Lipschitz Retracts, Selectors, and Extensions

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0. Introduction

We show, under mild hypotheses, that the intersection of a Lipschitz continuous set-valued mapping and a Lipschitz continuous ball-valued mapping is again Lipschitz continuous. This allows us to give simpler proofs, with improved Lipschitz constants, of several known results. In particular, we obtain:

(1) the result of Isbell [12] that, for any Banach space X, $\mathcal{K}(X)$ (i.e. the family of all closed, bounded, convex, nonempty subsets of X, equipped with the Hausdorff metric) is the range of a Lipschitz retract on any metric space containing it;

(2) a result of Skaletskii [29] implying that if $\mathcal{A}$ is a "reasonable" subset of $\mathcal{K}(X)$ with "uniformly normal structure" then there is a selector $F: \mathcal{A} \to X$ whose restriction to each bounded subset of $\mathcal{K}(X)$ is uniformly continuous;

(3) the result of Le Donne and Marchi [17] that there is a Lipschitz selector $s: \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}^n$ satisfying $s(A_0) = x_0$ for any previously specified $x_0 \in A_0 \in \mathcal{K}(\mathbb{R}^n)$; and

(4) the generalization of this due to Artstein [1], asserting that a Lipschitz selection on a subdomain can be extended to a Lipschitz selection on the entire domain of any set-valued mapping into $\mathcal{K}(\mathbb{R}^n)$.

A key ingredient of our proof is a simple 2-dimensional argument. Given a Banach space X, we define a function $\xi: [0,1) \to \mathbb{R}$ that depends only on the 2-dimensional subspaces of X. This function arises naturally in our work, and seems to be connected with the geometry of X. For example, ξ is minimal when X is a Hilbert space and maximal when X contains the 2-dimensional ℓ_1 space.

Our set-valued mappings will all be from some metric space S into the family of convex subsets of some Banach space X. This family of sets will always be considered in the Hausdorff metric (defined below), so it makes sense to talk about Lipschitz maps. (Actually, we hardly ever use the completeness assumption, so our results could easily have been formulated for

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normed spaces instead of Banach spaces.) To avoid trivialities we assume, unless stated otherwise, that X has dimension at least 2.

Restricting our attention to the case when X is finite-dimensional sometimes makes our task easier and sometimes not. Accordingly, Sections 1–3 contain results valid for the general case while results valid only in finite dimensions are postponed until Section 4. Restricting our attention to the case when X is a Hilbert space does lead to improved Lipschitz constants, which will therefore be stated separately. Of course, readers who need only know that a Lipschitz constant exists can ignore our estimates for them. In no case do we believe that these estimates are the best possible.

Section 1 includes the definition of ξ, some of its essential properties, and the intersection theorem mentioned previously. Proposition 2 implies that a reader who is not interested in the geometry of Banach spaces need not calculate ξ explicitly. It would suffice to replace $\xi(\beta)$ by $(1+\beta)/(1-\beta)$ throughout the paper. The reader interested only in the Euclidean case could make the stronger substitution $\xi(\beta)=1/\sqrt{1-\beta^2}$. It is also of interest to relate the behavior of ξ to some geometric properties of Banach spaces, such as uniform convexity and normal structure. We do this in a separate article [6], where we also use it to characterize Hilbert spaces.

The results of Section 1 are applied in Section 2 to give new, simpler proofs of the results of Isbell and Skaletskii, which avoid the integration technique used in the original proofs. Section 3 contains further technical results that were not necessary earlier. Some of them are necessary for the presentation of our finite-dimensional results, in particular the extension theorems, that appear in Section 4.

It might be helpful now to clarify our notation. By a *pseudometric* on a set S we mean a symmetric function $d: S\times S\to[0,\infty]$ satisfying the triangle inequality and vanishing on the diagonal. A *metric* is a pseudometric that vanishes only on the diagonal. We stress that the value ∞ is acceptable to us; this allows us to consider unbounded sets. In case d admits only finite values, we say that it is a finite (pseudo)metric. Balls in a pseudometric space are defined in the usual way, and may have radius ∞. The concept of Lipschitz mapping between pseudometric spaces should be clear; we denote the Lipschitz constant of a mapping f by L_f. For subsets $A,B\subseteq S$ and a point $x\in S$ we set $d(x,A)=\inf\{d(x,a): a\in A\}$, $d(A,B)=\inf\{d(a,b): a\in A, b\in B\}$, $h(A,B)=\sup\{d(a,B): a\in A\}$, and $d_H(A,B)=\max\{h(A,B),h(B,A)\}$. The first of these numbers is the usual distance between a point and a set, while the last is the well-known Hausdorff metric. We denote by $\mathcal{C}(X)$ the family of all closed convex nonempty subsets of X, and by $\mathcal{K}(X)$ the family of all compact convex nonempty subsets of X. Obviously, the definitions of $\mathcal{C}(X)$, $\mathcal{HC}(X)$, and $\mathcal{K}(X)$ make sense when X is merely a subset of a Banach space.

The support function of a subset $A\subset X$ is the mapping $h_A: X^*\to\mathbb{R}\cup\{\infty\}$ defined by $h_A(\gamma)=\sup_{a\in A}\gamma(a)$. If A is a bounded set, then the restriction of h_A to Γ, the unit ball of X^*, lies in the Banach space $\ell_\infty(\Gamma)$. The same remark applies to the restriction of h_A to the unit sphere of X^*. We will write

A^h for the restriction of h_A to Γ. This is not the usual notation for the support function, but it saves us from some cumbersome formulas later on. In case both A and B are bounded, the Hahn–Banach theorem tells us that $\|A^h - B^h\| = d_H(A, B)$. In other words, the natural embedding of $\mathcal{K}(X)$ into $\ell_\infty(\Gamma)$ is an isometry. We will also have cause to use the natural isometric embedding of X into $\mathcal{K}(X)$, that is, $x \mapsto \{x\}$.

1. The Modulus of Squareness and the Intersection Theorem

Observe that for any $x, y \in X$ with $\|y\| < 1 < \|x\|$, there is a unique $z = z(x, y)$ in the line segment $[x, y]$ with $\|z\| = 1$. We set

$$\omega(x, y) = \frac{\|x - z(x, y)\|}{\|x\| - 1}$$

and define $\xi = \xi_X : [0, 1) \to \mathbb{R}$ by

$$\xi(\beta) = \sup\{\omega(x, y) : \|y\| \le \beta < 1 < \|x\|\}.$$

We will call ξ the *modulus of squareness* of X. It seems to us that large values of this function tend to imply the presence of 2-dimensional subspaces whose unit balls are close to square. Our first result permits us, in a weak sense, to reverse the triangle inequality.

LEMMA 1. *Let y, z, x be (in that order) three colinear points in a normed space with $\|y\| < \|z\| < \|x\|$. Then*

$$1 \le \frac{\|x - z\|}{\|x\| - \|z\|} \le \xi\left(\frac{\|y\|}{\|z\|}\right).$$

The left inequality is the usual triangle inequality, while the right inequality is a trivial consequence of the definition of ξ. Serious results depend on having some estimates for ξ. It is clear that ξ is an increasing function, and that $\xi(0) = 1$. A priori, we do not even know if $\xi(\beta)$ is finite for positive β. Let us set $\xi_1(\beta) = (1+\beta)/(1-\beta)$ and $\xi_2(\beta) = (1-\beta^2)^{-1/2}$.

PROPOSITION 2.

(i) *For any Banach space X, $\xi_X \le \xi_1$.*
(ii) *If X is a Hilbert space, then $\xi_X = \xi_2$.*
(iii) *The function ξ is convex and absolutely continuous on* $[0, 1)$.

Proof. (i) Consider $\|y\| \le \beta < 1 < \|x\|$ with $z = z(x, y)$. Then $z = \alpha x + (1-\alpha) y$ for some $\alpha \in [0, 1)$, so re-arranging the triangle inequality yields

$$(1-\alpha)(\|x\| - \|y\|) \le \|x\| - \|z\|.$$

But then

$$\frac{\|x - z\|}{\|x\| - \|z\|} = \frac{1-\alpha}{\|x\| - \|z\|}\|x - y\| \le \frac{\|x\| + \|y\|}{\|x\| - \|y\|} = \xi_1\left(\frac{\|y\|}{\|x\|}\right) \le \xi_1(\beta).$$

(ii) Fix x with $\|x\|>1$. One sees easily that $\sup_{\|y\|\le\beta}\omega(x,y)$ is attained for some y with $\|y\|=\beta$, and that the line segment $[y,x]$ is contained in a line tangent to the sphere with center at the origin and radius β. In other words, $x-y$ is perpendicular to y. Hence we obtain

$$\sup_{\|y\|\le\beta}\omega(x,y)=\frac{(\|x\|^2-\beta^2)^{1/2}-(1-\beta^2)^{1/2}}{\|x\|-1}=f(\|x\|),$$

where f is a decreasing function. Thus $\xi(\beta)=\lim_{t\to 1}f(t)=(1-\beta^2)^{-1/2}$.

(iii) Since this result is not essential here (right continuity is only used incidentally in Theorem 4), we refer the reader to [6] for the proofs. We remark that proving left continuity is an easy exercise. □

The inequality $\xi\le\xi_1$ seems to be a much rediscovered result. An equivalent result appears in [35]. As far as we know, a related inequality was first used by Abel to show that if a power series has radius of convergence 1 and converges at some point z_0 on the unit circle, then the convergence is uniform on any set in which $|z_0-z|/(|z_0|-|z|)$ remains bounded [10, Thm. 50.3]. Setting $x=(1+\epsilon,0)$ and $y=(0,\beta)$ in $\ell_1(2)$ shows that the estimate $\xi\le\xi_1$ cannot be sharpened in general.

Note that the definition of ξ makes sense if the norm is replaced by a seminorm, and that Proposition 2 still holds in this case. Thus our results could be applicable in locally convex spaces. But rather than pursue this point now, we proceed with applications. First we need a simple lemma.

LEMMA 3. *Suppose that A is a convex subset of X which meets the ball $B(x,r_1)$. Then, for every $a\in A$ and every $r_2>r_1$, there is an $a'\in A$ satisfying $\|a'-x\|\le r_2$ and $\|a-a'\|\le(\|a-x\|-r_2)^+\xi(r_1/r_2)$.*

Proof. We may assume that x is the origin. The only nontrivial case is when $\|a\|>r_2$. Choose any $a''\in A\cap B(x,r_1)$ and define a' as the unique element of the line segment $[a,a'']$ with norm equal to r_2. Dividing everything by r_2 we see that $a'/r_2=z(a/r_2,a''/r_2)$. Hence

$$\|a/r_2-a'/r_2\|=(\|a/r_2\|-1)\,\omega(a/r_2,a'/r_2)\le(\|a/r_2\|-1)\,\xi(r_1/r_2),$$

as required. □

Our first theorem looks a bit cumbersome, but it is no harder to prove than any of the special cases we shall need later, so we state it in its full generality. It says that, under reasonable hypotheses, the intersection of two Lipschitz continuous set-valued functions is again Lipschitz. Special cases appear in [1], [4, Lemma 9.4.2], [9, Prop. 2.1], and [22, Lemma 1]. A similar idea appears in [24, Thm. 5.4].

THEOREM 4. *Let S be a metric space, let X be a Banach space, and let $f\colon S\to X$, $F\colon S\to\mathcal{C}(X)$, and $g\colon S\to\mathbb{R}\cup\{\infty\}$ be three Lipschitz mappings. Suppose that there is a $\gamma>1$ for which $g(x)\ge\gamma d(f(x),F(x))$ for every $x\in S$. Then the intersection map $G\colon S\to\mathcal{C}(X)$ defined by*

$$G(x)=F(x)\cap B(f(x),g(x))$$

is Lipschitz continuous, with its Lipschitz constant

$$L_G\le L_F+(L_f+L_g+L_F)\xi(1/\gamma).$$

Proof. The given inequality guarantees that $G(x)$ is nonempty. It is obviously closed and convex. Given $x,y\in S$ and $a\in G(x)$, we must find $b\in G(y)$ with $\|b-a\|$ at most epsilonically larger than $(L_F+(L_f+L_g+L_F)\xi(\beta))d$, where $\beta=1/\gamma$ and $d=d(x,y)$. We need only concern ourselves with the case where d, $g(x)$ and $g(y)$ are finite.

If $g(y)=0$ this is easy; we then have $G(y)=\{f(y)\}$ and

$$\begin{aligned}\|a-f(y)\|&\le\|a-f(x)\|+\|f(x)-f(y)\|-g(y)\\&\le g(x)-g(y)+L_fd\\&\le(L_g+L_f)d.\end{aligned}$$

Otherwise, choose $\epsilon>0$ with $\epsilon<g(y)-d(f(y),F(y))$. Certainly we can find a $b\in F(y)$ with $\|b-a\|<d_H(F(x),F(y))+\epsilon$. Note for later use that

$$\begin{aligned}\|b-f(y)\|-g(y)&\le\|b-a\|+\|a-f(x)\|+\|f(x)-f(y)\|-g(y)\\&<(L_f+L_F+L_g)d+\epsilon.\end{aligned}$$

Now set $r_1=d(f(y),F(y))+\epsilon$, $r_2=g(y)$, and $\beta'=r_1/r_2$. For ϵ small enough, β' will not be significantly larger than β. Since $F(y)$ meets $B(f(y),r_1)$, Lemma 3 gives us some $b'\in F(y)$ with $\|b'-f(y)\|\le g(y)$ and

$$\|b-b'\|\le(\|b-f(y)\|-g(y))^+\xi(\beta')\le((L_f+L_g+L_F)d+\epsilon)\xi(\beta').$$

But then $b'\in G(y)$ and

$$\|b'-a\|\le\|b-a\|+\|b-b'\|\le L_Fd+\epsilon+((L_f+L_g+L_F)d+\epsilon)\xi(\beta').$$

Letting $\epsilon\to0$, we conclude that G is Lipschitz with

$$L_G\le L_F+(L_f+L_g+L_F)\xi(\beta+).$$

Using the right continuity of ξ completes the proof. □

Easy examples show that the hypothesis $\gamma>1$ is essential in Theorem 4. Take $X=\ell_\infty(2)$ and $S=[0,1]$, with $f(x)=0$ and $g(x)=1$ being constant functions. Let $F(x)$ be the line segment joining $(1,1)$ with $(1+x,0)$. Then F is clearly a Lipschitz function, as is any constant ball-valued function, and $d(f(x),F(x))=g(x)$ for all x. However, the intersection function G is not even continuous at 0.

Theorem 4 has a number of interesting special cases. We will give only one now, then proceed straight to applications. Further special cases will be studied in Section 3.

COROLLARY 5. *Define* $G\colon X\to\mathcal{K}(A)$ *by* $G(x)=A\cap B(x,\gamma d(x,A))$, *where* $A\in\mathcal{C}(X)$ *and* $\gamma>1$ *are fixed. Then G is Lipschitz with* $L_G\le(1+\gamma)\xi(1/\gamma)$, *and satisfies* $G(a) = \{a\}$ *for every* $a \in A$. *We may choose* γ *so that* $L_G \le 8$ *in general;* $L_G < 3\frac{1}{3}$ *in the case where* X *is a Hilbert space.*

Proof. This is more or less obvious from Theorem 4. For the Lipschitz constant, put $\gamma = 3$ for a general Banach space, and $\gamma = 1.6$ for a Hilbert space. □

REMARK. All of the preceding results, and many of those which follow, can be restated for mappings with convex but not necessarily closed values. It is necessary only to replace the inequality constraints by strict inequalities.

2. Absolute Retracts and Uniformly Continuous Selectors

We prove here the theorems of Isbell and Skaletskii mentioned in the introduction. A metric space is said to be an *absolute retract* if it is the range of a continuous retract on any metric space that contains it as a closed subset. A routine application of Michael's selection theorem shows that any closed convex subset of a Banach space is an absolute retract.

We will call a given metric space a *Lipschitz absolute retract* (resp., a λ-*Lipschitz absolute retract*) if it is the range of a Lipschitz continuous retract (resp., with Lipschitz constant λ) on any metric space that contains it as a closed subset. The following result is well known and easy to prove (see [34, Chap. 3]).

LEMMA 6. (i) *For every index set* Γ, *the Banach space* $\ell_\infty(\Gamma)$ *is a* 1-*Lipschitz absolute retract, and contains an isometric copy of every metric space with weight* (*density character*) *less than or equal to the cardinality of* Γ.

(ii) *A metric space* Y *is a* λ-*Lipschitz absolute retract if and only if the following is true: For every* (*pseudo*)*metric space* S, *for every* $S_0 \subset S$, *and for every Lipschitz continuous function* $f: S_0 \to Y$, *there is a Lipschitz continuous extension* $g: S \to Y$, *with* $L_g \le \lambda L_f$.

(iii) *If the metric space* S_1 *is a* λ_1-*Lipschitz absolute retract, and if* $S_2 \subset S_1$ *is the range of a* λ_2-*Lipschitz retract on* S_1, *then* S_2 *is a* $\lambda_1\lambda_2$-*Lipschitz absolute retract.*

Now we would like to have some examples of Lipschitz absolute retracts. Isbell [12] showed that $\mathcal{K}(X)$ is a uniform absolute retract for any Banach space X, finite- or infinite-dimensional. Lindenstrauss [18] pointed out that Isbell's argument actually shows that $\mathcal{K}(X)$ is a 12-Lipschitz absolute retract. A slight improvement of this estimate was noted in [25, p. 114]. The main aim of this section is to show that $\mathcal{K}(X)$ is actually an 8-Lipschitz absolute retract, using a simpler proof than that of Isbell.

THEOREM 7. *For any Banach space* X *and any* $M \in \mathcal{C}(X)$, *the metric space* $\mathcal{K}(M)$ *is an* 8-*Lipschitz absolute retract.*

Proof. As usual, let Γ denote the unit ball of X^*. There is an obvious isometry between $\mathcal{K}(M)$ and $\mathcal{K}^h(M) = \{B^h : B \in \mathcal{K}(M)\} \subseteq \ell_\infty(\Gamma)$. Define U: $\mathcal{K}(\mathcal{K}^h(M)) \to \mathcal{K}^h(M)$ by

$$U(B) = \Bigl(\bigcup_{C^h \in B} C\Bigr)^h.$$

Easy calculations show that U is Lipschitz continuous with Lipschitz constant 1, and that $U(\{B^h\}) = B^h$ for every singleton $\{B^h\}$. Now define $R: \ell_\infty(\Gamma) \to \mathcal{K}^h(M)$ by $R = U \circ G$, where $G: \ell_\infty(\Gamma) \to \mathcal{K}(\mathcal{K}^h(M))$ is the map given by Corollary 5, with $A = \mathcal{K}^h(M)$. Then G is an 8-Lipschitz map that is (loosely speaking) the identity on M. Thus R is a retract from $\ell_\infty(\Gamma)$ onto $\mathcal{K}^h(A)$, with Lipschitz constant 8. The preceding lemma completes the proof. □

Combining this with Lemma 6(ii), we see that Lipschitz mappings into $\mathcal{K}(A)$, defined on any subset of a metric space, admit Lipschitz extensions to the whole space. A special case of this was first proved in [7].

We have seen [25] that, whenever X is infinite-dimensional, there is no uniformly continuous (let alone Lipschitz) selector $\mathcal{K}(X) \to X$. A reasonable question is then: Given a suitable subset $\mathcal{A}$ of $\mathcal{K}(X)$, is there a uniformly continuous selector $\mathcal{A} \to X$? Lindenstrauss [18, Thm. 8] showed that there is a uniformly continuous retract when X is uniformly convex and $\mathcal{A} = \{A \in \mathcal{K}(X) : \operatorname{diam} A \le r\}$ for some $r > 0$. Skaletskii [29] improved this by establishing the existence of a selector, whilst also weakening the first assumption to "X has uniformly normal structure". Here we will give a new simple proof of Skaletskii's theorem, without the convexity assumption imposed in [25, Thm. 7.2].

The radius of a set A relative to a point x is defined by $\operatorname{rad}(x, A) = \sup_{a \in A} \|x - a\|$. One then defines the radius of A by $\operatorname{rad} A = \inf_{x \in X} \operatorname{rad}(x, A)$. The *Jung constant* of X is $J(X) = \sup\{\operatorname{rad} A / \operatorname{diam} A : A \in \mathcal{K}(X),\ A \text{ infinite}\}$; X is said to have *uniformly normal structure* if $J(X) < 1$. It is well known that every uniformly convex space has uniformly normal structure. (A long-standing conjecture is that the converse is true, after renorming.) Following Skaletskii, we will say that a *subset* $\mathcal{A}$ of $\mathcal{K}(X)$ has uniformly normal structure if $\sup\{\operatorname{rad} A / \operatorname{diam} A : A \in \mathcal{A},\ A \text{ infinite}\} < 1$. If we consider $\mathcal{K}(X)$ to be embedded in $\mathcal{K}(\ell_\infty(\Gamma))$, it is a routine exercise to prove the identity $d(A^h, A) = \operatorname{rad} A$.

THEOREM 8. *Let $\mathcal{A}$ be a subset of $\mathcal{K}(X)$, with uniformly normal structure. Suppose also that $\mathcal{A}$ is an "ideal", that is, suppose*

$$[A \in \mathcal{K}(X),\ B \in \mathcal{A},\ A \subset B] \ \Rightarrow\ A \in \mathcal{A}.$$

Then there is a selector $F: \mathcal{A} \to X$ with the property that, for each $r > 0$, the restriction of F to $\{A \in \mathcal{A} : \operatorname{diam} A \le r\}$ is uniformly continuous.

Proof. For each $\alpha > 1$, we define $T_\alpha: \mathcal{A} \to \mathcal{A}$ by $T_\alpha(A) = A \cap B(A^h, \alpha \operatorname{rad} A)$. Theorem 4 guarantees that T_α is Lipschitz continuous. Note that A^h does not lie in X, but rather in the larger Banach space $\ell_\infty(\Gamma)$; this does not affect the applicability of Theorem 4.

Now we show that $\operatorname{diam} T_\alpha(A) \le \alpha \operatorname{rad} A$ for every $A \in \mathcal{A}$. For any $a, b \in T_\alpha(A)$, we have

$$\alpha \operatorname{rad} A \ge \|A^h - a\| \ge \sup_\gamma A^h(\gamma) - \gamma(a) \ge \sup_\gamma \gamma(b-a) = \|b-a\|,$$

as required. Since $\mathcal{A}$ has uniformly normal structure, we may choose α so that, for some fixed $k < 1$, $\operatorname{diam} T_\alpha(A) \le k \operatorname{diam} A$ for every $A \in \mathcal{A}$. Then $\operatorname{diam} T_\alpha^n(A) \le k^n \operatorname{diam} A \to 0$, and so $F(A) = \bigcap_{n=1}^\infty T_\alpha^n(A)$ is a singleton set. We may regard F as a selector from $\mathcal{A}$. Now

$$d_H(F(A), T_\alpha^n(A)) \le \operatorname{diam} T_\alpha^n(A) \le k^n \operatorname{diam} A,$$

and so $T_\alpha^n(A) \to F(A)$ uniformly on $\{A \in \mathcal{A} : \operatorname{diam} A \le r\}$. As a uniform limit of uniformly continuous functions, F must be uniformly continuous on $\{A \in \mathcal{A} : \operatorname{diam} A \le r\}$. □

It is clear that uniform continuity of a selector is not affected by renorming. Thus, if X is isomorphic to a Banach space with uniformly normal structure, then there is a selector $\mathcal{K}(X) \to X$ that is uniformly continuous on each of the sets $\{A \in \mathcal{K}(X) : \operatorname{diam} A \le r\}$.

The following consequence of Theorems 7 and 8 was pointed out to us by the referee. The meaning of *uniform absolute retract* should be obvious.

COROLLARY 9. *If X is isomorphic to a Banach space with uniformly normal structure, then its unit ball is a uniform absolute retract.*

The assumption of uniformly normal structure is not essential here. For example, the unit ball of the nonreflexive Banach space ℓ_∞ is also a uniform absolute retract. (According to [5] or [20], any Banach space with uniformly normal structure is reflexive.) However, there do exist Banach spaces whose unit balls are not uniform absolute retracts [18, Cor. 1, p. 282].

3. Computations, Observations, and Variations

This section contains technical results that might not be of interest to all readers. Some of them will be needed for the extension theorems in Section 4.

We begin by asking: Can the Lipschitz constant from Theorem 7 be improved? Let us write $\lambda_{\mathcal{K}}(X)$ for the smallest value of λ for which $\mathcal{K}(X)$ is a λ-Lipschitz absolute retract. (We use this notation rather than $\lambda(X)$, since the latter is already in widespread use for the projection constant.) For any infinite-dimensional classical Banach space X, in particular for any Hilbert space, we can show that $\mathcal{K}(X)$ is not a λ-Lipschitz absolute retract for any $\lambda < 2$. Probably this is true for every infinite-dimensional Banach space. Our proof of this requires a number of definitions, which may or may not be of wider interest. Recall that an *Auerbach basis* for an n-dimensional Banach space X is an algebraic basis $x_1, \ldots, x_n$ of norm-1 vectors whose coefficient functionals $f_1, \ldots, f_n$ also have norm 1. Auerbach's lemma [19, 1.c.3] asserts that every finite-dimensional Banach space has an Auerbach basis. Let us define the *boringness* of X by $b(X) = \sup\|\sum_{i=1}^n x_i\|$, where the supremum is taken over all Auerbach bases for X.

LEMMA 10. *For any finite-dimensional Banach space X,*

$$\lambda_{\mathcal{K}}(X) \geq \frac{2b(X)}{1+b(X)}.$$

Proof. Let $x_1, \ldots, x_n$ be any Auerbach basis for X. Put $A = \{\sum_{i=1}^n \alpha_i x_i : |\alpha_i| \leq 1 \text{ for all } i\}$, and for $1 \leq |k| \leq n$ set $A_k = \{\sum_{i=1}^n \alpha_i x_i \in A : \alpha_{|k|} = \operatorname{sgn} k\}$. Easy calculations show that $d_H(A_j, A_k) = 2$ whenever $j \neq k$. Let $S = S_0 \cup \{B\}$ be a metric space containing $S_0 = \{A_1, \ldots, A_n\}$ with $d(B, A_k) = 1$ for all k. Lemma 6(ii) gives us a mapping $R: S \to \mathcal{K}(X)$ that extends the identity $I: S_0 \to \mathcal{K}(X)$ and has Lipschitz constant $\lambda \leq \lambda_{\mathcal{K}}(X)$. We assume that $\lambda < 2$, as otherwise there is nothing to prove. Then $d_H(R(B), A_k) \leq \lambda$ for all k, which implies that $|f_k(x) \pm 1| \leq \lambda$ for all $x \in R(B)$ and both choices of sign. In other words, $R(B) \subseteq (\lambda - 1)A \subseteq A$. This forces

$$\begin{aligned}
\lambda &\geq \max_k d_H(R(B), A_k) \\
&\geq \max_k h(A_k, R(B)) \\
&\geq \max_k h(A_k, (\lambda-1)A) \\
&= h(A, (\lambda-1)A) \\
&= (2-\lambda)\sup_{a\in A}\|a\| \\
&= (2-\lambda)\sup_{\pm}\left\|\sum_{i=1}^n \pm x_i\right\|.
\end{aligned}$$

Since the choice of Auerbach basis was arbitrary and $(\pm x_i)$ is an Auerbach basis whenever (x_i) is, we have $\lambda \geq (2-\lambda)b(X)$. □

We call a Banach space *infinitely boring* if it admits a sequence of finite-rank projections P_n with $b(P_n X) \to \infty$ and $\|P_n\| \to 1$. Not all infinite-dimensional spaces have this property; in this context, the example of Pisier [23, Cor. 10.8] is rather interesting.

PROPOSITION 11.

(i) *If X is infinitely boring then $\lambda_{\mathcal{K}}(X) \geq 2$.*

(ii) *Any $\mathcal{L}_p$ space, $1 \leq p \leq \infty$, is infinitely boring.*

(iii) *Any Banach space that contains an isomorphic copy of c_0 is infinitely boring.*

(iv) *Any Banach space with a monotone basis is infinitely boring.*

Proof. (i) If $P: X \to Y$ is a projection, then $\lambda_{\mathcal{K}}(Y) \le \|P\| \lambda_{\mathcal{K}}(X)$.

(ii) For $p < \infty$, this follows from the easy estimate $b(\ell_p(n)) \ge n^{1/p}$. For $p = \infty$, it follows from the proof of (iii).

(iii) A result of James [19, 2.e.3] implies that X contains almost isometric copies of $\ell_\infty(n)$ for all n. Since $\ell_\infty(n)$ is the range of a norm-1 projection on every superspace, it suffices (by a routine approximation argument) to show that $b(\ell_\infty(n)) \to \infty$. For $n \ge 3$, set

$$x_1 = (-1, 1, 1, 1, \ldots),\ x_2 = (1, -1, 1, 1, \ldots),\ x_3 = (1, 1, -1, 1, \ldots), \ldots;$$

$$f_1 = \frac{1}{2n-4}(3-n, 1, 1, 1, \ldots),\ f_2 = \frac{1}{2n-4}(1, 3-n, 1, 1, \ldots), \ldots.$$

Then $x_1, \ldots, x_n$ form an Auerbach basis for $\ell_\infty(n)$, and $f_1, \ldots, f_n$ are the corresponding coefficient functionals. Furthermore,

$$b(\ell_\infty(n)) \ge \|x_1 - x_2 - x_3 - \cdots - x_n\| = n.$$

(iv) Let P_n be the projections associated with some monotone basis of X. Then each P_n has rank n and norm 1. If the sequence $b(P_n X)$ is not bounded, the conclusion is obvious. If it is bounded, a routine argument shows that X is isomorphic to c_0, and (iii) is applicable. □

If X is finite-dimensional (say, n-dimensional) then the argument from (iii) also shows that $\mathcal{K}(X)$ is never a 1-Lipschitz absolute retract, except in the degenerate case when $n = 1$. For

$$\begin{aligned} \lambda_{\mathcal{K}}(X) = 1 &\Rightarrow b(X) = 1 \\ &\Rightarrow \left\| \sum_{i=1}^{n} \pm x_i \right\| = 1 \text{ for all choices of sign, and at least one Auerbach basis} \\ &\Rightarrow X \text{ is isometric to } \ell_\infty(n) \\ &\Rightarrow b(X) = n. \end{aligned}$$

Thus X is 1-dimensional. We can also show that $\lambda_{\mathcal{K}}(X) > 1$ whenever X is infinite-dimensional, but we omit the proof.

Now we return to the study of intersections with balls. Recall that if we have a collection (finite or larger) of metric spaces (M_i, d_i), the product space can be metrized by the supremum metric $d((x_i), (y_i)) = \sup_i d_i(x_i, y_i)$.

A trivial consequence of Theorem 4 is that the function

$$(x, A) \mapsto A \cap B(x, \gamma d(x, A))$$

is Lipschitz on $X \times \mathcal{K}(X)$ for any given $\gamma > 1$. Fixing x (as in the proof of Theorem 8) then gives us a Lipschitz map $\mathcal{K}(X) \to \mathcal{K}(X)$; fixing A (as in Corollary 5) yields a Lipschitz map $X \to \mathcal{K}(X)$. We now pursue this idea in more detail.

COROLLARY 12. (i) *Let $\mathcal{A}$ be a subfamily of $\mathcal{C}(X)$, and equip $X \times \mathcal{A}$ with the sup metric. Suppose $g: X \times \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ is a Lipschitz continuous function, and that there is a $\beta < 1$ such that $\beta g(x, A) \ge d(x, A)$ for every $x \in X$*

and $A \in \mathcal{A}$. Then the mapping $T: X \times \mathcal{A} \to \mathcal{C}(X)$ defined by

$$T(x,A) = A \cap B(x, g(x,A))$$

is Lipschitz continuous with $L_T \le 1 + (L_g + 2)\xi(\beta)$.

(ii) *Let $\mathcal{A}$ be a subfamily of $\mathcal{C}(X)$, let $g: \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be a Lipschitz continuous function, and suppose there is a $\beta < 1$ and an $x \in X$ such that $\beta g(A) \ge d(x,A)$ for every $A \in \mathcal{A}$. Then the mapping $T: \mathcal{A} \to \mathcal{C}(X)$ defined by $T(A) = A \cap B(x, g(A))$ is Lipschitz continuous with $L_T \le 1 + (L_g + 1)\xi(\beta)$.*

(iii) *For every $x \in X$ there is a Lipschitz mapping $T = T_x: \mathcal{C}(X) \to \mathcal{K}(X)$ with $L_T \le 9$ such that $T(A) \subseteq A$ for every $A \in \mathcal{C}(X)$ and $T(A) = \{x\}$ whenever $A \ni x$. Furthermore, these mappings satisfy the "linearity" relation*

$$T_{x+y}(A+y) = T_x(A) + y.$$

In the case where X is a Hilbert space, each T can be chosen so that $L_T \le 1 + \sqrt{\frac{1}{2}(11 + 5\sqrt{5})} < 4.4$.

Proof. (i) Set $S = X \times \mathcal{A}$, $f(x,A) = x$, and $F(x,A) = A$ for each $(x,A) \in S$ in Theorem 4.

(ii) This follows equally easily from Theorem 4.

(iii) It suffices to set $g(A) = \gamma d(x,A)$ and $\mathcal{A} = \mathcal{C}(X)$ in part (ii), with $\gamma = \frac{1}{2}(\sqrt{5} + 1)$ if X is a Hilbert space, or $\gamma = 3$ for a general Banach space. □

As a special case of (iii), we see that for each $\gamma > 1$, the map $T_\gamma: \mathcal{C}(X) \to \mathcal{K}(X)$ defined by $T_\gamma(A) = A \cap B(0, \gamma d(0,A))$ is Lipschitz continuous with $L_T \le 1 + (\gamma + 1)\xi(1/\gamma)$, and has the so-called *zero property* [21], that is, $T_\gamma(A) = \{0\}$ whenever $0 \in A$.

When X is reflexive and strictly convex, one defines the *minimal selector* as the single-valued mapping that assigns to each $A \in \mathcal{C}(X)$ the unique element of A closest to the origin. As $\gamma \to 1$, $T_\gamma(A)$ might or might not converge in some sense to the minimal selector. In any case, the Lipschitz constants of T_γ need not remain bounded as $\gamma \to 1$, and simple examples [3, §1.7] show that the minimal selector is not always Lipschitz continuous.

With a more technical argument, the Lipschitz constant in (iii) can be reduced to 3 in the case of a Hilbert space, and this is the best possible [22]. The optimal choice for γ turns out to be $\sqrt{3}$.

In many cases (in particular, whenever X is reflexive), putting $\gamma = 1$ in Corollary 5 also gives us a well-defined (i.e. non-empty-valued) mapping. Sometimes—for example, in the strictly convex case—this leads to a single-valued function, the *metric projection* onto A. Even when A is a subspace, the metric projection need not be continuous (see [31] and the references therein). In the case of a Hilbert space, one can check that the metric projection is continuous, with Lipschitz constant 1. For uniformly convex and other spaces, see [33]. The next result recovers something in these situations; even when a Lipschitz selection cannot be obtained, one can at least find an interesting Lipschitz subset-valued function. We state it separately since it will be used later for Corollary 17.

COROLLARY 13. *Let S be a metric space and suppose that $F: S \to \mathcal{C}(X)$ and $f: S \to X$ are Lipschitz mappings. Given $\gamma > 1$, there exists a Lipschitz mapping $G = G_\gamma: S \to \mathcal{K}(X)$ such that:*

(i) *$G \subseteq F$, that is, $G(x) \subseteq F(x)$ for all $x \in S$;*

(ii) *$d_H(\{f(x)\}, G(x)) \le \gamma d(f(x), F(x))$ for all $x \in S$; in particular, $G(x) = \{f(x)\}$ whenever $f(x) \in F(x)$; and*

(iii) *$L_G \le L_F + (L_f + L_F)(\gamma+1)\xi(1/\gamma)$.*

Proof. Define $g(x) = \gamma d(f(x), F(x))$ and let G be given by Theorem 4. Then (i) is obvious and (ii) follows from the trivial inequalities

$$d_H(x, A \cap B(x,r)) \le r$$

and

$$d(f(x), F(x)) - d(f(y), F(y)) \le (L_f + L_F) d(x,y).$$

To prove (iii), note that $L_g \le \gamma(L_f + L_F)$. □

4. Finite-Dimensional Extensions and Selections

It is easy to see that any extension problem can be reformulated as a selection problem (for a function that is single-valued on the subset under consideration), so our previous results can be applied also to such problems. The results of this section depend on the existence of a Lipschitz selector— that is, a Lipschitz mapping $s: \mathcal{K}(X) \to X$ with the property that $s(A) \in A$ for each A. As we remarked earlier, there is no such map for any infinite-dimensional space X. Hence we restrict our attention here to the finite-dimensional case. We begin by recalling some information about the so-called *Steiner point* of a convex body. There exists exactly one continuous mapping $s: \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}^n$ with the properties that $s(A+B) = s(A) + s(B)$ for all $A, B \in \mathcal{K}(\mathbb{R}^n)$ and $s(TA) = T(s(A))$ for every rigid motion T of $\mathbb{R}^n$.

The mapping s can be represented by the formula

$$s(A) = n \int_{S^{n-1}} A^h(x) x \, d\sigma(x),$$

where $A \in \mathcal{K}(\mathbb{R}^n)$, A^h is its support function, and σ is the usual probability measure on the Euclidean sphere S^{n-1}. Shephard [28] first noticed that $s(A)$, now known as the Steiner point of A, always belongs to the relative interior of A. Equivalently, every bounded convex set contains the Steiner point of its closure. Furthermore, s is Lipschitz continuous, with Lipschitz constant equal to

$$c_n = n \int_{S^{n-1}} |\langle y, x \rangle| \, d\sigma(x) = \frac{2\Gamma(n/2+1)}{\sqrt{\pi}\,\Gamma((n+1)/2)},$$

where y is any element of S^{n-1} [8; 32]. The following tractable estimates will be used several times without comment:

$$\sqrt{2n/\pi} < c_n < \sqrt{2(n+1)/\pi}.$$

Every retract from $\mathcal{K}(\mathbb{R}^n)$ onto the natural copy of $\mathbb{R}^n$ has Lipschitz constant no less than c_n. Being a selector on $\mathcal{K}(\mathbb{R}^n)$, s is obviously a retraction onto $\mathbb{R}^n$. For further information about Steiner points, we refer the reader to [26] or [4, Chap. 9] and the references therein.

Building on these ideas, it was observed in [25, §6] that for every n-dimensional Banach space X there is a Lipschitz selector $s = s_X : \mathcal{K}(X) \to X$ with $L_s \le n$. For that particular selector, this estimate is sharp; it is attained when $X = \ell_1(n)$ or $\ell_\infty(n)$. With more work, a slightly better estimate can be obtained.

PROPOSITION 14.

(i) *For every n-dimensional normed space X, there is a Lipschitz selector $s = s_X : \mathcal{K}(X) \to X$ with $L_s \le c_n\sqrt{n}$.*

(ii) *If X has dimension 2, there is a Lipschitz selector $s = s_X : \mathcal{K}(X) \to X$ with $L_s \le 3/2$.*

Proof. (i) Recall the following result of John [13, Thm. 5.6(i)]: Given any n-dimensional normed space $(X, \|\cdot\|)$, there is a Euclidean norm $|\cdot|$ on X satisfying $|x| \le \|x\| \le \sqrt{n}|x|$ for all $x \in X$. In other words, the *Banach–Mazur distance* between X and $\ell_2(n)$ is no more than $\sqrt{n}$. Combining this with the properties of the Steiner selector, we see that (i) holds.

(ii) In two dimensions, a sharper estimate exists. As noted in [25, §6], there is a selector $s : \mathcal{K}(\ell_\infty(2)) \to \ell_\infty(2)$ with Lipschitz constant 1. According to Asplund [2], the Banach–Mazur distance of an arbitrary 2-dimensional space from $\ell_\infty(2)$ does not exceed 3/2. Thus follows (ii). □

The Euclidean case of the following result was stated without proof in [25, §5]. The case $n = 1$ is very well known; see for example [34, Thm. 13.16].

PROPOSITION 15. *Let X be an n-dimensional Banach space, and let $f : S_0 \to X$ be a Lipschitz mapping defined on a subset S_0 of a pseudometric space S. Then there exists a Lipschitz mapping $g : S \to X$ that extends f, with $L_g < \frac{8}{5} n^{7/8} L_f$. If X is Euclidean, we have $L_g \le c_n L_f$.*

Proof. In the Euclidean case, g can be defined by the formula

$$g(x) = n \int_{S^{n-1}} \sup_{y \in S_0} (\langle f(y), u\rangle - d(y,x)) u \, d\sigma(u).$$

(We inadvertently omitted the factor n in [25, p. 125].) For $x \in S_0$, the first factor in the integrand is simply $\langle f(x), u\rangle$. Transitivity of S^{n-1} under the rotation group shows that g is an extension of f. Estimating the Lipschitz constant is straightforward.

It is well known that, for $X = \ell_\infty(n)$, we can arrange $L_g = L_f$ simply by applying the 1-dimensional result componentwise. Auerbach's lemma (stated at the beginning of §3) implies that the Banach–Mazur distance of a general n-dimensional Banach space from $\ell_\infty(n)$ is at most n; this then gives a simple proof of the slightly weaker estimate $L_g \le nL_f$. For the general case, we can also use the stronger result of Szarek [30]: There is an absolute constant k such that the Banach–Mazur distance of an arbitrary n-dimensional space from $\ell_\infty(n)$ does not exceed $kn^{7/8}$. Tinkering with Szarek's proof shows that k is not large. More precisely, one can take $c = 1/2$ in his Lemma 6, and $\delta = \epsilon/3$ in his Proposition 5. This leads to $c_0 = 1/3\sqrt{6}$ and $d(X, \ell_\infty(n)) \leq 3^{3/8}n^{7/8}$. Combining this with the previous observation completes the proof. □

We note that for small values of n, say $3 \leq n \leq 162$, working via the Euclidean case gives an extension g satisfying the sharper estimate $L_g \leq \sqrt{n}c_n L_f$. In particular, we have $L_g \leq \frac{5}{6}nL_f$ for all $n \geq 5$. For $n = 2$, working via $\ell_\infty(2)$ gives a better estimate, $L_g \leq \frac{3}{2}L_f$.

A special case of the following result (namely, when A is fixed) appears in [17]. Again, we assume that X is n-dimensional.

COROLLARY 16. *For every $x \in X$, there is a selector $s_x: \mathcal{C}(X) \to X$ such that $s_x(A) = x$ for every A containing x. Each of these selectors has Lipschitz constant less than $8n$, or less than $3\sqrt{n}$ if X is Euclidean. Furthermore, they satisfy the "linearity" relation $s_{x+y}(A+y) = s_x(A)+y$.*

Proof. Using Corollary 12(iii) and Proposition 14(i) gives the general result for $n \neq 2$. For $n = 2$, use Proposition 14(ii). For the Euclidean case, use also the result of Ornelas [22] mentioned after Corollary 12. □

The proof of the next result is similar, using Corollary 13.

COROLLARY 17. *Let S be a metric space and X an n-dimensional Banach space. Suppose that $F: S \to \mathcal{C}(X)$ and $f: S \to X$ are Lipschitz mappings. Fix $\gamma > 1$. Then there exists a Lipschitz selection $s = s_\gamma: S \to X$ such that:*

(i) *$s(x) \in F(x)$ for all $x \in S$;*
(ii) *$\|f(x) - s(x)\| \leq \gamma d(f(x), F(x))$ for all $x \in S$; in particular, $s(x) = f(x)$ whenever $f(x) \in F(x)$; and*
(iii) *$L_s \leq c_n d_n(L_F + (L_f + L_F)(\gamma+1)\xi(1/\gamma))$, where $d_n = 1$ in the Euclidean case and $d_n = \sqrt{n}$ in general.*

We can now state the main theorem of this section.

THEOREM 18. *Let X be an n-dimensional Banach space, S a pseudometric space, and S_0 a subset of S. Suppose that $F: S \to \mathcal{C}(X)$ and $g: S_0 \to X$ are Lipschitz mappings, with $g(x) \in F(x)$ for every $x \in S_0$. Then there is a Lipschitz mapping $f: S \to X$ that is simultaneously an extension of g and a selection of F. Furthermore, f may be chosen so that $L_f \leq 10n^{15/8}L_g + 8nL_F$. If X is Euclidean, then $L_f \leq 3nL_g + 4\sqrt{n}L_F$.*

Proof. Combining Proposition 15 and Corollary 17 gives us a suitable function f. To verify these estimates for its Lipschitz constant, use the inequality $c_n < \sqrt{2(n+1)/\pi}$ for large values of n, and the exact value of c_n for small values of n. □

If we use the version of Proposition 15 resulting from John's theorem (instead of Szarek's theorem) in the preceding proof, we obtain the simpler estimate $L_f < 6n^2 L_g + 8nL_F$. For low dimensions, this is clearly stronger.

The Euclidean case of Theorem 18, with a larger Lipschitz constant, was also proved implicitly by Dommisch [11] and explicitly by Artstein [1], using a similar geometric argument.

We note that all of the results in this section are false for every infinite-dimensional space X. To see this for the last theorem, put $S = \mathcal{K}(X)$ and $S_0 = X$ and let F and g be the identity functions. The existence of a Lipschitz function f as in Theorem 18 would imply the existence of a Lipschitz selector $\mathcal{K}(X) \to X$, which is impossible if X is infinite-dimensional [25].

Nevertheless, some analogs of Proposition 15, showing the Lipschitz extendability of mappings into infinite-dimensional spaces, can be proved under certain "finiteness" assumptions on the domain of the mappings. In particular, this is true either if S_0 is finite, or if S is a (subset of a) finite-dimensional normed space—see [16] and the references therein.

Finally, retracts that are absolute but not Lipschitz (in the sense defined above) are also of interest; we now use the previous results to exhibit some. Let S be a pseudometric space and fix $x_0 \in S$. We consider the Banach space $\mathrm{Lip}(S, X)$ of all Lipschitz mappings from S into X equipped with norm $\|f\|_{x_0} = L_f + \|f(x_0)\|$.

Corollary 19. *Let X be a finite-dimensional Banach space, let $F: S \to \mathcal{C}(X)$ be a Lipschitz mapping, and let $\mathfrak{F}$ be the set of all Lipschitz selections of F. Then there is a Lipschitz retract $R: \mathrm{Lip}(S, X) \to \mathfrak{F}$ that satisfies the consistency conditions $(Rf)(x) = (Rg)(x)$ whenever $f(x) = g(x)$ and $(Rf)(x) = f(x)$ whenever $f(x) \in F(x)$.*

Proof. In the notation of Corollary 12, we set $(Rf)(x) = s(T_{f(x)}F(x))$. (Recall that s denotes the Steiner point of a finite-dimensional compact convex set.) The first consistency condition is now clear, and the rest is obvious. □

Of course, this $\mathfrak{F}$ is an absolute retract by the remarks at the beginning of Section 2. We cannot conclude that $\mathfrak{F}$ is a Lipschitz absolute retract, since $\mathrm{Lip}(S, X)$ is not (in general) a Lipschitz absolute retract.

To see this, note that if a dual Banach space Y is a uniform absolute retract then it is injective (i.e., the range of a continuous linear projection on any superspace). For Y will be a uniform retract on some space $\ell_\infty(\Gamma)$, so the work of Lindenstrauss [18] implies that Y^{**} will be isomorphic to a complemented subspace of the injective space $\ell_\infty(\Gamma)^{**}$. As a dual space, Y is complemented in Y^{**} and hence injective. (See also [25, p. 118] for more details of this type of argument.)

Now let S be the Hilbert cube. Johnson showed in [15, Prop. 2.2] that $\mathrm{Lip}(S, \mathbb{R})$ is not injective, and in [14, Cor. 4.2] that $\mathrm{Lip}(S, \mathbb{R})$ is a dual space. Hence $\mathrm{Lip}(S, \mathbb{R})$ is not a Lipschitz absolute retract, nor even a uniform absolute retract.

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