Extremal Problems for Quadratic Differentials

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1. Introduction

The Extremal Problem on the Unit Disk

Let $\mathbb{D}$ denote the unit disk and σ a finite set of four or more points on $\partial\mathbb{D}$. Then the Banach space Q_σ of all functions φ, holomorphic in $\mathbb{C}\backslash\sigma$, for which $\varphi(z)\,dz^2$ is real along $\partial\mathbb{D}\backslash\sigma$ and for which the L_1-norm

$$\|\varphi\| = \int\int_{\mathbb{D}} |\varphi|\,dx\,dy < \infty,$$

has positive dimension. Let the homotopy types γ_j of cross-cuts in $\bar{\mathbb{D}}\backslash\sigma$ be given. By reflection across the boundary of $\mathbb{D}$ $(z \mapsto (\bar{z})^{-1})$, the cross-cuts γ_j become closed curves g_j on the Riemann sphere. We assume that the family of these closed curves on $\bar{\mathbb{C}}\backslash\sigma$ is *admissible* in the following sense:

(i) the curves g_j are nonintersecting Jordan curves;
(ii) no two of the closed curves g_j is homotopic in $\bar{\mathbb{C}}\backslash\sigma$; and
(iii) none of the curves g_j is homotopically trivial or homotopic to a single point of σ.

The system of cross-cuts γ_j is called admissible if the corresponding reflected system of closed Jordan curves g_j is admissible. The Banach space Q_σ is a real subspace of the complex vector space of holomorphic, quadratic differentials with finite norm on the Riemann surface $\bar{\mathbb{C}}\backslash\sigma$. Since there is a global parameter z for the Riemann surface $\mathbb{D}$, identifying $\varphi(z)$ with $\varphi(z)\,dz^2$ provides an isomorphism between functions and quadratic differentials. Using this identification, we will refer to elements of Q_σ as *quadratic differentials.*

Any quadratic differential on a Riemann surface S induces a vector of heights of homotopy classes of simple, closed curves on S. The height of a closed curve γ is the infimum of the integrals

$$\int_{\tilde{\gamma}} |\text{Im}(\varphi(z)^{1/2}\,dz)|,$$

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where $\tilde{\gamma}$ is any closed curve homotopic to γ. The conformal structure of the Riemann surface S together with this vector of heights determines the holomorphic quadratic differential [HM; G; MS]. Quadratic differentials in Q_σ extend to Jenkins–Strebel differentials on $\bar{\mathbb{C}}\backslash\sigma$. This is because any noncritical, horizontal arc α along which $\varphi(\alpha(t))\alpha'(t)^2 > 0$ and which is a cross-cut of $\mathbb{D}$ reflects to a closed, horizontal trajectory $\tilde{\alpha}$ on $\mathbb{D}$. To say that elements φ of Q_σ are Jenkins–Strebel differentials means that the families of homotopic noncritical, horizontal trajectories of φ together with the critical trajectories partition $\mathbb{D}$ into strips bounded by the critical trajectories. On $\bar{\mathbb{C}}$, the strips joined with their reflections under j, where $j(z) = (\bar{z})^{-1}$, become annuli. The critical trajectories are those that meet zeros or poles of φ. Each of the strips determines the homotopy class of a cross-cut γ_j. Each strip has a height b_j, which is the vertical distance, measured in the metric $|\text{Im}(\varphi^{1/2}(z)\,dz)|$, between the horizontal boundary components of the strip. It is known [St] that for Jenkins–Strebel differentials these heights b_j determine the heights of all simple, closed curves on the surface $\bar{\mathbb{C}}\backslash\sigma$. Conversely, the vector of heights of all simple, closed curves determines the numbers b_j. When we refer to the heights of a quadratic differential, we usually mean the vector of heights of all homotopy classes of simple, closed curves. However, in the special case of a Jenkins–Strebel differential, by the heights of φ we will sometimes mean the heights of the Jenkins–Strebel annuli for φ.

Figure 2 (at the end of this article) illustrates the horizontal trajectory structure of a quadratic differential in Q_σ, where σ consists of six points on $\partial\mathbb{D}$. The disk is partitioned into three strip domains together with the critical trajectories of φ, which consist of three prongs emanating from the zero of φ.

To state our extremal problem, we take as given the finite set σ on the boundary of $\mathbb{D}$ and also the quadratic differential φ in Q_σ; φ determines an admissible family of cross-cuts γ_j together with heights b_j. In addition, we also assume there is given a compact subset E of $\mathbb{D}$ with finitely many components, each of which is simply connected. Let $\mathfrak{F}$ be the family of all univalent mappings f from $\mathbb{D}\backslash E$ into $\mathbb{D}$ such that $f(\partial\mathbb{D}) = \partial\mathbb{D}$. Although mappings f in $\mathfrak{F}$ are not defined on all of $\mathbb{D}$, each f induces a homotopy type $f(\gamma_j)$ in $\bar{\mathbb{D}}\backslash f(\sigma)$. This is because, given any cross-cut γ_j, we can select a homotopic cross-cut $\tilde{\gamma}_j$ that lies in $\mathbb{D}\backslash E$.

Given such an f and an admissible system of cross-cuts γ_j on $\bar{\mathbb{D}}\backslash\sigma$, we obtain an admissible system $f(\gamma_j)$ on $\bar{\mathbb{D}}\backslash f(\sigma)$. Together with the positive numbers b_j, this system determines a Jenkins–Strebel quadratic differential φ_f in $Q_{f(\sigma)}$.

In this article, we pose the problem of maximizing $\|\varphi_f\|$ by varying f in $\mathfrak{F}$. We show that the maximum exists and is realized by a uniquely determined pair (f_0, φ_0). If we let $f(E)$ be the set $\mathbb{D}\backslash f(\mathbb{D}\backslash E)$, then $f_0(E)$ consists of horizontal "slits" for the quadratic differential φ_0. Generically, these slits consist of arcs of horizontal trajectories of φ_0. In exceptional cases, some components of $f(E)$ may run through a zero of φ_0. Such a situation is depicted in Figure 2.

The Extremal Problem on a Surface of Finite Topological Type

Since our extremal problem involves considering the varying Riemann surfaces $\bar{\mathbb{C}}\setminus f(\sigma)$ as moving through the Teichmüller space of $\bar{\mathbb{C}}\setminus\sigma$, it is natural to reformulate the problem in the terms of Teichmüller theory. Accordingly. let S be a Riemann surface obtained from a compact Riemann surface by deleting finitely many holes and punctures. Such a surface is said to have *finite topological type.* If S has no holes, we say it has finite *analytic* type. If S has some holes, it can be realized as half of a doubled surface S^d with an anticonformal involution j such that $S^d = S \cup \text{border}(S) \cup j(S)$. The border of S is thereby realized as a finite set of simple, closed analytic curves in S^d.

The Teichmüller space, Teich(S), consists of equivalence classes of quasiconformal mappings g from S into a variable Riemann surface S_g. Two mappings g_0 and g_1, mapping S into S_0 and S_1 (respectively), are equivalent if there is a conformal mapping c from S_1 onto S_0 and a homotopy h_t mapping S to S_0 such that $h_0 = g_0$ and $h_1 = c \circ g_1$ and if, for every point p in any component of the boundary of S, $h_0^{-1} \circ h_t(p)$ is in the same component. This Teichmüller space is called a *reduced* Teichmüller space because, for p in the border of S, we permit p to differ from $h_0^{-1} \circ h_t(p)$ as long as it remains in the same component of the boundary. It follows that g_1 and g_0 represent the same element of Teich(S) if there is a conformal map c from S_1 to S_0 such that $g_0^{-1} \circ c \circ g_1$ preserves the free homotopy classes of a set of curves that mark a basis for the fundamental group of S.

Let $\varphi = \varphi(z)\, dz^2$ be a holomorphic quadratic differential on S of finite norm with the property that $\varphi(z)\, dz^2$ is real-valued on the border of S. From the heights mapping theorem [HM; MS; G; GM] we know there is a corresponding holomorphic quadratic differential φ_f on S_f whose heights on S_f are equal to the corresponding heights of φ on S.

Let E be a closed subset of S and let $\mathfrak{F}$ be the family of pairs (f, S_f), where f is holomorphic and univalent on $S\setminus E$ and maps to the variable Riemann surface S_f. We assume that every component of E is simply connected and that the function f induces an isomorphism from the fundamental group of S onto the fundamental group of S_f. This last assumption ensures that every element (f, S_f) of $\mathfrak{F}$ determines a point in Teich(S), even though f is defined only on $S\setminus E$.

For (f, S_f) in $\mathfrak{F}$, define $f(E)$ to be $S_f \setminus f(S\setminus E)$ and let

$$M_f = \|\varphi_f\| = \iint_{S_f} |\varphi_f|.$$

Consider the extremal problem

$$M = \sup\{M_f\colon (f, S_f) \in \mathfrak{F}\}. \tag{1}$$

The following "slit mapping theorem" provides the solution to this extremal problem.

THEOREM 1 (Existence). *Suppose S is a Riemann surface of finite topological type, and that φ is a holomorphic quadratic differential on S for which $\varphi(z)\,dz^2$ is real-valued on the border of S. Let E be a closed subset of the interior of S. Suppose also that E has finitely many components each of which is simply connected. Then there exists a point $\tau \in \mathrm{Teich}(S)$ represented by an element (f, S_f) of $\mathfrak{F}$ such that M_f realizes the supremum in* (1). *For this point τ, the mapping $f\colon S\backslash E \to S_f$ has the property that each component of $f(E)$ is an arc of a horizontal trajectory of φ_f or a connected union of arcs of horizontal trajectories and critical points of φ_f.*

We also have a uniqueness result.

THEOREM 2 (Uniqueness). *With all the same hypotheses as in Theorem 1, the point in $\mathrm{Teich}(S)$ represented by (f, S_f) is uniquely determined. In particular, if (g, S_g) is another element of the family $\mathfrak{F}$ and if the components of $g(E)$ are each contained in horizontal arcs of φ_g or in finite unions of horizontal arcs of φ_g and critical points of φ_g, then $h = g\circ f^{-1}$ extends to a conformal mapping from S_f onto S_g and $\varphi_g(h(z))\,h'(z)^2 = \varphi_f$.*

2. Smoothing the Contours

In the classical slit mapping theorem for a finitely connected plane domain $\Omega = \bar{\mathbb{C}}\backslash E$, one first simplifies the problem of finding a slit mapping by assuming that each component of the obstacle set E has an analytic boundary curve. This harmless assumption is justified by several preliminary applications of the Riemann mapping theorem. If $E = E_1 \cup E_2 \cup \cdots \cup E_n$, one first uses the Riemann mapping theorem to map the complement of E_1 to the complement of the unit disk. By this step, Ω is changed to a conformally equivalent domain where the component of the complement corresponding to the E_1 is bounded by a closed analytic curve, so that one may as well assume from the beginning that E_1 has this form. One can apply the Riemann mapping theorem n times in succession to obtain a domain conformal to Ω such that every component of the complement is bounded by an analytic curve.

In order to achieve this type of simplification in our setting, we need a canonical way to embed $S\backslash E$ by a mapping ι into a Riemann surface S' of finite analytic type so that $S'\backslash\iota(S\backslash E)$ consists of disks with analytic boundary curves. The infinite Nielsen extension provides one way to realize such an embedding. Since we do not need the details of the construction, we refer to [B2] for the definition of Nielsen extension. We let S_N be the infinite Nielsen extension of $S\backslash E$ that attaches to $S\backslash E$ a punctured disk corresponding to each component of E. We then let S' be the union of S_N together with these n punctures. There is an injection ι of $S\backslash E$ into S'. The mapping ι embeds $S\backslash E$ into S' in such a way that ι induces an isomorphism from the fundamental group of S onto the fundamental group of S'. The set $E' = S'\backslash\iota(S\backslash E)$ is a new obstacle set whose boundary is made up of n analytic curves.

To the holomorphic quadratic differential φ on S with finite norm there corresponds the measured foliation $|dv| = |\mathrm{Im}(\varphi^{1/2}\,dz)|$ on S. Since ι induces an isomorphism of fundamental groups, there is a corresponding measured foliation $|dv'|$ on S' with the same corresponding heights that $|dv|$ has on S; by the theorem of Hubbard and Masur [HM], there is a quadratic differential φ' on S' with the same corresponding heights.

Let $\mathcal{G}$ be the family of pairs (g, S_g), where g is holomorphic and univalent on $S'\backslash E'$ and maps to a variable Riemann surface S_g. The mapping ι from $S\backslash E$ to $S'\backslash E'$ induces an isomorphism from the family $\mathfrak{F}$ to the family $\mathcal{G}$ defined by the formula

$$f \mapsto g \circ \iota^{-1}.$$

Moreover, any extremal configuration for the family $\mathfrak{F}$ and the quadratic differential φ on S is also an extremal configuration for the family $\mathcal{G}$ and the quadratic differential φ' on S'. We have verified the following result.

PROPOSITION 1. *In proving Theorem 1, it suffices to consider obstacle sets E whose components are bounded by closed analytic curves.*

3. Boundedness of the Norm

The objective in this section is to show that for a fixed Riemann surface S of finite analytic type, for a fixed compact set E that is a union of simply connected components contained in S, and for a fixed quadratic differential φ of finite norm on S, the set of numbers $M_f = \|\varphi_f\|$, where (f, S_f) is in $\mathfrak{F}$, is bounded.

We begin with two lemmas. The first gives a lower bound on the widths of a finite set of curves measured with respect to all quadratic differentials with norm equal to 1. The second gives an upper bound on the Teichmüller distance in terms of the distortion of extremal lengths of this finite set of curves. We are indebted to Howard Masur for these observations; they appear in abbreviated form in [GM].

For a simple, closed curve β on S and a quadratic differential $\varphi(z)\,dz^2$ on S, we define the width of β with respect to φ to be

$$\mathrm{width}_\varphi(\beta) = \inf\left\{\int_{\tilde\beta} |\mathrm{Re}(\varphi(z)^{1/2}\,dz)|\right\},$$

where the infimum is taken over all curves $\tilde\beta$ freely homotopic in S to β.

LEMMA 1. *Let S be a Riemann surface of finite analytic type. Then there is a finite set of simple, closed curves β_j on S and a positive number c such that, for all holomorphic quadratic differentials φ with $\iint_S |\varphi|\,dx\,dy = 1$, for at least one of the curves β_j we have*

$$\mathrm{width}_\varphi(\beta_j) \geq c.$$

Proof. In the set $\{\beta_1, \ldots, \beta_N\}$ we include the partitioning curves in some pants decomposition of S. We also include pairs of curves crossing each partitioning curve, the first obtained from the second by a Dehn twist about the partitioning curve which they cross. The set of curves has the property that if $\text{width}_\varphi(\beta_j) = 0$ for every j with $0 \leq j \leq N$ then φ is identically equal to zero. This is because these widths are the heights of the measured foliation determined by the measure $|\text{Re}(\varphi^{1/2})|$ and, if the heights on this finite set of curves are all equal to zero, then the measure class of the measured foliation is equal to zero.

Suppose the constant c does not exist. Then there is a sequence of quadratic differentials φ_n, all of norm 1 and converging to a quadratic differential φ_0 of norm 1, such that all of the widths with respect to φ_n of all of the curves β_j converge to zero. Since the $\text{width}_{\varphi_n}(\beta_j)$ converges to $\text{width}_{\varphi_0}(\beta_j)$, φ_0 is a quadratic differential of norm 1 that has zero width along each curve β_j. This is a contradiction. □

For a simple, closed curve β on the Riemann surface S, let $\Lambda(\beta, S)$ denote the extremal length of the family of all curves on S that are freely homotopic to β. Let (g, S_g) be a point in Teich(S) and let $K_0(g)$ be the dilatation of the unique extremal quasiconformal mapping from S to S_g that is homotopic to g. From Teichmüller's theorem, we know that if g is the extremal mapping in its homotopy class then the Beltrami coefficient $g_{\bar{z}}/g_z$ has the form $k|\varphi|/\varphi$, where $0 \leq k < 1$ and $\|\varphi\| = 1$. Moreover, k is uniquely determined, and if $0 < k < 1$ then φ is uniquely determined. Corresponding to φ of norm 1 on S there is a terminal differential $\tilde{\varphi}$ of norm 1 on S_g such that the extremal mapping g takes the horizontal and vertical trajectories of φ onto the horizontal and vertical trajectories of $\tilde{\varphi}$, expanding in the horizontal direction by the factor of $K_0^{1/2}$ and contracting in the vertical direction by the factor of $K_0^{-1/2}$.

Since extremal length is a supremum over all choices of conformal metrics on a Riemann surface, normalized to have total mass 1, of squares of infima of lengths of curves in the same homotopy class, we have the following string of inequalities:

$$\Lambda(g(\beta_j), S_g) \geq \left(\int_{g(\beta_j)} |\tilde{\varphi}|^{1/2}\right)^2 \geq \left(\int_{g(\beta_j)} |\text{Re}(\tilde{\varphi}^{1/2}\, dz)|\right)^2$$
$$= K_0\left(\int_{\beta_j} |\text{Re}(\varphi^{1/2}\, dz)|\right)^2,$$

where we choose β_j suitably in its homotopy class. By choosing the appropriate curve β_j, we find that

$$K_0 = c^{-2}\Lambda(g(\beta_j), S_g),$$

which we summarize in the following lemma.

LEMMA 2. *There exist a positive constant c and a finite set of simple, closed curves β_j on S such that, for all points (g, S_g) in* Teich(S),

$$K_0(g) \le \frac{1}{c^2} \sup_{1 \le j \le N} \{\Lambda(g(\beta_j), S_g)\}.$$

The preceding lemmas enable us to prove the main result of this section, which gives a bound for the supremum in (1).

THEOREM 3. *Assume S is a Riemann surface of finite analytic type, and let E be a compact subset of S that is a union of simply connected components. Let φ be a quadratic differential of finite norm on S and let (f, S_f) be an element of the family $\mathfrak{F}$, where f is a univalent holomorphic mapping from $S - E$ into S_f inducing an isomorphism of the fundamental group of S onto the fundamental group of S_f. Let M_f be the norm of the unique holomorphic quadratic differential on S_f whose heights are equal to the corresponding heights of φ on S. Then the set of numbers M_f, for all (f, S_f) in $\mathfrak{F}$, is bounded.*

Proof. Note that although f is not defined on E, f determines a homotopy class of mappings from S to S_f because the components of E are simply connected. From the basic inequality for extremal length [G], we know that $M_f \le K_0 \|\varphi\|$, where K_0 is the norm of the extremal quasiconformal mapping g from S to S_f in the same homotopy class as f. On the other hand, if we pick β_j representing homotopy classes so that the β_j lie in $S - E$, then

$$K_0(g) \le \frac{1}{c^2} \sup_{1 \le j \le N} \{\Lambda(f(\beta_j), S_f)\}.$$

But

$$\Lambda(f(\beta_j), S_f) \le \Lambda(f(\beta_j), S_f \backslash f(E)) = \Lambda(\beta_j, S \backslash E).$$

Since each of these numbers is finite and there are a finite number of indices $1 \le j \le N$, we obtain the desired bound. □

The conclusion of this theorem remains valid with the following alterations in the hypotheses. S need only be a Riemann surface of finite topological type and φ a quadratic differential of finite norm with the property that $\varphi(z)\,dz^2$ is real-valued on the border of S. Elements (f, S_f) of the family $\mathfrak{F}$ consist of a univalent, holomorphic mapping f from $S \backslash E$ into a variable Riemann surface S_f. The mappings f have the further property that they map the border of S to the border of S_f. Under these hypotheses, the mappings f automatically extend to mappings between subsurfaces of the doubled Riemann surfaces S^d and S_f^d. The quadratic differentials φ and φ_f also extend to these doubled surfaces. Any quadratic differential on S^d with finite norm that is symmetric with respect to the anticonformal involution j is determined by its height on S. Moreover, S^d and S_f^d are surfaces of finite analytic type. We obtain the following generalization of Theorem 3.

THEOREM 4. *Assume S is a Riemann surface of finite topological type, and let E be a compact subset of the interior of S that is a union of simply con-*

nected components. Let φ be a quadratic differential of finite norm on S for which $\varphi(z)\,dz^2$ is real-valued on the border of S. Let (f, S_f) be an element of the family $\mathfrak{F}$, where f is a univalent holomorphic mapping from $S\backslash E$ into S_f inducing an isomorphism of the fundamental group of S onto the fundamental group of S_f. Moreover, assume f maps the border of S to the border of S_f. Let M_f be the norm of the unique holomorphic quadratic differential on S_f whose heights are equal to the corresponding heights of φ on S. Then the set of numbers M_f, for all (f, S_f) in $\mathfrak{F}$, is bounded.

4. A Variational Lemma

A point in Teich(S) is represented by a pair (g, S_g), where g is a quasiconformal mapping from S onto a variable Riemann surface S_g. The pair (g, S_g) determines a Beltrami coefficient μ on S given by the formula

$$\mu(z) = g_{\bar{z}}(z)/g_z(z).$$

The Beltrami coefficient μ determines g up to postcomposition by a conformal mapping. Therefore, μ determines the point in Teichmüller space represented by (g, S_g), and we may speak equivalently either of the Teichmüller class of (g, S_g) or of the Teichmüller class of μ.

A quadratic differential φ on S and an element of Teich(S) represented by a pair (g, S_g) determines uniquely, by the heights mapping principle, a quadratic differential φ_g on S_g. The height of any closed curve γ on S measured with respect to $|\mathrm{Im}(\varphi^{1/2}(z)\,dz)|$ is equal to the height of $g(\gamma)$ on S_g measured with respect to $|\mathrm{Im}(\varphi_g^{1/2}(z)\,dz|$.

The following formula for the variation of $M_g = \|\varphi_g\|$ is given in [G, p. 217]:

$$\log M_g = \log M + 2\,\mathrm{Re}\,\frac{1}{\|\varphi\|}\iint_S \mu\varphi\,dx\,dy + o(\|\mu\|_\infty).$$

5. Tangent Curves to Teichmüller Space

The tangent space to Teichmüller space can be realized as a factor space of the tangent space at the origin to the open unit ball of Beltrami coefficients in L_∞. In this section we give a variational lemma that is a necessary and sufficient condition for a Beltrami differential to be a tangent vector to a curve of trivial Beltrami coefficients. This lemma, although not always prominently exhibited, is the building block for most of the existence theorems of Teichmüller theory [A1; B1; R; G].

We will apply the lemma to the reduced Teichmüller space of the Riemann surface $R = S\backslash E$, where E is a finite union of simply connected components each of which is bounded by an analytic curve. We let $Q(R)$ be the finite-dimensional normed space of holomorphic quadratic differentials ψ on R for which

$$\|\psi\| = \iint_R |\psi(z)|\, dx\, dy < \infty$$

and for which $\psi(z)\,dz^2$ is real on the boundary of R. $Q(R)$ is the space of finite-norm holomorphic quadratic differentials on the double R^d of R which are invariant under the natural anticonformal involution j such that $R^d = R \cup \text{border}(R) \cup j(R)$.

A Beltrami differential μ on R is called *infinitesimally trivial* if

$$\operatorname{Re}\left(\iint_R \mu\psi\right) = 0$$

for all ψ in $Q(R)$. A curve of Beltrami coefficients $\mu_t = f_{\bar{z}}^t / f_z^t$ is called *trivial* if the pair $(f^t, f^t(R))$ represents the trivial element of the reduced Teichmüller space Teich(R) for each t, where the trivial element of Teich(R) is the one represented by the identity mapping from R to R.

The variational lemma can now be stated.

LEMMA 3. *There exists a curve of trivial Beltrami coefficients μ_t tangent to ν if and only if*

$$\operatorname{Re}\left(\iint_R \nu\psi\, dx\, dy\right) = 0$$

for all ψ in $Q(R)$.

REMARK. The statement that μ_t is tangent to ν is taken to mean that

$$\|\mu_t(z) - t\nu(z)\|_\infty / t \to 0 \quad \text{as } t \to 0.$$

A proof of Lemma 3 can be found in [G, Thm. 6, p. 107].

6. Schiffer and Beltrami Variations

In this section we show how certain Beltrami variations are equivalent to Schiffer variations. A Schiffer variation is obtained by attaching a disk along a slit. The equivalence will mean that existence of the Beltrami variation implies the existence of the Schiffer variation. On the other hand, the Schiffer variation yields a curve of mappings univalent in the complement of the obstacle set. Thus, if one is given a conformal invariant for a plane domain or a domain contained in a Riemann surface, and if one can show that the tangent vector to Beltrami variation paired with the cotangent vector for the conformal invariant gives a positive value, then one can conclude the existence of a curve of mappings holomorphic and univalent in the complement of the obstacle set along which the value of the conformal invariant is increased.

Before describing these two types of variation of conformal structure, we prove a preliminary lemma concerning a way of introducing a new conformal structure on a given Riemann surface S. Assume that W is a quasidisk contained in the interior of S, and that we are given an orientation-preserving quasisymmetric mapping h_0 from the boundary of W to the boundary of a quasidisk W_0 contained in the w-plane. We form the Riemann surface S_0 by deleting W from S and sewing, in its place, the quasidisk W_0 according to the boundary correspondence given by h_0. Formally, this means that S_0 is the disjoint union of $S \backslash W$ and W_0 factored by the equivalence defined by letting a point p in the boundary of W be identified with the point $h_0(p)$ in the boundary of W_0.

We define $f_0: S \to S_0$ by letting f_0 be the identity on the complement of W and letting f_0 be equal to any quasiconformal extension of h_0 mapping the interior of W to the interior of W_0. Since W is simply connected, the homotopy class of f_0 is determined independently of which extension is taken. If we declare that a chart for S_0 is any local homeomorphism of the form $z \circ f_0^{-1}$, where z is an arbitrary chart for S, then S_0 becomes a Riemann surface which we will think of as a variation of S.

Starting with the same Riemann surface S, the same quasidisk W, and another quasisymmetric mapping h_1 from the boundary of W to the boundary of another quasidisk W_1, we construct similarly the mapping $f_1: S \to S_1$. The following lemma states that the variation of the Riemann surface of S created by this recipe depends essentially on only the quasisymmetric mapping on the boundary of the quasidisk W, not on which particular quasiconformal extension is chosen in the interior of this quasidisk.

LEMMA 4. *If there exists a conformal mapping c from W_1 onto W_0 such that $h_0^{-1} \circ c \circ h_1$ is the identity on the boundary of W, then (f_0, S_0) and (f_1, S_1) represent the same point in* Teich(S).

Proof. We must find a conformal mapping C from S_1 onto S_0 such that $f_0^{-1} \circ C \circ f_1$ is homotopic to the identity by a homotopy that fixes the ideal boundary points of S. We let C be the identity on the complement of W and be equal to c on W_1. This is continuous on all of S_1 and also conformal because quasicircles are removable for conformal maps. The homotopy condition is satisfied because $f_0^{-1} \circ C \circ f_1$ is equal to the identity in the complement of V. □

We now describe what is called the "Schiffer interior variation by attaching a cell" [S]. This variation can be thought of as the process of cutting a buttonhole into a shirt along a short vertical slit, stretching the slit into a circular hole and then inserting a disk into this hole. If we think of conformal equivalence as defining geometrical shape, then this procedure changes the geometrical shape of the entire shirt. As the size of the buttonhole increases from zero to a small positive number, we obtain a variation of the geometrical structure.

To be mathematically precise, let S be a Riemann surface, V a simply connected open subset of S, and z a local parameter defined on V for which $z(V) = U$ is the unit disk in the complex plane $\mathbb{C}$. In U, let I be the vertical slit joining $2i\epsilon$ to $-2i\epsilon$ and let $J = z^{-1}(I)$. For each point z in the interior of I we associate two points z_+ and z_-, which are the prime ends corresponding to the approach to z from the right or from the left side of I. We let I_+ and I_- be the sets of points z_+ and z_- and let J_+ and J_- be the corresponding sets on S.

Let $w(z)$ be the univalent function defined in $U \setminus I$ by the formula $z = w - \epsilon^2 w^{-1}$. Note that the circle $w = \epsilon e^{i\theta}$ in the w-plane corresponds in the z-plane to $\{z = 2i\epsilon \sin\theta : 0 \le \theta \le 2\pi\}$. Let D_ϵ be the disk of radius ϵ centered at the origin in the w-plane. The Schiffer interior variation S_ϵ of S is formed by attaching D_ϵ to S along the arc J. A point $w = \epsilon e^{i\theta}$ on the boundary of D_ϵ with $-\pi/2 < \theta \le \pi/2$ is identified with p_+ if $w(z(p_+)) = w$. Similarly, $w = \epsilon e^{i\theta}$ with $\pi/2 < \theta \le 3\pi/2$ is identified with p_- if $w(z(p_-)) = w$. Let $V_\epsilon = (V \setminus J) \cup D_\epsilon$, and let

$$\tilde{w} = \begin{cases} w \circ z & \text{on } V \setminus J, \\ w & \text{on } D_\epsilon. \end{cases}$$

Charts for S_ϵ on open sets in $S \setminus J$ are the same as charts for S. On V_ϵ, which is an open set containing D_ϵ, we take $\tilde{w}$ as the chart. This collection of charts makes S_ϵ into a Riemann surface because the correspondence $z = w - \epsilon^2 w^{-1}$ is holomorphic on the overlapping domain $V \setminus J$, which is the common domain of definition of the charts $\tilde{w}$ and z.

Define a Beltrami variation $S_{\mu(\epsilon)}$ of S by deleting from S the interior of the quasidisk W whose boundary is the set of p such that $|w(z(p))| = 2\epsilon$ and attaching a new disk. The new disk has local parameter $\zeta = z + \frac{1}{4}\bar{z}$ for z in W. Note that the boundary of W corresponds to an ellipse in the z-plane and to a circle with radius 2ϵ in the w-plane. The Beltrami coefficient $\mu(\epsilon)$ takes the constant value 1/4 in W with respect to the local parameter z, and the size of W depends on ϵ. Both S_ϵ and $S_{\mu(\epsilon)}$ are obtained by a deformation inside a simply connected subset, and they therefore determine well-defined curves in Teich(S) that depend on the parameter ϵ.

THEOREM 5. *In the Teichmüller space* Teich(S), *the marked Riemann surfaces $S_{\mu(\epsilon)}$ and S_ϵ represent the same point.*

Proof. We apply Lemma 4 to the quasidisk W described in the previous paragraph. Let W_1 be the disk defined by $|w| \le 2\epsilon$ in the w-plane, and for p in the boundary of W let $h_1(p) = w(z(p))$. Let W_0 be a disk in the ζ-plane and let $h_0(p) = \zeta(z(p))$. To apply the lemma we must show there is a conformal map c such that $h_0^{-1} \circ c \circ h_1$ is the identity on the boundary of W. This is equivalent to showing that $h_0 \circ h_1^{-1}$ is the restriction of a conformal mapping on the boundary of W_1. Since $|w|^2 = 4\epsilon^2$ on that boundary,

$$w - \frac{\epsilon^2}{w} = w - \frac{1}{4}\bar{w} = z.$$

Therefore, for points on that boundary,

$$\zeta = z + \frac{1}{4}\bar{z} = \left(w - \frac{1}{4}(\bar{w}) + \frac{1}{4}\left(\bar{w} - \frac{1}{4}w\right)\right) = \frac{15}{16}w.$$

We conclude that S_ϵ and $S_{\mu(\epsilon)}$ represent the same point in Teich(S). □

7. Proof of Existence

By Theorem 3 and 4, we know that the set of numbers $M_f = \|\varphi_f\|$, where (f, S_f) is in $\mathfrak{F}$, is bounded. Select a sequence (f_n, S_n) such that M_{f_n} approaches the supremum of the values of M_f. By using normal families, we obtain a point (f_0, S_0) such that M_{f_0} is as large as possible. Let $E_0 = S_0 \backslash f_0(S \backslash E)$, and let φ_{f_0} be the holomorphic quadratic differential on S_0 with the same corresponding heights that φ has on S.

In order not to carry forward excess notation, for the remainder of this section we assume $S = S_0$, $\varphi = \varphi_{f_0}$, and $E = E_0$. Thus, it is no longer true that components of E are each bounded by closed analytic curves. On the other hand, for no univalent function f mapping $S \backslash E$ into a variable Riemann surface S_f and inducing an isomorphism of fundamental groups is it possible for $\|\varphi_f\|$ to be larger than $\|\varphi\|$.

We divide the proof of Theorem 1 into several steps.

Step 1: The set E has measure zero. If not, one can find a Beltrami differential μ that is identically zero on $S \backslash E$ but for which $\iint \mu\varphi \, dx\, dy > 0$. Thus, by the variational formula of Section 4, we obtain a curve of quasiconformal mappings f^t with Beltrami coefficients $t\mu$ that are conformal on $S \backslash E$ and for which M_f, where $f = f^t$, is larger than $\|\varphi\|$.

Step 2: φ is in $Q(S \backslash E)$ and, in particular, $\varphi(z)\, dz^2$ is real-valued on the border of $S \backslash E$. Assume that φ is not in $Q(S \backslash E)$. Let $L_1(S)$ denote the space of integrable quadratic differentials on S. Since φ is an element of $L_1(S)$ that is not in $Q(S \backslash E)$, by the Hahn–Banach theorem there is an L_∞ Beltrami differential ν such that $\iint \nu\varphi \, dx\, dy > 0$ and such that $\iint \nu\psi \, dx\, dy = 0$ for all ψ in $Q(S \backslash E)$. This ν is infinitesimally trivial for the Teichmüller space Teich$(S \backslash E)$. By the variational lemma (Lemma 3), there exists a curve of Beltrami coefficients μ_t tangent to ν that are defined for small positive values of t and are trivial as elements of the Teichmüller space Teich$(S \backslash E)$. This means that if f^t are quasiconformal mappings with Beltrami coefficient μ_t mapping S into Riemann surfaces S_t, then there is a conformal mapping c_t from $S \backslash E$ to $f_t(S \backslash E)$ such that c_t and f_t are homotopic. Moreover, by the variational formula of Section 4, the value of M_{f_t}, for small enough values of t, is larger than $\|\varphi\|$. This contradiction proves Step 2.

Step 3: Every point in E is either a critical point of φ or has a neighborhood U such that $U \cap E$ is a piece of a horizontal or vertical trajectory of φ. Suppose a point p in E is not a critical point of φ. Then there is a neighborhood U of p and a choice of local parameter ζ vanishing at p such that $\varphi\, dz^2 = d\zeta^2$ in U. But, by Step 2, we know that φ is an element of $Q(S \backslash E)$. Thus, in $U \cap E$ the parameter ζ gives a quadratic differential $d\zeta^2$ that is a local expression for an element of $Q(S \backslash E)$. Therefore, the part of the border of $S \backslash E$ that is contained in U is an analytic curve in the variable ζ. Since $d\zeta^2$ is real-valued on this curve, in the ζ-plane it must be either a vertical or horizontal line segment passing through the origin.

Step 4: No piece of vertical trajectory is contained in E. Assume that the obstacle set E contains a connected piece of a vertical trajectory. By scaling we can assume that, in the ζ-plane, the vertical trajectory corresponds to a piece of the vertical line segment joining $-i$ to i, and that the local parameter ζ gives a homeomorphism from an open set on the Riemann surface to the square with side length equal to 4 and centered at the origin. Inside this box, consider the vertical line segment α of length 4ϵ parameterized by $2\epsilon\sin(\theta)$. Under the correspondence

$$\zeta = w - \epsilon^2 w^{-1},$$

this vertical line segment is mapped to a circle centered at the origin with radius ϵ. Form the new surface S_ϵ, which is the Schiffer variation of S, by attaching a disk of radius ϵ to the boundary of this circle.

By Theorem 5, S_ϵ represents the same point in the Teichmüller space as the quasiconformal deformation $S_{\mu(\epsilon)}$. On the surface S, the Beltrami coefficient $\mu(\epsilon)$ expressed in terms of the local parameter z is equal to 1/4. Because of the variational formula for M_f in Section 4, we can use the Beltrami variation to find a curve that increases the value of M_f. Because this variation is equivalent to a Schiffer variation, we obtain a curve of univalent functions f^t defined on $S\backslash E$ for which M_{f^t} increases, which leads to a contradiction.

On putting all of these steps together, we have completed the proof of Theorem 1. □

8. Proof of Uniqueness

We assume (f, S_f) is an element of the family $\mathfrak{F}$ that maximizes (1). By definition, for any other element of (g, S_g) in $\mathfrak{F}$, we have

$$\|\varphi_g\| \le \|\varphi_f\|.$$

Now suppose that $g(E) = E_g$ consists entirely of finitely many arcs of horizontal trajectories of φ_g and, in addition, possibly some critical points of φ. We will show that then

$$\|\varphi_f\| \le \|\varphi_g\|$$

and that $g \circ f^{-1}$ extends to a conformal mapping from S_f onto S_g.

Let γ be a simple, closed polygonal curve on S_f whose segments are, alternately, regular arcs of vertical and regular arcs of horizontal trajectories of φ_f. Moreover, assume that γ is quasitransversal to the horizontal foliation for φ_f. This means that if $\beta_1\alpha\beta_2$ are three successive segments on γ, then β_2 departs from the side of α opposite to the side at which β_1 enters α. With this assumption (see [G]), the height of γ with respect to φ_f is realized by γ, that is,

$$\int_\gamma |\mathrm{Im}(\varphi_f(z)^{1/2}\,dz)| = \mathrm{height}_{\varphi_f}[\gamma]. \tag{2}$$

Let $h = g \circ f^{-1}$, and consider the quadratic differential defined on $S_f - E_f$ given by $\tilde{\varphi} = \varphi_g(h(z))h'(z)^2$. This differential is holomorphic on $S_f - E_f$, but possibly not continuous on S_f. Since E_f has measure zero, $\tilde{\varphi}$ is defined almost everywhere on S_f and is in L_1 because $\|\tilde{\varphi}\| = \|\varphi_g\| \le \|\varphi_f\|$.

Because h is possibly discontinuous on E_f, $h \circ \gamma$ is also possibly discontinuous. But since h is conformal and defined on the complement of E_f and since E_f consists of a union of finitely many analytic arcs, $h \circ \gamma$ has only finitely many intervals of discontinuity. For $\gamma = \gamma(t)$, let $I = [t_0, t_1]$ be one of these intervals. When γ crosses E_f transversely, this interval reduces to a point. There is a point P in some component of E_g and another point Q in the same component such that, as t converges to t_0 from below, $h \circ \gamma(t)$ converges to P and, as t converges to t_1 from above, $h \circ \gamma(t)$ converges to Q. Because they are in the same component of E_g, we can connect P and Q by a curve α_j lying in E_g. We pick a different α_j for each interval of discontinuity of $h(\gamma)$. Let $\tilde{\gamma}$ be $h(\gamma)$ with these additional curves α_j adjoined for every point of discontinuity of $h(\gamma)$. We claim that

$$\int_\gamma |\mathrm{Im}\{\varphi_f^{1/2}(z)\,dz\}| \le \int_\gamma |\mathrm{Im}\{\tilde{\varphi}^{1/2}(z)\,dz\}|. \tag{3}$$

By the definition of φ_f and φ_g, we know that

$$\mathrm{height}_{\varphi_f}[\gamma] = \mathrm{height}_{\varphi_g}[\tilde{\gamma}]. \tag{4}$$

On the other hand, since the height is an infimum over all curves in the same free homotopy class, we have

$$\mathrm{height}_{\varphi_g}[\tilde{\gamma}] \le \int_{h(\gamma)} |\mathrm{Im}\{\varphi_g^{1/2}(z)\,dz\}| + \sum \int_{\alpha_j} |\mathrm{Im}\{\varphi_g^{1/2}(z)\,dz\}|, \tag{5}$$

where the last summation is over a set of curves α_j lying in components of E_g. By assumption, E_g consists of horizontal trajectories of φ_g plus some additional finite set of points, and therefore this summation is equal to zero.

Putting inequalities (2), (4), and (5) together and using the definition of $\tilde{\varphi}$, we obtain inequality (3) for arbitrary φ_f-polygonal closed curves γ quasitransverse to the horizontal trajectories of φ_f. On the other hand, from inequality (3) together with the arguments used to prove the first and second minimal norm properties in [G, Thm. 1 & Thm. 9], we conclude that

$$\|\varphi_f\| \le \iint_{S_f} |\varphi_f^{1/2}(z)\,\tilde{\varphi}^{1/2}(z)|\,dx\,dy.$$

By Schwarz's inequality, this is less than or equal to

$$\|\varphi_f\|^{1/2}\|\tilde{\varphi}\|^{1/2} = \|\varphi_f\|^{1/2}\|\varphi_g\|^{1/2} \le \|\varphi_f\|.$$

However, one has equality in Schwarz's inequality only if $|\varphi_f|$ is a multiple of $|\tilde{\varphi}|$ and, from the normalizations and the fact that these quadratic differentials are holomorphic, we find that $\varphi_g(h(z))h'(z)^2 = \varphi_f(z)$. By continuation, it follows that h extends to a conformal mapping from S_f onto S_g. □

9. Slit Mappings

Consider a multiply connected plane domain Ω bounded by a finite number n of boundary contours, $n \geq 3$. We can select two components of the complement and, by uniformization, we can find a conformal mapping, $z \mapsto w$, from Ω onto the part of the annular region between two concentric circles $|w| = 1$ and $|w| = R$ such that, as $|w|$ approaches 1, z approaches one of the components and, as $|w|$ approaches R, z approaches the other component. Then the other components of the complement of Ω correspond to closed sets contained in the annulus.

There are two theorems for this classical situation, which are called the circular and radial slit mapping theorems. The circular slit mapping theorem states that Ω can be mapped to such an annulus, with the smallest possible value of R. In that case the complement of the image of Ω consists of the interior of the unit disk, the exterior of the disk of radius R, and $n-2$ concentric circular slits. Moreover, up to postcomposition by an affine mapping of the plane, the realization of Ω in this form is unique.

The radial slit mapping theorem is realized by maximizing R, and the $n-2$ components of the complement correspond in the w-plane to radial slits. Once again, the realization of Ω is unique up to postcomposition by an affine mapping. Both of these theorems are easy consequences of our Theorems 1 and 2, where we use the quadratic differential dz^2/z^2 for the radial slit mapping theorem and we use $-dz^2/z^2$ for the circular slit mapping theorem.

The vertical and horizontal slit mapping theorems are obtained by applying our techniques to the quadratic differentials $-dz^2$ and dz^2 and to large squares S_n containing the given obstacle set E. On each of the large squares S_n, the differential dz^2 has finite norm equal to the area of S_n. By taking an exhaustion of the plane by these squares and using the distortion theorem, we obtain a normal family of suitably normalized mappings $f_n: S_n \to R_n$, where R_n are larger and larger rectangles. The mappings f_n take the horizontal and vertical sides of S_n to the horizontal and vertical sides of R_n, respectively. The sequence f_n converges uniformly on compact sets to a mapping f from the complement of the obstacle set in the plane to the complement of a finite number of vertical slits. Since the result is well known, we omit the details and comment only that our version of the uniqueness theorem does not apply here because the quadratic differential dz^2 on $\mathbb{C}$ has infinite norm.

A more interesting slit mapping theorem is obtained by considering the quadratic differential

$$\varphi(z)\, dz^2 = \frac{p(z)\, dz^2}{q(z)},$$

where $q(z)$ and $p(z)$ are polynomials, the degree of $q(z)$ is at least three more than the degree of $p(z)$, and $q(z)$ has only simple zeros. The obstacle set E is a closed bounded set contained in $\mathbb{C} - \{\text{the zeroes of } q(z)\}$. If E is invariant under the conjugation $j(z) = \bar{z}$ and if $p(z)$ and $q(z)$ have real coefficients, then the extremal configuration will also be invariant under j. In this example we see that the slits of our slit mapping theorem can take complicated forms. In particular, a component of the obstacle set can correspond under the extremal mapping to a union of arcs that emanate from a singularity of φ along the n prongs meeting at this singularity (see Figure 2).

Another example comes from starting with a Riemann surface S of finite topological type and with nonempty border. Then S determines the doubled Riemann surface S^d together with an anticonformal involution j such that $S^d = S \cup \{\text{the border of } S\} \cup j(S)$, where the border of S is the fixed point set of j. We assume that the obstacle set E consists of a finite number of closed simply connected components contained in the interior of S. Moreover, we assume that there is a set of n given points $\{x_k : 1 \le k \le n\}$, $n \ge 1$, on the border of S. Let $Q(S^d - \{x_k : 1 \le k \le n\})$ be the space of integrable holomorphic quadratic differentials on $S^d - \{x_k : 1 \le k \le n\}$ that are invariant under j. Assume g is the genus of S^d and $3g-3+n > 0$, which guarantees that the dimension of $Q(S^d - \{x_k : 1 \le k \le n\})$ is positive. Fix a nonzero quadratic differential φ in $Q(S^d - \{x_k : 1 \le k \le n\})$, and consider the family $\mathfrak{F}$ of all functions f that:

(a) are holomorphic and univalent on $S - E$;
(b) map $S - E$ into a variable bordered Riemann surface S_f;
(c) take the border of S to the border of S_f; and
(d) induce an isomorphism from the fundamental group of S onto the fundamental group of S_f.

The family $\mathfrak{F}$ and the obstacle set E extend by symmetry to the double S^d, and Theorems 1 and 2 are applicable. The extremal in (1) of the introduction is realized by a symmetric mapping f and a Riemann surface S_f^d with an involution j_f; f satisfies the symmetry condition $f \circ j = j_f \circ f$. The quadratic differential φ_f and the image obstacle set $f(E) = E_f$ are also symmetric with respect to j_f.

As a special case, we can take S to be the unit disk D, j to be the anticonformal involution across $|z| = 1$, and S^d to be the extended complex plane $\bar{\mathbb{C}}$. The n given points are then situated on $|z| = 1$ where $n \ge 4$. We will call the unit disk with these n distinguished points a *conformal polygon*. To pose the extremal problem, we are also given the obstacle set E, which is a compact subset of the open unit disk with only finitely many components, and a quadratic differential. The quadratic differential φ is holomorphic; its only poles are simple and are situated at the n given points; and $\varphi(z)\,dz^2$ is real-valued along the circumference of the unit circle $z = \exp(i\theta)$.

The variable bordered Riemann surface S_f is realized by the unit disk $|w| = 1$ with n variable distinguished points on the boundary. The family $\mathfrak{F}$ is the family of all conformal mappings of $D \backslash E$ into D that map $|z| = 1$ onto $|w| = 1$ and distinguished points onto variable distinguished points. The extremal in (1) maps $D \backslash E$ into D, and $f(E)$ consists of slits on the horizontal trajectories of φ_f.

10. Trajectories around the Obstacle Set

The data for the extremal problem in (1) are the given Riemann surface S with a quadratic differential φ on S and the compact subset E of S. The unique extremal solving (1) is realized by the Riemann surface S_f, the conformal mapping f from $S\backslash E$ into S_f, and the quadratic differential φ_f on S_f, where each component of $f(E)$ consists of parts of horizontal trajectories of φ_f.

We wish to compare the properties of the differential

$$\psi(z)\, dz^2 = \varphi_f(f(z))\, f'(z)^2\, dz^2,$$

which is defined on $S\backslash E$, to the originally given quadratic differential φ defined on S.

1. Only a finite number of horizontal trajectories of ψ terminate at E, whereas, in general, full strips of trajectories of φ pass through E.
2. Branches of the natural parameter $\Psi(z) = \int \psi(z)^{1/2}\, dz$ can be selected that map all of $S\backslash E$ onto a union of horizontally slit rectangles, whereas branches of the natural parameter $\Phi = \int \varphi(z)^{1/2}\, dz$ map $S\backslash E$ onto a union of rectangles with arbitrary deleted continua.
3. For any curve γ contained in $S\backslash E$, the height of the homotopy class of γ in $S\backslash E$ with respect to ψ is equal to the height of the homotopy class of γ in S with respect to φ.
4. For any curve γ contained in $S\backslash E$ that is homotopic in S with fixed endpoints to a horizontal segment α of ψ, the ψ-length of γ is at least as large as the ψ-length of α.
5. $\|\psi\| > \|\varphi\|$ unless $\psi = \varphi$.

The significance of these properties of ψ is most easily understood when the orignally given Riemann surface S is a conformal polygon P, that is, when P is the unit disk D with the set σ of $n \geq 4$ marked vertices on its boundary. We call any interval of the boundary of D joining two adjacent vertices a *side* of the polygon. Two cross-cuts of D are homotopic if they join the same two sides of the polygon.

The holomorphic quadratic differential $\varphi(z)\, dz^2$ belonging to P must be real-valued on the boundary of D and may have at most simple poles, all of which occur at the vertices of the polygon. Its natural parameter maps the unit disk to a finite number of strips S_i, which determine homotopy classes of cross-cuts. Let the width and height of each of these strips S_i measured in the natural parameter for φ be a_i and b_i. We summarize the properties of φ and the pull-back $f^*\varphi_f$ of the extremal differential φ_f in the following theorem.

THEOREM 6. *Assume we are given the polygon P, the holomorphic quadratic differential φ belonging to P, and a compact subset E of the interior of P that consists of finitely many simply connected components. Then there* *exists a holomorphic quadratic differential* ψ *defined on* $P \backslash E$ *with the following properties.*

1. *Only a finite number of the horizontal trajectories of* ψ *terminate at the set* E.
2. *Branches of the natural parameter* $\Psi(z) = \int \psi(z)^{1/2}\, dz$ *map all of* $P \backslash E$ *onto a union of horizontally slit rectangles* R_i *whose vertical sides correspond to parts of some of the sides of the polygon* P.
3. *The rectangles* R_i *join the same sides of* P *as the strips* S_i, *and the height* b_i *of each strip* S_i *is equal to the height of* R_i.
4. *If a cross-cut* α *of* P *joins the same sides that are joined by the strip* S_i, *and if this cross-cut lies in* $P \backslash E$, *then its length measured in the* $|\psi|^{1/2}$-*metric is at least as long as the width* a_i' *of the rectangle* R_i.
5. $\|\psi\| = \sum a_i' b_i \geq \sum a_i b_i = \|\varphi\|$, *and there is strict inequality unless* $\psi = \varphi$.

Proof. All of these properties follow from considering the extremal quadratic differential φ_f on S_f and the results of Theorems 1 and 2. □

Figures 1, 2, and 3 illustrate the case where $n = 6$ and E has two components. In Figure 1, we see the obstacle set E with no relationship to the horizontal trajectory structure of φ. In Figure 2, the components of E have been

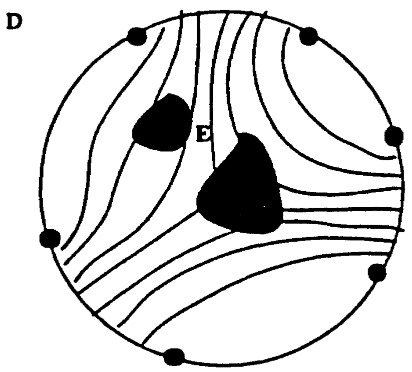

Figure 1

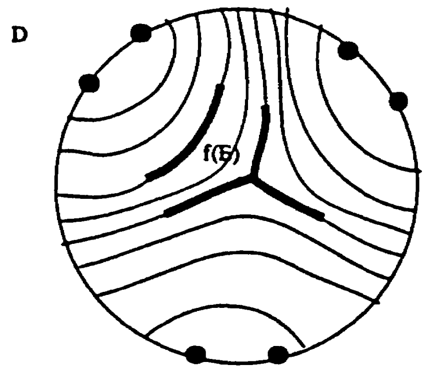

Figure 2

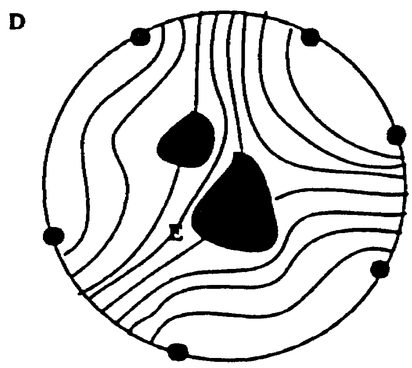

Figure 3

compressed into horizontal segments of φ_f, and one component is comprised of the union of three prongs emanating from a singularity. In Figure 3, we see the trajectory structure of $\psi = f^*\varphi_f$, with only finitely many horizontal trajectories of ψ leading into E. Otherwise, the trajectories of ψ go around the components of E.

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