, of $[g_1,g_2]$. The results of [9, Chap. VII, esp. Cor. 3.13] show that the coefficients of these expansions have the right properties. Finally, using the Galois action of $(\mathbb{Z}/|G|\mathbb{Z})^*$ in the theory of modular forms, we can extend the definition to the rest of the elements of TG. □

Using Λ, we can reduce our study of the structure Ell^*_G to the study of the structure of the factors $\mathrm{Ell}^*(\Gamma([g_1,g_2]))$. The subgroups $\Gamma([g_1,g_2])$ are always congruence subgroups. The choice of $[g_1,g_2]$ defines a homomorphism of groups $\mathbb{Z}\times\mathbb{Z}\to G$, given by $(n,m)\to g_1^n g_2^m$; this homomorphism defines an exact sequence of abelian groups $0\to L\to\mathbb{Z}^2\to\langle g_1,g_2\rangle\to 0$, where $\langle g_1,g_2\rangle$ is the subgroup of G generated by $\{g_1,g_2\}$. The isotropy group is equal to the subgroup of $\Gamma_0(2)$ that preserves L.

Condition (2) of Definition 3.2 implies (see the introduction to [9]) that we must work in the setting of algebraic geometry. Our references for this section are [9] and [16].

DEFINITIONS AND FACTS 3.20. Let S and S' be schemes over $\mathbb{Z}[1/2]$.

(1) An *elliptic curve* $E \xrightarrow{p} S$ over S (see [9, Def. 1.0]) is a proper and flat morphism of relative dimension at most 1 and constant Euler–Poincaré

characteristic 0, together with a section $s: S \to E$. An elliptic curve admits a unique structure of a group scheme.

(2) The sheaf $\omega_{E|S} \to S$ is defined as $\omega_{E|S} = p_*(\Omega_{E|S})$, where $\Omega_{E|S} \xrightarrow{p} E$ is the invertible sheaf of relative differentials.

(3) Let $[n]: E \to E$, $n \in \mathbb{N}$, be the map induced by multiplication by n in the group scheme structure on the elliptic curve. Then, if n is invertible in S, the map $[n]$ is étale. We shall let $E[n] = \ker[n]$.

(4) An elliptic curve $E \to S$ is *universal* if, given any other elliptic curve $F \to S'$, there exists a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & E \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\beta} & S \end{array}$$

such that α induces an isomorphism $F \to E \times_S S'$.

(5) Let A be an abelian group. An *A-structure* on an elliptic curve $E \to S$ is a morphism of abstract groups $\phi: A \to E$ such that the effective Cartier divisor D_A of degree $\#A$ defined by $D_A = \sum_{a \in A} [\phi(a)]$ is a subgroup of $E \mid S$.

(6) A *modular form* f of level A and weight k is a rule that assigns to each triple $(E \mid \mathrm{spec}(R), \phi, \omega)$ formed by an elliptic curve $E \mid \mathrm{spec}(R)$ over the spectrum of a ring R together with an A-structure ϕ on E and a basis ω of $\omega_{E|\mathrm{spec}(R)}$ an element of R in such a way that:

- (i) the element $f(E \mid \mathrm{spec}(R), \phi, \omega)$ depends only on the R-isomorphism class of the triple $(E \mid \mathrm{spec}(R), \phi, \omega)$;
- (ii) f is homogeneous of degree $-k$ in the third variable; and
- (iii) the formation of f commutes with arbitrary extensions of scalars.

EXAMPLE 3.21. We shall be interested in five types of level structures as follows.

1. $\Gamma(n)$-*structures* are defined by a group homomorphism $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}n\mathbb{Z} \xrightarrow{\phi} E[n]$ that is an A-structure in the sense that $E[n] = \sum[\phi(i,j)]$.
2. $\Gamma_0(n)$-*structures* are defined by an isogeny $\alpha: E \to E'$ of degree n such that locally f.f.f.p. (faithfully flat of finite presentation) $\ker \alpha$ admits a generator.
3. $\Gamma_1(n)$-*structures* are A-structures for the cyclic group $\mathbb{Z}/n\mathbb{Z}$.
4. *Jacobi structures* $\mathfrak{M}_J(E \mid S)$ is the set of pairs (α, ω) with α a $\Gamma_0(2)$-structure on $E \mid S$ and ω an $\mathcal{O}_S$-basis of $\omega_{E|S}$.
5. $\langle g_1, g_2 \rangle$-*structures* are A-structures for the abelian group $\langle g_1, g_2 \rangle$. The moduli problem $\mathfrak{M}_J$ is closely related to Jacobi quartics.

LEMMA 3.22. *Let $(E \to S, \alpha)$ be an elliptic curve with a $\Gamma_0(2)$-structure α: $E \to E'$ and let $U \to S$ be an étale open set $U = \mathrm{spec}\, A$, where $\omega_{E \times_S U}$ admits a $\mathcal{O}_U$-basis ω and where there is an element $P \in E[2]$ such that, over U, the isogeny α corresponds to $E \xrightarrow{\pi} E/\{e, P\} = \tilde{E}$. In this case $\tilde{P} = \pi(E[2] - \{e, P\})$ is a point of order 2 in $\tilde{E}$. Then, if $\hat{E}|_U \equiv \mathrm{Spf}(A[[T]])$ is the formal completion of $E|_U$ along the zero section s, with T a local parameter adapted to*

ω (see [16, p. 69]), *then there are unique elements* $x \in \mathcal{O}(-(e+P)+\pi^*(\tilde{P}))|_U$ *and* $y \in \mathcal{O}([2]^*(P)-2(\pi^*(\tilde{P})))$ *such that if* $\beta: E|_U \to E|_U$ *is translation by* P *then:*

(i) $x\beta = -x$ *and* $x(-a) = -x$;
(ii) $y\beta = -y$ *and* $y(-a) = y(a)$;
(iii) $x \sim T(1+\textit{higher terms})$ *and* $y \sim (1+\textit{higher terms})$;
(iv) *there exist* $\delta, \epsilon \in \mathcal{O}_S$ *such that* $y^2 = 1+\delta x^2+\epsilon x^4$.

Proof. The result can be proved in the same way as Proposition 2 of [11]. The construction of [11] deals with the case $S = \operatorname{spec} k$ where k is a field of characteristic different from 2, but it can be easily modified (working as in [16, Chap. 2]). □

COROLLARY 3.23. *The moduli problem* $\mathcal{M}_J$ *is represented by the universal Jacobi quartic*

$$E_J = \operatorname{spec}(\mathbb{Z}[\tfrac{1}{2}][\delta, \epsilon, \Delta^{-1}][X, Y]/(Y^2-1+2\delta X^2-\epsilon X^4))$$

over $\mathcal{M}_J = \operatorname{spec}\mathbb{Z}[\tfrac{1}{2}][\delta, \epsilon, \Delta^{-1}]$, *where* $\Delta = \delta(\epsilon-\delta^2)^2$.

Modular forms of level A are related to the existence of a pair $(E_A \to \mathcal{M}_A, \Phi)$ that represents the moduli problem of A-structures. If such a pair exists, then the ring of modular forms of level A can be defined as $\bigoplus_{n\geq 0}\Gamma(\mathcal{M}_A, \omega^{\otimes n})$.

PROPOSITION 3.24. *The simultaneous moduli problems* (A-structure, $\mathcal{M}_J$), *where A-structure is one of the structures of Example 3.21, are representable.*

Proof. The existence of $E_{AJ} \to \mathcal{M}_{AJ}$ for the simultaneous moduli problem (A-structure, $\mathcal{M}_J$), where A-structure is $\Gamma(n), \Gamma_1(n), \Gamma_0(n)$ and $n \geq 3$ is an odd natural number, is a consequence of [16, Lemma 3.2.3, result (4.3.4), & Thm. 5.1.1]. If $\Gamma(g_1, g_2)$ is the isotropy group of a pair $(g_1, g_2) \in TG$, then there exists n odd such that $\Gamma(n) \subset \Gamma(g_1, g_2)$. The group $\Gamma(g_1, g_2)$ acts on the moduli problem of $\Gamma(n) \cap \Gamma_0(2)$-structures. Using the theory of [16, Chap. 7], one can show that the moduli problem ($\langle g_1, g_2\rangle$-structures, $\mathcal{M}_J$) is representable by a curve $E_{\Gamma(g_1,g_2),J} \to \mathcal{M}_{\Gamma(g_1,g_2),J}$ and that $M_{\Gamma(n),J} \to \mathcal{M}_{\Gamma(g_1,g_2),J}$ is a $\Gamma(g_1, g_2)/\Gamma(n)$-torsor. □

REMARK 3.25. From the results of [16] (especially (4.3.4) and (5.1.1)), it follows that the Jacobi quartics corresponding to the universal curves $E_{\Gamma(g_1,g_2),J}$ that represent the simultaneous moduli problem ($\langle g_1, g_2\rangle$-structures, $\mathcal{M}_J$) are obtained from E_J by change of basis; that is, $E_{\Gamma(g_1,g_2),J} = E_J \times_{\mathcal{M}_0} \mathcal{M}_{\Gamma(g_1,g_2)}$.

Because we are including a choice of a basis ω of $\omega_{E|S}$ in the definition of the moduli problem ($\Gamma(g_1, g_2), \mathcal{M}_J$), it follows by Definition 3.20(6) that the ring of modular forms of level Γ is included in $\mathcal{O}(\mathcal{M}_{\Gamma(g_1,g_2),J})$. This ring is equal to $\mathcal{O}(\mathcal{M}_\Gamma)[\delta, \epsilon, \Delta^{-1}]$. One can therefore reduce its study to that of the moduli $\mathfrak{M}_\Gamma$. Let us begin with the case of $\Gamma_1(n)$-structures. In this case the moduli is a subscheme [16, Prop. 1.10.13] of $E_J[n]$, the set of n-torsion points of the universal Jacobi quartic E_J. The ring $\mathcal{O}(E_J[n])$ is by Remark 3.25 (see also [14]) equal to

$$(\mathfrak{M}_J[X,Y]/\langle Y^2-1+2\delta X^2+\epsilon x^4\rangle)/\langle T_n(X), YG_n(X)-F_n^2(X)\rangle,$$

where the polynomials T_n, G_n, F_n are defined by the equalities

$$[n]X = X^{n^2}F_n(X^{-1})F_n^{-1}(X), \tag{3.26}$$

$$[n]Y = YG_n(X)F_n^{-2}(X), \tag{3.27}$$

and $T_n(X) = X^{n^2}F_n(X^{-1})$.

PROPOSITION 3.28. *The ring $\mathcal{O}(E_J[n])$ is isomorphic to*

$$R_n = \mathbb{Z}[\tfrac{1}{2}][\delta, \epsilon, \Delta^{-1}][X]/\langle T_n(X)\rangle.$$

Proof. The explicit description of the polynomials T_n and G_n given in [14] shows that they are coprime; hence $G_n([X])$, where $[X]$ is the equivalence class of X, is a unit in R_n. Putting $Y = F_n^2(X)G_n^{-1}(X)$ yields the isomorphism. □

The ring $\mathcal{O}(\mathfrak{M}_{\Gamma_1(n)J})$ is a quotient ring of $\mathcal{O}(E_J[n])$. Geometrically speaking, $\mathcal{O}(E_J[n])$ represents the points of order $\leq n$ while $\mathcal{O}(\mathfrak{M}_{\Gamma_1(n)J})$ represents the points of *exact* order n. The explicit description of $\mathcal{O}(\mathfrak{M}_{\Gamma_1(n)J})$ is

$$\mathbb{Z}[\tfrac{1}{2}][\delta, \epsilon, \Delta^{-1}][x], \tag{3.29}$$

where $[x]$ is now a root of the polynomial $\Phi_n(X)$ defined in [14, p. 447]. This polynomial divides $T_n(X)$. Let us consider the analogy with nth roots of unity. In this case the polynomial $T_n(X)$ corresponds to X^n-1 and the polynomial $\Phi_n(X)$ to the cyclotomic polynomial. The coefficient functions of the x-coordinates of the torsion points (i.e., the roots of T_n) correspond in the analytical setting to the functions s_{ij} defined in (2.13). In particular they are modular forms, so we see that the inclusion of the modular functions in $\mathcal{O}(\mathfrak{M}_{\Gamma(g_1,g_2),J})$ is an equality. Also we see that the root x in (3.29) can be taken to be the function $s_{10}(\tau)$. The moduli of Γ_n structures is a subscheme of $E_J[n]\times_{\mathfrak{M}_0} E_J[n]$, and the coefficient ring is a quotient of the tensor product $\mathcal{O}(\Gamma_1(n))\otimes_{R_0}\mathcal{O}(\Gamma_1(n)) = R_0[x_1, x_2]$; more precisely, it is isomorphic to

$$R_0(s_{10}(\tau), s_{01}(\tau)) \tag{3.30}$$

or to any other pair $s_{ij}(\tau), s_{km}(\tau)$ such that the mod n reductions

$$(\tilde{i}, \tilde{j}), (\tilde{k}, \tilde{m}) \in \mathbb{Z}_n\times\mathbb{Z}_n$$

form a basis of $\mathbb{Z}_n\times\mathbb{Z}_n$.

REMARK 3.31. Let us remark that the ring defined in (3.30) contains also the functions $s'_{jk}(\tau)$. This follows from the proof of Proposition 3.28. Using (2.15), we see that the ring (3.30) contains also all the functions $s_{jk}(\tau)$.

The final conclusion is that in any case we can generate the ring of modular functions using the x-coordinates of the torsion points of the quartic.

Proof of Proposition 3.18. The moduli problems $\Gamma(g_1, g_2)$, J are flat (this is due to the fact that the problem $\Gamma(n)$ is flat [16]). Therefore $\mathrm{Ell}^*(\Gamma(g_1, g_2))$ is flat over Ell^*. The proposition follows from Proposition 3.12 and the existence of the morphism Λ. □

Comparison between Ell_G^* *and* $\mathrm{Ell}^*(BG)$

Let us suppose now that $G = \mathbb{Z}/p\mathbb{Z}$, where p is an odd prime, and let BG be the classifying space. In this case an element of $\theta \in \mathrm{Ell}_G^*$ is completely determined by its values on the classes $([0],[0])$, $([0],[1]) \in G \times G = TG$. Therefore, by the isomorphism λ,

$$\mathrm{Ell}_G^* \sim \mathrm{Ell}^* \oplus \mathrm{Ell}^*(\Gamma_1(p)) \sim \mathrm{Ell}^* \oplus \mathrm{Ell}^*[X]/\Phi_p(X) \sim \mathrm{Ell}^*[X]/T_p(X).$$

On the other hand, it is known [13] that

$$\mathrm{Ell}^*(BG) \sim \mathrm{Ell}^*[[X]]/[p]X.$$

This ring is different from Ell_G^*. For example, it is not a finitely generated Ell^*-module. However, one can see that it can be obtained using an I-adic completion of Ell_G^* [10]. The situation is analogous to the relation between the rings $R(G) \sim \mathbb{Z}[X]/X^p - 1$ and $K(BG) = \mathbb{Z}[[X]]/(X-1)^p - 1$.

4. The Homotopy-Theoretic Twisted Elliptic Genus

In this section we shall extend the domain of definition of the geometric twisted elliptic genus Φ_G to the homotopy-theoretic equivariant bordism ring MSO_*^G. The ring MSO_*^G is the coefficient ring of a stable G-equivariant homology theory [8] $MSO_*^G(X)$ which we shall now describe. Let (X, A) be a pair formed by a G-CW complex X and a G-CW subcomplex A. Then, for each real orthogonal representation V of G of dimension $|V|$, there exists a suspension homomorphism

$$\sigma(V) : \Omega_n^G(X, A) \to \Omega_{n+|V|}^G(D(V) \times X, (D(V) \times A) \cup (S(V) \times X)), \tag{4.1}$$

where $D(V)$ (resp. $S(V)$) is the unit disk (resp. the unit sphere) in V. The suspension $\sigma(V)$ is defined, for $(M, \partial M) \to (X, A)$ a representative of a cycle, by

$$\{(M, \partial M) \to (X, A)\} \\ \to \{(M \times D(V), \partial(M \times D(V))) \to (X \times D(V), (D(V) \times A) \cup (S(V) \times X))\}.$$

In general, $\sigma(V)$ is not an isomorphism (unless G acts trivially on V).

Let $\mathfrak{U}$ be an orthogonal representation of G that contains, an infinite number of times, each finite-dimensional representation of G. We shall order the set $F\mathfrak{U}$ of finite-dimensional G subspaces of $\mathfrak{U}$ by $V < W$ if V is isomorphic to some G-submodule of W. Using this order and that $\sigma(V \oplus W) = \sigma(V)\sigma(W)$, we can define a structure of a direct system in $\{\Omega_*^G(X \times D(V), (D(V) \times A) \cup (S(V) \times X))\}$.

DEFINITION 4.2. The *homotopy-theoretic equivariant bordism group* [8, p. 72] of the pair (X, A) is the graded group $MSO_*^G(X, A)$ defined by the equality

$$MSO_n^G(X, A) = \varinjlim_{V \in F\mathfrak{U}} \Omega_{n+|V|}^G(D(V) \times X, D(V) \times \bigcup S(V) \times X). \tag{4.3}$$

REMARK 4.4. The way in which the theory $MSO_*^G(X, A)$ has been defined corresponds to the definition of [8] only when the order of the group is odd, since then the universal equivariant orientation in the sense of [8] is determined by an orientation-preserving action of G (see e.g. [7, Sec. 6]).

Let us suppose that $G = \langle g, h \rangle$ and $gh = hg$. Let $(g_1, g_2) \in TG$. From (4.3) we see that, in order to extend the domain of definition of Φ_G to MSO_G^*, it suffices to define, for each $V \in F\mathfrak{U}$, a morphism

$$\Phi_G^V : \tilde{\Omega}_{n+|V|}^G(\Sigma(V)) \to \mathrm{Ell}_n^G,$$

where $\Sigma(V) = D(V)/S(V)$, in a way compatible with the suspension maps (4.1).

Let $(M, \partial M) \to (D(V), S(V))$ be a representative of a class in $\Omega_*^G(D(V), S(V))$. In the definition of $\Phi_G^V(M, \partial M)(g_1, g_2, \tau)$ we must consider two cases.

Case 1: Suppose that $G = \langle g_1, g_2 \rangle$ and let $V = V_0 \oplus V_1$ be the orthogonal decomposition of V given by $V_0 = \{v \in V \mid gv = v \; \forall g \in G\}$ and $V_1 = V_0^{\perp}$. Then $\sigma(V_0) : \Omega^G(D(V_1), S(V_1)) \to \Omega^G(D(V), S(V))$ is an isomorphism. Suppose that $(N, \partial N) \xrightarrow{p} (D(V_1, S(V_1))$ represents the class of $\sigma^{-1}(\;V_0)(M, \partial M)$, and let $V_1 = \bigoplus_{ij} n_{ij} V_{ij}$ be the decomposition of V into irreducible factors. By hypothesis, $D(V_1)^{g_1, g_2} = 0$ so $N^{g_1, g_2} \subset p^{-1}(0)$. Let $TN_{N^{g_1, g_2}} = TF \oplus NF$ be the decomposition into the part TF tangent to the fiber of $p : N \to D(V)$ and the normal part NF of the restriction of the tangent bundle of N to N^{g_1, g_2}. Then we define

$$\Phi_G^V(M, \partial M)(g_1, g_2, \tau) = \prod_{ij} s_{ij}^{-n_{ij}}(\tau) \left\langle \frac{\Phi([TF])}{\Phi([\dim TF])}, [N^{g_1, g_2}] \right\rangle. \tag{4.5}$$

The conventions in (4.5) are the same as in Definition 2.7, and the functions $s_{ij}(\tau)$ are those defined in (2.13).

Case 2: Suppose now that $H = \langle g_1, g_2 \rangle \neq G$, and let V' and $(M', \partial M')$ be the representation V and the manifold M with the H action. Then we define

$$\Phi_G^V(M, \partial M)(g_1, g_2, \tau) = \Phi_H^{V'}(M', \partial M')(g_1, g_2, \tau), \tag{4.6}$$

where the right-hand side is defined as in Case 1.

We can now handle the general case.

PROPOSITION 4.7. *Suppose that, for any* $H \in$ objects $\mathfrak{Z}G$ (see Proposition 3.12), *there exists an extension* $\Phi_H: MSO_*^H \to \mathrm{Ell}_*^H$ *of the geometric twisted elliptic genus* Φ_H. *Suppose also that if* $K \xrightarrow{\alpha} H$ *is a morphism in* $\mathfrak{Z}G$ *then* $\alpha_{\mathrm{Ell}}\Phi_H = \Phi_K \alpha_{MSO}$, *where the morphisms* α_{Ell} *and* α_{MSO} *are the morphisms induced by restriction or conjugation morphisms defined by the Mackey functor structures of* Ell_*^G *and* MSO_*^G. *Then there exists a unique extension of* Φ_G *compatible with the restriction and conjugation morphisms of the Mackey functor structures of* MSO^G *and* Ell^G.

Proof. By hypothesis, an extension of the geometric twisted genus Φ_G must fit into a commutative diagram

$$\begin{array}{ccc} MSO_*^G & \xrightarrow{\Phi_G} & \mathrm{Ell}_*^G \\ \text{rest}\downarrow & & \text{rest}\downarrow \\ \varprojlim_{\langle g_1, g_2\rangle} MSO_*^{\langle g_1, g_2\rangle} & \longrightarrow & \varprojlim_{\langle g_1, g_2\rangle} \mathrm{Ell}_*^{\langle g_1, g_2\rangle}, \end{array} \tag{4.8}$$

where the limits are taken over the category $\mathfrak{Z}G$ and the bottom row is obtained from the extensions of Φ_H. The result follows because, by Proposition 3.12, the restriction in the right-hand column is an isomorphism. □

REMARK 4.9. The functions $\sigma_{ij}(\tau)$ are what we obtain in Witten's twisted class if we take E to be the trivial vector bundle $V_{ij} \times X \to X$. Since they are units, Witten's twisted class is well-defined.

PROPOSITION 4.10. *If* V *and* W *are two elements of* $F\mathfrak{U}$ *such that* $V \cup W = 0$, *then* $\Phi_G^{V\otimes W}\sigma(W) = \Phi_G^V$.

Proof. It clearly suffices to consider the case $G = \langle g_1, g_2\rangle$ and to check that both sides give the same result when we evaluate them in (g_1, g_2, τ). In this case, the result follows immediately from the defintion. □

Let

$$\Delta_{MSO} = [\mathbb{H}P^2]([\mathbb{C}P^2]^2 - [\mathbb{H}P^2])^2 \in MSO_* \subset MSO_*^G.$$

Then $\Phi_G(\Delta_{MSO}) = (\Delta) \in \mathrm{Ell}_*^G$. Since Δ is invertible in Ell_*^G, the twisted elliptic genus admits a factorization

$$\begin{array}{ccc} MSO_*^G & \xrightarrow{\Phi_G} & \mathrm{Ell}_*^G \\ \downarrow & & \| \\ MSO_*^G[1/\Delta_{MSO}] & \xrightarrow{\Phi_G} & \mathrm{Ell}_*^G. \end{array}$$

The same is true, of course, for cobordism and Ell_G^*. By standard arguments of change of rings we have the equality

$$MSO_G^*(X) \otimes_{MSO_G} \mathrm{Ell}_G^* = MSO_G^*\left[\frac{1}{\Delta_{MSO}}\right](X) \otimes_{MSO_G[1/\Delta_{MSO}]} \mathrm{Ell}_G^*.$$

We can therefore use $MSO_G^*[1/\Delta_{MSO}](X)$ instead of $MSO_G^*(X)$ in our computations.

PROPOSITION 4.11. *For each finite G-CW complex X, the functor*

$$H \to MSO_*^H\left[\frac{1}{\Delta_{MSO}}\right](X) \quad \left(\textit{resp. } H \to MSO_H^*\left[\frac{1}{\Delta_{MSO}}\right](X)\right)$$

has the structure of a Mackey (resp. Green) functor such that the natural transformations

$$MSO_*^H(X) \to MSO_*^H\left[\frac{1}{\Delta_{MSO}}\right](X) \quad \textit{and} \quad MSO_H^*(X) \to MSO_H^*\left[\frac{1}{\Delta_{MSO}}\right](X)$$

are transformations of Mackey (resp. Green) functors.

The proof is easy and is left to the reader. From now on, for any subgroup H of G let $mso_H^*(X) = MSO_H^*(X)[\Delta_{MSO}^{-1}] \otimes \mathbb{Z}[1/|G|]$, and let $mso^*(X) = MSO^*(X)[\Delta_{MSO}^{-1} \otimes \mathbb{Z}[1/|G|]$. Because $|G|$ is a unit in Ell_G^*, it follows that $\mathrm{Ell}_G^*(X) = mso_G^*(X) \otimes_{mso_G^*} \mathrm{Ell}_G^*$.

It is not difficult to show that the geometric twisted elliptic genus $\Phi_G: \Omega_G^* \to \mathrm{Ell}_G^*$ is a natural transformation of Green functors. The stabilization procedure used in order to pass from Ω_*^G to MSO_*^G is compatible with the Mackey functor structure. It is also easy to see that each one of the homomorphisms $\Phi_{\langle g_1, g_2 \rangle}^V$ is compatible with the Mackey functor structure in the sense that $\mathrm{rest}_H^G(\Phi_G^V) = \Phi_H^{\mathrm{rest}\, V}$ and $\mathrm{ind}_H^G(\Phi_H^V) = \Phi_G^{\mathrm{ind}\, V}$; a similar formula holds for the conjugation. The restriction and induction on the "representation variable" V correspond to the restriction and induction on representation theory.

PROPOSITION 4.12. *The homotopy-theoretic twisted elliptic genus* Φ_G: $mso_G^* \to \mathrm{Ell}_G^*$ *is a natural transformation of Mackey functors.*

PROPOSITION 4.13. *The homotopy-theoretic twisted elliptic genus* Φ_G: $mso_G^* \to \mathrm{Ell}_G^*$ *is an epimorphism.*

Proof. Using Proposition 4.12 and Proposition 3.12, we see that it suffices to consider the case $G = \langle g_1, g_2 \rangle$. In this case it suffices to prove, using the isomorphism Λ, that if $\langle g_1, g_2 \rangle = G$ then, for all $\theta \in \mathrm{Ell}^*(\Gamma(g_1, g_2))$, there exists $[M] \in mso_G^*$ such that $\Phi_G([M]) = \theta$.

By the structure theorem for Abelian groups we can suppose that $G = \mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/c'\mathbb{Z}$ where $c' \mid c$. Let us suppose first that $c' = c$. In this case

$$\mathrm{Ell}^*(\Gamma_0(c) \cup \Gamma_0(2)) = \mathrm{Ell}^*[s_{jk}(\tau)], \tag{4.14}$$

where $s_{jk}(\tau)$ are the functions defined in (2.13). The result follows from the fact that the functions $s_{jk}(\tau)$ can be obtained by applying the homotopy twisted elliptic genus to the Euler class of the irreducible representation V_{jk} of weight (j, k) of $\mathbb{Z}_c \times \mathbb{Z}_c$.

We can now consider the general case. In this case

$$\Gamma(g_1, g_2) = \left\{ \begin{pmatrix} a & b \\ e & d \end{pmatrix} \in \Gamma_0(2) \mid \begin{pmatrix} a & b \\ e & d \end{pmatrix} \equiv \begin{pmatrix} 1 & jc' \\ 0 & 1 \end{pmatrix} \bmod c \right\},$$

where $j = 1, \cdot, c/c'$. The ring $\mathrm{Ell}^*(\Gamma(g_1, g_2))$ is in this case equal to

$$\mathrm{Ell}^*(\Gamma(c)\cap\Gamma_0(2))^H, \tag{4.15}$$

where $H=\Gamma(g_1,g_2)/\Gamma(c)\cap\Gamma_0(2)$. The morphism $\left(\begin{smallmatrix} a & b\\ e & d\end{smallmatrix}\right)\to b$ induces an isomorphism $H\sim \mathbb{Z}/(c/c')\mathbb{Z}$. On the other hand,

$$R(\mathbb{Z}/c\mathbb{Z}\times\mathbb{Z}/c'\mathbb{Z})=R(\mathbb{Z}/c\mathbb{Z}\times\mathbb{Z}/c'\mathbb{Z})^H,$$

where the action of H is induced by its action on G. The result follows from the previous case and the fact that the homotopy-theoretic elliptic genus is H equivariant. □

Let us record now two auxiliary results that we will need in the next section.

PROPOSITION 4.16. *Let H be a subgroup of G, and let I_H be the kernel of the homotopy-theoretic twisted elliptic genus $\Phi_H\colon m\mathrm{so}_H^*\to\mathrm{Ell}_H^*$. Then $I_H=\mathrm{rest}_H^G(I_G)m\mathrm{so}_H^*$.*

Proof. The inclusion $\mathrm{rest}_H^G(I_G)m\mathrm{so}_H^*\subset I_H$ follows from Proposition 4.12. We shall prove the other inclusion by induction on H. If H is the trivial subgroup, then the result is obvious. Suppose then that the result is true for any proper subgroup of H. Let $N_H(G)$ be the normalizer of H in G and let $P_H=(1/|N_H(G)|)\sum_{g\in N_H(G)}c_g$. Then $P_H\colon m\mathrm{so}_H^*\to m\mathrm{so}_H^*$ is a projector, and applying it to I_H we obtain a direct sum decomposition $I_H=I_H^{\mathrm{inv}}+I'_H$ with $I_H^{\mathrm{inv}}=P_H(I_H)$ and $I'_H=(1-P)I_H$. Let $[M]=[M_1]+[M_2]\in I_H^{\mathrm{inv}}+I'_H$. Then

$$\mathrm{rest}_H^G\,\mathrm{ind}_H^G([M])=\mathrm{rest}_H^G\,\mathrm{ind}_H^G([M_1])+\mathrm{rest}_H^G\,\mathrm{ind}_H^G([M_2]). \tag{4.17}$$

Since the homotopy-theoretic twisted elliptic genus is a transformation of Green functors, the left-hand side of this formula is in $\mathrm{rest}_H^G(I_G)$. Let us study now the right-hand side. Recall that the induction and the conjugation homomorphisms commute and therefore $0=\mathrm{ind}_H^G(P_H([M_2]))=P_H(\mathrm{ind}_H^G([M_2])]$; however, since $N_H\subset G$, P_H acts on $m\mathrm{so}_G^*$ as the identity. This implies that $\mathrm{rest}_H^G\,\mathrm{ind}_H^G[(M_2])=0$. By the Mackey axiom,

$$\mathrm{rest}_H^G\,\mathrm{ind}_H^G([M_1])=\sum_{g\in H\backslash G/H}\mathrm{ind}_{H\cap H^g}^H\,\mathrm{rest}_{H\cap H^g}^{H^g}\,c_g([M_1]).$$

There are two possibilities: either $H^g=H$ (in which case $g\in H\backslash N_H(G)/H$); or $H\cap H^g$ is properly contained in H. We can therefore write

$$\mathrm{rest}_H^G\,\mathrm{ind}_H^G([M_1])=W_1+W_2,$$

where

$$W_1=\sum_{g\in H\backslash N_H(G)/H}\mathrm{ind}_{H\cap H^g}^H\,\mathrm{rest}_{H\cap H^g}^{H^g}\,c_g([M_1])$$

and

$$W_2=\sum_{\substack{g\in H\backslash G/H\\ H^g\neq H}}\mathrm{ind}_{H\cap H^g}^H\,\mathrm{rest}_{H\cap H^g}^{H^g}\,c_g([M_1]).$$

As $[M_1]\in I_H^{\mathrm{inv}}$, $W_1=k[M_1]$ where k is a number that divides $|G|$. On the other hand, by the inductive hypothesis each one of the terms $\mathrm{rest}_{H\cap H^g}^{H^g}\,c_g([M_1)]$

in the sum that defines W_2 is in $\mathrm{rest}^G_{H\cap H^g}(I_G)m\mathrm{so}^*_{H\cap H^g}$ and, by Proposition 4.12 combined with condition G2 of Definition 3.6, it follows that W_2 is in $\mathrm{rest}^G_H(I_G)m\mathrm{so}^*_H$. □

COROLLARY 4.18. *The isomorphism* $\kappa\colon m\mathrm{so}^*_H(X)\to m\mathrm{so}^*_G(G/H\times X)$ *induces, for any finite G-CW complex X, an isomorphism* $\kappa\colon \mathrm{Ell}^*_H(X)\to \mathrm{Ell}^*_G(G/H\times X)$.

As an application of the theory of Green functors let us establish a flatness lemma for oriented equivariant cobordism. The technique can, in principle, also be used for any stable equivariant cohomology theory.

PROPOSITION 4.19. *The ring* $r\mathrm{so}^*_G$ *is a flat* $m\mathrm{so}^*$ *module.*

Proof. We shall prove the proposition by induction on $|G|$. The case $|G|=1$ is trivial. Let us suppose then that the result is true for any group H such that $|H|<|G|$. The decomposition $A(G)\otimes\mathbb{Z}[1/|G|]=\bigoplus e_H(A(G)\otimes\mathbb{Z}[1/|G|])$ induces a decomposition $m\mathrm{so}^*_G=\bigoplus_{(H)} e_H m\mathrm{so}^*_G$. It is not difficult to show that this decomposition is a decomposition of $m\mathrm{so}^*$ modules, so it suffices to show that each one of the factors is a flat $m\mathrm{so}^*$ module. By Lemma 2.2 of [13], the restriction rest^G_H induces an isomorphism $e_H m\mathrm{so}^*_G\simeq e_H\{m\mathrm{so}^*_H\}^{WH}$. The $m\mathrm{so}^*$ module $e_H\{m\mathrm{so}^*_H\}^{WH}$ is a direct summand of $m\mathrm{so}^*_H$, since it can be obtained applying the projectors e_H and $(1/|N_H(G)|)\sum_{g\in N_H(G)} c_g$. Therefore, if H is a proper subgroup of G, then by the inductive hypothesis the $m\mathrm{so}^*$ module $e_H\{m\mathrm{so}^*_H\}^{WH}$ is flat. Let us consider now the factor $e_G m\mathrm{so}^*_G$. By Theorems (3.6) and (4.7) of [1], there is an isomorphism $e_G m\mathrm{so}^*_G\simeq m\mathrm{so}^*$. (In [1] it is shown only that it is an isomorphism of $A(G)$ modules, but this isomorphism is obtained as a composition of a restriction with a "restriction to fixed point spectrum" and is therefore also an $m\mathrm{so}^*$-modules homomorphism.) Therefore, the theorem is also true in this case. □

5. Statement and Proofs of the Main Results

In this section we shall prove Theorem 1.8.

PROPOSITION 5.1. *Let X be a finite G-CW complex. Then the functor* $H\to \mathrm{Ell}^*_H(X)$ *has a natural structure of Green functor.*

Proof. Because the homotopy-theoretic twisted elliptic genus is an epimorphism, $\mathrm{Ell}^*_G(X)$ is isomorphic to $m\mathrm{so}^*_G(X)/I_G m\mathrm{so}^*_G(X)$, where $I_G\subset m\mathrm{so}^*_G$ is the kernel of the homotopy-theoretic twisted elliptic genus. We want to show that the Green functor structure of $H\to m\mathrm{so}^*_H(X)$ induces a Green functor structure on $X\to \mathrm{Ell}^*_H(X)$. For this it suffices to show that the restriction, conjugation, and induction homomorphisms of the Green functor structure of $m\mathrm{so}^*_G$ preserve the ideals $I_G m\mathrm{so}^*_G(X)$. The result for restriction and induction is an immediate consequence of Proposition 4.12 and the fact that they are morphisms of algebras.

We need to show that also the induction morphisms ind_K^H pass to the quotient. This result follows from Proposition 4.16 and condition G2 in Definition 3.6 of a Green functor. □

COROLLARY 5.2. *Let X be a finite G-CW complex. Then the natural transformation*

$$m\mathrm{so}_G^*(X) \to \mathrm{Ell}_G^*(X)$$

is a natural transformation of Green functors.

THEOREM 5.3. *There exists a natural equivalence of functors*

$$\mathrm{Ell}_G(X) \to \bigoplus_{\langle g_1, g_2\rangle \in \mathcal{CC}} [\mathrm{Ell}^*(X^{g_1, g_2}) \otimes_{\mathrm{Ell}^*} \mathrm{Ell}^*_{\langle g_1, g_2\rangle}]^{W\langle g_1, g_2\rangle}_{S(\langle g_1, g_2\rangle)},$$

where the sum is taken over a complete set $\mathcal{CC}$ of representatives of conjugation classes of subgroups of the form $\langle g_1, g_2\rangle$, $W(\langle g_1, g_2\rangle)$ is the Weyl group of $\langle g_1, g_2\rangle$, and the localization is with respect to the set $S(\langle g_1, g_2\rangle)$, which is the image of the ideal $q(\langle g_1, g_2\rangle, 0) \subset A(G)$ under the natural homorphism $A(G) \to \mathrm{Ell}_G^*$. Both functors are functors from finite G-CW complexes to graded rings.

PROPOSITION 5.4. *The functor*

$$X \to \bigoplus_{(g_1, g_2) \in TG/G} [\{\mathrm{Ell}^*(X^{g_1, g_2}) \otimes_{\mathrm{Ell}^*} \mathrm{Ell}^*_{\langle g_1, g_2\rangle}]^{W\langle g_1, g_2\rangle}_{S(\langle g_1, g_2\rangle)} \tag{5.5}$$

is an equivariant cohomology theory.

The proof of this result follows from an argument entirely similar to the argument used to prove Propostion 2.2. The key point is the flatness of $\mathrm{Ell}^*_{\langle g_1, g_2\rangle}$, which was established in Proposition 3.18.

Clearly, Theorem 1.8 is an immediate consequence of these results.

Let

$$\mathrm{Ell}_G(X) \simeq \bigoplus_{H \in G} e_H \mathrm{Ell}_G^*(X) \tag{5.6}$$

be the decomposition of $\mathrm{Ell}_G^*(X)$ induced by the Burnside ring module structure of equivariant elliptic cohomology (see (3.9)).

PROPOSITION 5.7. *There is a canonical isomorphism*

$$e_H\left(m\mathrm{so}_G^*(X) \bigotimes_{m\mathrm{so}_G^*} \mathrm{Ell}_G^*\right) \simeq e_H(m\mathrm{so}_G^*(X)) \bigotimes_{e_H m\mathrm{so}_G^*} e_H \mathrm{Ell}_G^*.$$

The proof is straightforward given that multiplication by elements of the Burnside ring commutes with multiplication by elements in $m\mathrm{so}_G$.

The action of the Burnside ring on Ell_G^* has been described in Proposition 3.12, so let us now describe the "topological" part.

PROPOSITION 5.8. *Let $i: A(G) \to m\mathrm{so}_G^*(X)$ be the natural ring homomorphism defined by the Green functor structure of $m\mathrm{so}_G^*(C)$. Let (H) be a conjugacy class of subgroups of G and let H be a fixed representative in (H). Then there exists a natural isomorphism of functors*

$$e_H(m\mathrm{so}_G^*(X)) \simeq \{(m\mathrm{so}^*(X^H) \otimes_{m\mathrm{so}^*} m\mathrm{so}_H^*)_{(S(H))}\}^{WH},$$

where $S(H) = i(q(H,0))$ *and* $m\mathrm{so}^*(X^H) \otimes_{m\mathrm{so}^*} m\mathrm{so}^*_{H(S(H))}$ *is the localization of the $A(G)$ module $m\mathrm{so}^*(X^H) \otimes_{m\mathrm{so}^*} m\mathrm{so}_H^*$ at $S(H)$.*

Proof. We shall prove the proposition in two steps.

Step 1: By Lemma 2.2 of [13] and the isomorphism $\kappa: m\mathrm{so}_H^*(X) \to m\mathrm{so}_G^*(G/H \times X)$, there is an isomorphism

$$\begin{aligned} e_H m\mathrm{so}_G^*(X) &\simeq e_H m\mathrm{so}_G^*(G/H \times X^H)^{WH} \\ &\simeq e_H m\mathrm{so}_H^*(X^H)^{WH}. \end{aligned}$$

By Lemma 4.7 of [17], it follows that $e_H m\mathrm{so}_H^*(X^H)^{WH} \simeq m\mathrm{so}_H^*(X^H)^{WH}_{S(H)}$.

Step 2: Let X be a finite CW-complex with a trivial H action. We have two natural homomorphisms i and p^*. The first one,

$$i: m\mathrm{so}^*(X) \xrightarrow{i} \Omega_H^*(X) \otimes \mathbb{Z}[1/|G|] \to m\mathrm{so}_H^*(X),$$

is the natural inclusion obtained by regarding a manifold $M \to X$ as a G-manifold with the trivial action. The second one, $p^*: m\mathrm{so}_G^* \to m\mathrm{so}_G^*(X)$, corresponds to $p: X \to pt$. These homomorphisms induce, by multiplication, a group homomorphism $r(X): m\mathrm{so}^*(X) \otimes_{\mathbb{Z}[1/|G|]} m\mathrm{so}_G^* \to m\mathrm{so}_G^*(X)$, where $(m\mathrm{so}^*(X) \otimes_{\mathbb{Z}[1/|G|]} m\mathrm{so}_G^*)$ has the graded tensor product structure. Since $m\mathrm{so}_G^*(X)$ is graded commutative, r is a ring homomorphism. This homomorphism induces an homomorphism $r'(X): (m\mathrm{so}_G^*(X) \otimes_{m\mathrm{so}^*} m\mathrm{so}_G^*) \to m\mathrm{so}_G^*(X)$. By Proposition 4.19, $m\mathrm{so}_G^*$ is a flat $m\mathrm{so}^*$ module, so the functor $X \to (m\mathrm{so}^*(X) \otimes_{\mathbb{Z}} m\mathrm{so}_G^*)$ is an equivariant cohomology theory. The functor $X \to m\mathrm{so}_G^*(X)$ is also a generalized cohomology theory. It is easy to see r' is a natural transformation between both cohomology theories that is an isomorphism when $X = pt$. The comparison theorem implies that r' is an isomorphism of cohomology theories. □

Combining all the previous propositions we obtain the following isomorphism. (For simplicity, from now on we shall omit from the formulas the localization with respect to $S(\langle g_1, g_2 \rangle)$.)

$$e_H \mathrm{Ell}_G^*(X) \simeq \{m\mathrm{so}^*(X^H) \otimes_{m\mathrm{so}^*} m\mathrm{so}_H^*\}^{WH} \bigotimes_{[m\mathrm{so}_H^*]^{WH}} \{\mathrm{Ell}_H^*\}^{WH}. \qquad (5.9)$$

The set of WH-invariant elements of $\{m\mathrm{so}^*(X^H) \otimes_{m\mathrm{so}^*} m\mathrm{so}_H^*\}$ can be obtained taking the average with respect to the WH action. It follows that

$$\{m\mathrm{so}^*(X^H) \otimes_{m\mathrm{so}^*} m\mathrm{so}_H^*\}^{WH} \simeq \{m\mathrm{so}^*(X^H)\}^{WH} \otimes_{m\mathrm{so}^*} \{m\mathrm{so}_H^*\}^{WH}. \qquad (5.10)$$

Combining this isomorphism with the commutative diagram

$$\begin{array}{ccc} m\mathrm{so}^* & \xrightarrow{i} & \{m\mathrm{so}_H^*\}^{WH} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi_H} \\ \mathrm{Ell}^* & \xrightarrow{i} & \{\mathrm{Ell}_H^*\}^{WH}, \end{array}$$

where i are the obvious inclusions, we obtain from (5.10) an isomorphism

$$e_H \mathrm{Ell}_G^*(X) \simeq \{\mathrm{Ell}^*(X^H)\}^{WH} \otimes_{\mathrm{Ell}^*} \mathrm{Ell}_H^{WH}. \tag{5.11}$$

Repeating the same argument used to establish (5.10), we conclude that there is a natural isomorphism of functors

$$\{\mathrm{Ell}^*(X^H)\}^{WH} \otimes_{\mathrm{Ell}^*} \mathrm{Ell}_H^{WH} \to \{\mathrm{Ell}^*(X^H) \otimes_{\mathrm{Ell}^*} \mathrm{Ell}_H^*\}^{WH}.$$

Theorem 5.3 follows immediately.

COROLLARY 5.12. *If G acts trivially on X, then* $\mathrm{Ell}_G^*(X) \simeq \mathrm{Ell}^*(X) \otimes_{\mathrm{Ell}^*} \mathrm{Ell}_G^*$.

6. Equivariant Euler Characteristics

Graded Fields

A *graded field* $\mathbb{F} = \bigoplus_i F_i$ is a graded ring in which every nonzero homogeneous element is invertible.

PROPOSITION 6.1. *Let $\mathbb{F}$ be a graded field and let G be a finite group that acts on $\mathbb{F}$ via automorphisms of degree 0. Then:*

(1) *every graded $\mathbb{F}$-module* $\mathbf{M}$ *is free; and*

(2) *if $\mathbb{F}^G = \{\sigma \in \mathbb{F} \mid g\sigma = \sigma \ \forall g \in G\}$, then $\mathbb{F}^G$ is a graded field and*

$$\mathrm{rank}_{\mathbb{F}^G} \mathbb{F} = |G|.$$

Proof. The first assertion is a standard fact of graded algebra. The key points are that from any set of generators one can find a set of homogeneous generators. It is then easy to show, using the same argument of linear algebra, that a minimal set of homogeneous generators is a basis. The proof of the second statement is similar to the proof of Theorem 3 of [4, Chap. 5, Sec. 7.5]. □

If Γ is a congruence subgroup of $\Gamma_0(2)$, then we shall denote by $\mathbb{F}(\Gamma)$ the graded field of fractions of $\mathrm{Ell}^*(\Gamma) \otimes \mathbb{C}$. Let $\Gamma_G = \Gamma(|G|) \cap \Gamma_0(2)$ and $\mathbb{F}_G^* = \mathbb{F}^*(\Gamma_G)$.

These graded fields are obtained by taking *meromorphic modular forms* instead of holomorphic ones, and appear naturally in the theory of automorphic forms [26, Def. 2.1]. We need to work with them since they capture the action of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which cannot be seen if one works with fields of modular functions. For example, we see from condition (1) in Definition 3.2 that elements of degree 0 can not distinguish between (g_1, g_2) and (g_1^{-1}, g_2^{-1}). This difference introduces a crucial factor of 2 and destroys the relation between Ell_G^* and modular forms of high level.

Proof of Theorem 1.12. Let $\mathbb{F}_c^* = \mathbb{F}^*(\Gamma_0(2))$. There are two advantages in tensoring by $\mathbb{C}$. The first one is that we can use the classical theory of modular forms to compute the numbers involved. The other advantage is that we can limit ourselves to the study of the $\Gamma_0(2) \times G$ action, since the Galois symmetry disappears (cf. [20, Chap. 6]).

We shall consider first the case $X = pt$. In this case, since $\mathrm{Ell}_G^{\mathrm{odd}} = 0$, we need only compute $\mathrm{rank}_{\mathbb{F}_c^*}(\mathrm{Ell}_G \otimes_{\mathrm{Ell}^*} \mathbb{F}_c^*)^{\mathrm{even}}$. Let us fix $[g_1, g_2] \in TG$ representing a simultaneous conjugacy class $\overline{(g_1, g_2)}$, and let $\Gamma([g_1, g_2]) \subset \Gamma_0(2)$ be the isotropy group of $[g_1, g_2]$. Using the results of [9, Part VII], one can prove that $\mathrm{Ell}^*(\Gamma(g_1, g_2))^{\Gamma_0(2)} = \mathrm{Ell}^*$. It follows that $\mathbb{F}(\Gamma(g_1, g_2))^{(\Gamma_0(2))} = \mathbb{F}_c^*$, and using Proposition 6.1 we find that

$$\mathrm{rank}_{\mathbb{F}_c(\Gamma([g_1,g_2]))^{\Gamma_0(2)}} \mathbb{F}_c(\Gamma([g_1, g_2])) = [\Gamma_0(2), \Gamma([g_1, g_2])] = \#[g_1, g_2],$$

where $\#[g_1, g_2]$ is the cardinality of the set $\Gamma_0(2)[g_1, g_2]$. If $C_{g_1,g_2}(G) = C_{g_1}(G) \cap C_{g_2}(G)$, and if S is a set containing one representative $[g_1, g_2]$ in each orbit $\overline{(g_1, g_2)}$ of $\Gamma_0(2) \times G$ on TG, then

$$\begin{aligned}\mathrm{rank}_{\mathbb{F}_c^*} \mathrm{Ell}_G^{\mathrm{even}} \otimes \mathbb{F}_c^* &= \sum_{[g_1,g_2] \in S} \mathrm{rank}_{\mathbb{F}_c^*} \mathrm{Ell}^*(\Gamma[g_1, g_2]) \otimes_{\mathrm{Ell}} \mathbb{F}_c^* \\ &= \frac{1}{|G|} \sum_{(g_1,g_2) \in TG} |C_{g_1,g_2}(G)| = \frac{1}{|G|}(\#TTG). \qquad (6.2)\end{aligned}$$

THEOREM 6.3. *There is a natural isomorphism*

$$\mathrm{Ell}_G^*(X) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^* \xrightarrow{\psi} \bigoplus_{[g_1,g_2] \in TG/G} [\mathrm{Ell}^*(X^{g_1,g_2}) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*]^{C_{g_1,g_2}(G)},$$

where the sum is over a set of representatives of simultaneous conjugacy classes in TG and $[\cdots]^{C_{g_1,g_2}(G)}$ denotes the $C_{g_1,g_2}(G)$-invariant part.

Proof. Let us choose a pair $(g_1, g_2) \in TG$ and let $H \subset G$ be the subgroup generated by g_1 and g_2. Then we have a homomorphism

$$\mathrm{Ell}_G^*(X) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^* \xrightarrow{\mathrm{rest}_G^H \otimes \mathrm{id}} \mathrm{Ell}_H^*(X) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*. \qquad (6.4)$$

The inclusion $X^{g_1,g_2} \xrightarrow{i} X$ of H-spaces induces a homomorphism

$$\mathrm{Ell}_H^*(X) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^* \xrightarrow{i^*} \mathrm{Ell}_H^*(X^H) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*. \qquad (6.5)$$

The space X^H is a trivial H-space, so by Corollary 5.12 there exists a natural isomorphism

$$\mathrm{Ell}_H^*(X^H) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G = [\mathrm{Ell}^*(X^H) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*] \otimes_{\mathrm{Ell}^*} [\mathrm{Ell}_H^* \otimes \mathbb{F}_G]. \qquad (6.6)$$

Composing (6.4), (6.5), and (6.6), we obtain a homomorphism

$$\mathrm{Ell}_G^*(X) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^* \to [\mathrm{Ell}^*(X^H) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*] \otimes_{\mathrm{Ell}^*} [\mathrm{Ell}_H^*(pt) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*]. \qquad (6.7)$$

Since we are taking restriction of elements in $\mathrm{Ell}_G^*(X)$, it follows that (6.7) factors through the $C_{g_1,g_2}(G)$-invariant elements. Evaluation on (g_1, g_2) induces a homomorphism $\mathrm{Ell}_H^* \otimes_{\mathrm{Ell}} \mathbb{F}_G^* \to \mathbb{F}_G^*$. Using this homomorphism, we obtain from (6.7) a well-defined homomorphism

$$\psi_{g_1,g_2}: \mathrm{Ell}_G^*(X) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^* \to (\mathrm{Ell}^*(X^H) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*)^{C_{g_1,g_2}(G)}. \tag{6.8}$$

We define $\psi = \bigoplus \psi_{g_1,g_2}$. The same argument of Proposition 2.2 shows that both sides of (6.8) are equivariant cohomology theories, and it is easy to see that ψ is a transformation between them. For the case $X = pt$ we know that ψ is an isomorphism. To check that this is true in the general case we need only check [2] that this is true when X is of the form G/H, where H is any subgroup of G. In this case, by Corollary 4.18,

$$\mathrm{Ell}_G^*(G/H) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^* \simeq \mathrm{Ell}_H^*(pt) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*. \tag{6.9}$$

Because we are working with modules over a graded field, in order to show that they are isomorphic it suffices to show that they have the same rank. The rank of $\mathrm{Ell}_H^*(pt) \otimes \mathbb{F}_G^*$ as a $\mathbb{F}_G^*$-graded module is equal to $(1/|H|)(\#TTH)$. The right-hand side of (6.8) is

$$\bigoplus_{[g_1,g_2] \in TG/G} [\mathrm{Ell}^*((G/H)^{g_1,g_2}) \otimes_{\mathrm{Ell}^*} \mathbb{F}_G^*]^{C_{g_1,g_2}(G)}. \tag{6.10}$$

If $[g_1, g_2] \cap (H \times H) = \emptyset$, then $(G/H)^{g_1,g_2} = \emptyset$ so there is no contribution of the class $[g_1, g_2]$ to (6.10). If $[g_1, g_2] \cap (H \times H) \neq \emptyset$, then we can choose a representative $(h_1, h_2) \in H \times H$ of the conjugacy class $[g_1, g_2]$, and then $(G/H)^{h_1,h_2}$ is the set of classes gH such that $h_1 g = g h_1'$ and $h_2 g = g h_2'$ for some pair $(h_1', h_2') \in TH$. The rank of the submodule of $C_{g_1,g_2}(G)$-invariant elements of $\mathrm{Ell}^*((G/H)^{g_1,g_2}) \otimes \mathbb{F}_G$ is equal to $\#\{(G/H)^{g_1,g_2}/C_{g_1,g_2}(G)\}$.

The function $gH \to (g^{-1}h_1 g, g^{-1}h_2 g)$ induces a surjective map $(G/H)^{g_1,g_2} \xrightarrow{s} ([g_1, g_2] \cap TH)/H$, where H acts on $[g_1, g_2] \cap TH$ by conjugation. One can see that $s(gH) = s(g'H)$ if and only if gH and $g'H$ are in the same $C_{g_1,g_2}(G)$ orbit; therefore, $\#\{(G/H)^{g_1,g_2}/C_{g_1,g_2}(G)\} = \#\{([g_1, g_2] \cap TH)/H.\}$ Note that TH is equal to the disjoint union of the sets $([g_1, g_2] \cap TH)$ and that the action of H by simultaneous conjugation is compatible with the decomposition. Then we have

$$\sum_{[g_1,g_2]} \mathrm{rank}_{\mathbb{F}_G^*}(\mathrm{Ell}((G/H)^{g_1,g_2})^{C_{g_1,g_2}(G)} \otimes \mathbb{F}_G^*)$$
$$= \sum_{(g_1,g_2) \in TG/G} \#\{(G/H)^{g_1,g_2}/C_{g_1,g_2}(G)\} = \#\{TH/H\} = \frac{1}{|H|}(\#TTH).$$

This completes the proof of Theorem 6.3. □

Since $\mathrm{Ell}^*(X) \otimes_{\mathrm{Ell}} \mathbb{F}_G^*$ and $H^*(X, \mathbb{F}_G^*)$ are complex oriented cohomology theories, with the same coefficient ring over $\mathbb{Q}$ algebras there exists an isomorphism α between them. If we define $H^q(X, \mathbb{F}_G^*) = \bigoplus_{k=0}^{\infty} H^k(X, \mathbb{C}) \otimes \mathbb{F}_G^{n-k}$, then α induces isomorphisms between the odd and even parts. We can therefore replace $\mathrm{Ell}^*(X, \mathbb{F}_G^*)$ by $H^*(X, \mathbb{F}_G^*)$ in our computations. From the Lefschetz fixed-point theorem, it follows that

$$\chi(H^*(X^{g_1,g_2}, \mathbb{F}_G^*)^{C_{g_1,g_2}(G)}\} = \frac{1}{|C_{g_1,g_2}(G)|} \sum_{g_3 \in C_{g_1,g_2}(G)} \chi(X^{g_1,g_2,g_3}).$$

Then

$$\chi(\mathrm{Ell}_G(X)) = \frac{1}{|G|} \sum_{[g_1,g_2] \in TG/G} \left(\frac{|G|}{|C_{g_1,g_2}(G)|} \sum_{g_3 \in C_{g_1,g_2}(G)} \chi(X^{g_1,g_2,g_3}) \right)$$

$$= \frac{1}{|G|} \sum_{(g_1,g_2) \in TG} \sum_{g_3 \in C_{g_1,g_2}(G)} \chi(X^{g_1,g_2,g_3}) = \chi_{\mathrm{Ell}}(X). \qquad \square$$

7. Added in Proof

Correction to the Description of the Rings of Modular Forms

It is our aim to make some corrections to the description of the moduli of elliptic curves with level structures. We need to make these corrections because Proposition 3.28 is false.

The moduli of Γ_n structures is a subscheme of $E_J[n] \times_{\mathfrak{M}_0} E_J[n]$. The coefficient ring $\mathcal{O}(\Gamma_n)$ is a localization of the tensor product

$$MU_*^G(E_J[n]) \otimes_{R_0} MU_*^G(E_J[n]) = R_0[x,y] \otimes_{R_0} R_0[x,y] \simeq R[x_1, x_2, y_1 y_2],$$

where x is a root of the polynomial $T_n(X)$ and y is a root of the polynomial $YG_n(X) - F_n^2(X)$. For each pair $(a,b) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ such that $(a,b) \neq 0$ we have an element

$$S_{(a,b)} \in \mathcal{O}(\mathfrak{M}_J(n))$$

defined by $S_{(a,b)}(P,Q) = x(aP + bQ)$, where $aP + bQ \in E_J[n]$ is obtained using the group structure of $E_J[n]$ and x is the restriction of the X-coordinate of the universal Jacobi quartic; the elements $S_{(a,b)}$ correspond to the functions $s_{(i,j)}$ of (2.13). A pair (P,Q) is in $\mathfrak{M}_J(n)$ if and only if $S_{(a,b)}(P,Q) \neq 0$ for all the pairs (a,b); hence

$$\mathcal{O}(\mathfrak{M}_J(n)) = R_0[x_1, x_2][\{S_{(a,b)}^{-1}\}_{(a,b) \neq (0,0)}].$$

It is easy to see that $x_1 = S_{(1,0)}$ and $x_2 = S_{(0,1)}$.

Using the addition law for the Jacobi quartic [20], one can obtain all the elements y_1, y_2 from the set of functions $\{S_{(a,b)}, S_{(a,b)}^{-1}\}$. The explicit formula is:

$$y_1 = \frac{1}{2}\left[\frac{(1 - \epsilon^2 S_{(1,0)}^4) S_{(2,0)}}{S_{(1,0)}}\right] \quad \text{and} \quad y_2 = \frac{1}{2}\left[\frac{(1 - \epsilon^2 S_{(0,1)}^4) S_{(0,2)}}{S_{(0,1)}}\right];$$

we therefore have the equality

$$\mathcal{O}(\mathfrak{M}_J(n)) = R_0[S_{(a,b)}, S_{(a,b)}^{-1}]. \tag{7.1}$$

Correction to the Proof of Proposition 4.13

We need to show that also the inverses $[s_{jk}(\tau)]^{-1}$ are in the image of the homotopic twisted genus. These inverses can be obtained by applying the twisted genus to the elements represented by the cycles

$$(pt, \emptyset) \to (D(V_{jk}), S(V_{jk})),$$

where V_{jk} is the irreducible representation of $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ of weight (j, k). □

References

[1] S. Araki, *Equivariant stable homotopy theory and idempotents of the Burnside ring,* Publ. Res. Inst. Math. Sci. 18 (1982), 1193–1212.

[2] M. F. Atiyah and G. B. Segal, *On equivariant Euler characteristics,* J. Geom. Phys. 6 (1989), 671–677.

[3] M. F. Atiyah and I. Singer, *The index of elliptic operators IV,* Ann. of Math. (2) 93 (1971), 119–138.

[4] N. Bourbaki, *Algèbre,* Éléments de mathematique, Chapter 5, Masson, Paris, 1985.

[5] J. L. Brylinski, *Representations of loop groups, Dirac operators on loop space and modular forms,* Topology 29 (1990), 461–480.

[6] P. Conner and E. Floyd, *Differentiable periodic maps,* 2nd ed., Lecture Notes in Math., 738, Springer, Berlin, 1979.

[7] S. Costenoble, P. May, and S. Waner, *Equivariant orientation theory,* preprint, 1989.

[8] S. Costenoble and S. Waner, *Equivariant orientations and G-bordism theory,* Pacific J. Math. 140 (1989), 63–84.

[9] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques,* Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math., 349, pp. 143–316, Springer, Berlin, 1972.

[10] J. A. Devoto, *An algebraic description of the elliptic cohomology of the classifying spaces,* I.H.E.S. preprint, 1994.

[11] J. Franke, *On the construction of elliptic cohomology,* Max-Planck Institut preprint no. 92-3.

[12] M. Hopkins, N. Kuhn, and D. Ravenel, *Morava K-theories of classifying spaces and generalized characters for finite groups,* Algebraic topology (San Feliu de Guixals, 1990), Lecture Notes in Math., 1509, pp. 186–209, Springer, Berlin, 1992.

[13] ———, *Generalized group characters and complex oriented cohomology theories,* preprint, 1990.

[14] J. Igusa, *On the transformation theory of elliptic functions,* Amer. J. Math. 81 (1959), 436–452.

[15] N. Katz, *p-adic properties of modular schemes and modular forms,* Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math., 350, pp. 69–191, Springer, Berlin, 1973.

[16] N. Katz and B. Mazur, *Arithmetic moduli of elliptic curves,* Ann. of Math. Stud., 108, Princeton Univ. Press, Princeton, NJ, 1985.
[17] C. Kosniowski, *Equivariant cohomology and stable cohomotopy,* Math. Ann. (1974), 83–104.
[18] P. Landweber, *Supersingular elliptic curves and congruences for Legendre polynomials,* Elliptic curves and modular forms in algebraic topology (P. Landweber, ed.), Lecture Notes in Math., 1326, pp. 69–93, Springer, Berlin, 1988.
[19] P. Landweber, D. Ravenel, and R. Stong, *Periodic cohomology theories defined by elliptic curves,* preprint.
[20] S. Lang, *Elliptic functions,* 2nd ed., Springer, Berlin, 1987.
[21] D. Mumford, *Tata lectures on theta I,* Progr. Math., 28, Birkhäuser, Boston, 1983.
[22] G. Segal, *Classifying spaces and spectral sequences,* Inst. Hautes Études Sci. Publ. Math. 34 (1968), 105–112.
[23] ———, *Equivariant K-theory,* Inst. Hautes Études Sci. Publ. Math. 34 (1968), 129–151.
[24] ———, *Elliptic cohomology,* Séminaire Bourbaki 1987/88, Astérisque 161/162 (1988), Exp. No. 695, 4, 187–201 (1989).
[25] J. P. Serre, *Linear representations of finite groups,* Springer, Berlin, 1977.
[26] G. Shimura, *Introduction to the arithmetic theory of authomorphic forms,* Princeton Univ. Press, Princeton, NJ, 1971.
[27] R. Stong, *Notes on cobordism theory,* Princeton Univ. Press, Princeton, NJ, 1968.
[28] T. tom Dieck, *Transformation groups and representation theory,* Lecture Notes in Math., 766, Springer, Berlin, 1979.
[29] E. Witten, *The index of the Dirac operator in loop space,* Elliptic curves and modular forms in algebraic topology (P. Landweber, ed.), Lecture Notes in Math., 1326, pp. 161–181, Springer, Berlin, 1988.
[30] T. Yoshida, *Idempotents of Burnside rings and Dress induction theorem,* J. Algebra 80 (1983), 90–105.

International Centre for Theoretical Physics
Trieste 34100
Italy

devoto@cfgauss.uni-math.gwdg.de

Current address
Math. Institut
Georg-August Univesität
Bunsentraße 3-5
Göttingen
Germany