Nontriviality of the Abel–Jacobi Mapping for Varieties Covered by Rational Curves

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1. Introduction

The existence and structure of families of rational curves on projective varieties has played a key role in Mori's program to classify higher-dimensional complex varieties. One of the early results in this program showed that if X is a Fano variety (i.e., the anticanonical bundle $-K_X$ of X is ample) then X is covered by rational curves (see [6]). In this paper it will be shown that if a smooth projective variety X is covered by rational curves, then these rational curves, together with their degenerations, generate the middle-dimensional primitive cohomology of X via the "cohomological" Abel–Jacobi mapping. When X is a threefold this result will be reinterpreted to give a surjectivity result for the Abel–Jacobi mapping of Griffiths into the intermediate Jacobian of X. In particular, it follows that for Fano threefolds the images of the families of rational curves C on X with $(-K_X \cdot C) \leq 4$ generate the intermediate Jacobian of X. This result validates the general principle espoused by Clemens in [4] that the intermediate Jacobian of a threefold X which is covered by rational curves is generated by algebraic cycles on X.

The literature on this subject is vast, but to the author's knowledge the results just described have only been obtained for low-degree rational curves on special Fano varieties. For example, the result is known for the families of lines on several generic complete-intersection Fano threefolds [2; 14; 5] and for generic hypersurfaces of degree n in $\mathbb{P}^n$ [12]. It is also known for conic bundles [1] and for the families of conics on a generic quartic threefold and sextic double solid [11; 3]. In each case, the arguments used to obtain nontriviality of the Abel–Jacobi mapping use specific facts about the family of curves in question or their degenerations, and for this reason do not carry over to more general situations.

The cohomological Abel–Jacobi mapping is defined as follows. Let X be a smooth complex projective variety of dimension n, and let F be a smooth projective variety parameterizing a family of proper subvarieties of dimension d on X. Let $E = \{(C,x) \in F \times X : x \in C\}$, and let $p: E \to F$ and $q: E \to X$ be the natural projections. The cohomological Abel–Jacobi mapping is the morphism of Hodge structures of type $(-d, -d)$ defined by the composition

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$$H^*(X,\mathbb{C}) \xrightarrow{q^*} H^*(E,\mathbb{C}) \xrightarrow{p_*} H^{*-2d}(F,\mathbb{C}).$$

Note that, when E is not smooth, $p_*: H^*(E,\mathbb{C}) \to H^{*-2d}(F,\mathbb{C})$ is defined via a desingularization $\pi: \tilde{E} \to E$ of E and given by the composition

$$H^*(E,\mathbb{C}) \xrightarrow{\pi^*} H^*(\tilde{E},\mathbb{C}) \xrightarrow{(p\circ\pi)_*} H^{*-2d}(F,\mathbb{C}),$$

where $(p\circ\pi)_*$ is the Poincaré dual of $(p\circ\pi)_*$ on homology. It can easily be checked that this is independent of the choice of desingularization, since any two desingularizations of E can be mutually dominated by a third. Here we will be interested only in the case where F is a family of curves whose general member is a reduced irreducible rational curve (i.e., the image of $\mathbb{P}^1$ under a nontrivial morphism into X). If D is an ample divisor on X then we denote by $H^*(X,\mathbb{C})^\circ$ the primitive cohomology of X with respect to the divisor D. The main result is as follows.

THEOREM 1.1. *Let X be a proper smooth variety of dimension n that is covered by a family of rational curves, and let D be an ample divisor on X. Then there exist families of rational curves on X parameterized by a smooth, possibly disconnected, nonequidimensional projective variety F of dimension $\leq n-1$, as well as a diagram*

$$\begin{array}{ccccc} E & \xrightarrow{\phi} & F\times\mathbb{P}^1 & \xrightarrow{\pi} & F \\ {\scriptstyle q}\downarrow & & & & \\ X & & & & \end{array}$$

such that $E \to F$ is flat with generic fiber a rational curve, ϕ is birational, and the cohomological Abel-Jacobi mapping

$$(\pi\circ\phi)_* q^*: H^n(X,\mathbb{C})^\circ \to H^{n-2}(F,\mathbb{C})$$

is injective. Furthermore, each component of F parameterizes either the original covering family of rational curves or degenerations of these curves.

To see that it is necessary to include families occurring as degenerations of the curves in the covering family, let X be a smooth cubic threefold and fix a generic line L in X. Consider a generic projection centered along L, $\pi_L: X-L \to \mathbb{P}^2$, and let E be the variety obtained by blowing up X along L. This gives a family $\pi_L: E \to \mathbb{P}^2$ of conics on X that are coplanar with L. By [13, Prop. 1.25], the fibers of π_L are smooth except along a smooth plane curve $C \subseteq \mathbb{P}^2$ of degree 5, where the conics degenerate into the sum of two distinct lines. Since $H^1(\mathbb{P}^2,\mathbb{C}) = 0$, the covering family of conics does not generate any of the cohomology of X. On the other hand, the lines occurring as degenerations of the smooth conics in this family are parameterized by an étale cover $\tilde{C}$ of C of degree 2, and Theorem 1.1 asserts in this case that the cohomological Abel-Jacobi mapping $H^3(X,\mathbb{C}) \to H^1(\tilde{C},\mathbb{C})$ is injective.

This work is an expanded version of some results in the author's dissertation completed at the University of Utah. The author would like to thank Herb Clemens for suggesting this problem and for many helpful conversations.

2. Proof of the Main Theorem

To clarify the exposition, we will divide the proof of Theorem 1.1 into several steps.

Step 1: Since X is covered by a family of rational curves, there is a quasi-projective variety W' of dimension $n-1$ and a morphism

$$W' \times \mathbb{P}^1 \to X$$

such that the image of $w \times \mathbb{P}^1$ in X is a rational curve in X for each $w \in W$. This induces a map $W' \to \mathrm{Hom}(\mathbb{P}^1, X)$. Let W be the image of W' in $\mathrm{Hom}(\mathbb{P}^1, X)$. Since $\mathrm{Hom}(\mathbb{P}^1, X)$ is represented as an open subset of $\mathrm{Hilb}(\mathbb{P}^1 \times X)$ and the connected components of $\mathrm{Hilb}(\mathbb{P}^1 \times X)$ are projective, the closure F' of W in $\mathrm{Hilb}(\mathbb{P}^1 \times X)$ is projective. Then there exists a closed subscheme $E' \subseteq F' \times \mathbb{P}^1 \times X$ such that E' is flat over F' and hence also a diagram

$$\begin{array}{ccccc} E' & \xrightarrow{\phi'} & F' \times \mathbb{P}^1 & \xrightarrow{\pi'} & F' \\ {\scriptstyle q'}\downarrow & & & & \\ X. & & & & \end{array}$$

By construction, ϕ' restricted to $(\phi')^{-1}(W \times \mathbb{P}^1)$ is an isomorphism onto $W \times \mathbb{P}^1$, so that ϕ' is birational. Let $U = mD$ be a sufficiently ample divisor on X such that U intersects each curve parameterized by F' properly, and let F be a desingularization of $(q')^{-1}(U)$. By base extending, we can replace F' by F, E' by $E = F \times_{F'} E'$, and ϕ', π', q' by the corresponding maps ϕ, π, q, respectively. Then $\pi \circ \phi : E \to F$ is flat and $\pi : F \times \mathbb{P}^1 \to F$ has a section s given by the composition

$$F \xrightarrow{\mathrm{id} \times j} E = F \times_{F'} E' \xrightarrow{\phi} F \times \mathbb{P}^1,$$

where $\mathrm{id}: F \to F$ is the identity map and $j: F \to q^{-1}(U) \subseteq E'$. This gives a diagram as in the statement of the theorem with $\dim F = n-1$ and such that $F \times \mathbb{P}^1 \to F$ has a section s, where $s(F)$ is mapped onto a multiple of D.

Step 2: Consider the composition

$$H^n(X, \mathbb{C})^\circ \xrightarrow{q^*} H^n(E, \mathbb{C}) \xrightarrow{\phi_*} H^n(F \times \mathbb{P}^1, \mathbb{C}).$$

Let $p: \tilde{E} \to E$ be a desingularization of E. Then $q \circ p: \tilde{E} \to X$ is a surjective map of proper smooth varieties of dimension n, so that $p^* \circ q^*$ and hence q^* is injective. As in the introduction, ϕ_* is defined by the composition

$$H^n(E, \mathbb{C}) \xrightarrow{p^*} H^n(\tilde{E}, \mathbb{C}) \xrightarrow{(\phi \circ p)_*} H^n(F \times \mathbb{P}^1, \mathbb{C}).$$

Then ϕ_* is surjective, since $\phi_* \circ \phi^*$ is the identity on $H^n(F \times \mathbb{P}^1, \mathbb{C})$. However, it is possible for the composition $\phi_* \circ q^*$ to have a nonzero kernel $K_1 \subseteq H^n(X, \mathbb{C})^\circ$. Suppose this is the case.

Let $W \subseteq F$ be an open set such that $\phi^{-1}: W \times \mathbb{P}^1 \to E$ is a morphism and $Y = F - W$ is a divisor on F. Let $Y' = (\pi \circ \phi)^{-1}(Y)$ and let $W' = (\pi \circ \phi)^{-1}(W) \cong W \times \mathbb{P}^1$. To simplify notation, we will drop the coefficient group $\mathbb{C}$ in all cohomology groups. By Poincaré–Lefschetz duality, $H^n(E, Y') \cong H_c^n(W')$ and $H^n(F \times \mathbb{P}^1, Y \times \mathbb{P}^1) \cong H_c^n(W \times \mathbb{P}^1)$, where the "$c$" denotes cohomology with compact support. With these identifications, the long exact sequences in cohomology for the pairs (E, Y') and $(F \times \mathbb{P}^1, Y \times \mathbb{P}^1)$ give a commutative diagram:

$$\begin{array}{ccccc} & & H^n(X)^\circ & \searrow & \\ & & q^* \downarrow & & \\ H_c^n(W') & \longrightarrow & H^n(E) & \longrightarrow & H^n(Y') \\ \sim \downarrow & & \phi_* \downarrow & & \\ H_c^n(W \times \mathbb{P}^1) & \longrightarrow & H^n(F \times \mathbb{P}^1) & \longrightarrow & H^n(Y \times \mathbb{P}^1). \end{array}$$

In what follows, some standard facts from the theory of mixed Hodge structures will be used. These results can be found in [7; 8; 9]. The notation $\mathrm{Gr}_k(H^n(M))$ will be used to denote the kth graded piece of the mixed Hodge structure on $H^n(M)$, where M is any complex variety. The image of $H^n(F \times \mathbb{P}^1)$ in $H^n(W \times \mathbb{P}^1)$ is $\mathrm{Gr}_n(H^n(W \times \mathbb{P}^1))$, the lowest-weight piece of the mixed Hodge structure on $H^n(W \times \mathbb{P}^1)$. By duality, this says that

$$\mathrm{Gr}_n(H_c^n(W \times \mathbb{P}^1)) \to H^n(F \times \mathbb{P}^1)$$

is injective. Since X is smooth and proper, $H^n(X)^\circ$ has a Hodge structure of pure weight n and so $H^n(X)^\circ$ maps injectively into $\mathrm{Gr}_n(H^n(E))$. Referring to the previous diagram, if ω is a nonzero cohomology class in $K_1 = \ker(\phi_* \circ q^*)$ then $q^*\omega \in \mathrm{Gr}_n(H^n(E))$ is nonzero and could not be in the image of $\mathrm{Gr}_n(H_c^n(W'))$, since otherwise $\phi_*(q^*\omega) \neq 0$ by the injectivity of the composition

$$\mathrm{Gr}_n(H_c^n(W')) \to \mathrm{Gr}_n(H_c^n(W \times \mathbb{P}^1)) \to H^n(F \times \mathbb{P}^1).$$

Thus $K_1 \subseteq H^n(X)^\circ$ is mapped injectively into $H^n(Y')$.

Let $\tilde{Y}'$ be the normalization of Y'. Then the composition

$$K_1 \to \mathrm{Gr}_n(H^n(Y')) \to \mathrm{Gr}_n(H^n(\tilde{Y}'))$$

is injective. Since $\tilde{Y}'$ is normal, the generic fiber of $\tilde{Y}' \to Y$ must be nonsingular and hence a disjoint union of $\mathbb{P}^1$s. For each connected component $\tilde{Y}_i'$ of $\tilde{Y}'$, let Y_i denote the irreducible component of Y dominated by $\tilde{Y}_i'$ and let

$$\begin{array}{ccc} \tilde{Y}_i' & \searrow & \\ \downarrow & & \\ V_i & \longrightarrow & Y_i \end{array}$$

be the Stein factorization of $\tilde{Y}_i' \to Y_i$. Then $\tilde{Y}_i' \to V_i$ has connected fibers that are generically irreducible rational curves. It follows that, after a base extension, there is an open subset $W_i \subseteq V_i$ and a morphism $W_i \times \mathbb{P}^1 \to \tilde{Y}_i'$ such that the image of $w \times \mathbb{P}^1$ under the composition

$$W_i \times \mathbb{P}^1 \to \tilde{Y}_i' \to Y' \subseteq E \xrightarrow{q} X$$

is a rational curve on X for each $w \in W_i$.

Step 3: This is the situation we started with in Step 1, except that now each W_i is of dimension $n-2$. Repeating the argument of Step 1 for each W_i and setting F and E equal to the disjoint unions of the resulting F_i and E_i, one obtains a diagram as in the statement of the theorem. The only missing ingredient in continuing this process is the injectivity of

$$H^n(X)^\circ \to H^n(E).$$

However, this map restricted to K_1 is injective. To see this, notice that throughout the construction of Step 1 the family of curves in X being parameterized does not change. Thus in the end we obtain some projective variety E and a morphism $E \to X$ such that the image of E and the image of $\tilde{Y}'$ in X are the same variety $\bar{Y} \subseteq X$. Since $K_1 \to \mathrm{Gr}_n(H^n(\tilde{Y}'))$ is injective and $\tilde{Y}' \to X$ factors through $\bar{Y}$, the map $K_1 \to \mathrm{Gr}_n(H^n(\bar{Y}))$ is injective. Now $E \to \bar{Y}$ is surjective and $\bar{Y}$ is proper, so $\mathrm{Gr}_n(H^n(\bar{Y})) \to \mathrm{Gr}_n(H^n(E))$ is injective. Thus $K_1 \to \mathrm{Gr}_n(H^n(E))$ is injective.

Knowing this, one can replace $H^n(X)^\circ$ with K_1 and repeat the argument of Step 2 with the lower-dimensional F and E constructed above. In this case we set

$$K_2 = \ker(\phi_* q^* : K_1 \to H^n(F \times \mathbb{P}^1, \mathbb{C}))$$

and construct some E and F of still lower dimension such that the mapping $K_2 \to \mathrm{Gr}_n(H^n(E))$ is injective. Continuing in this way, E will eventually have dimension $< n/2$ so that $K_r = 0$ for some r. This gives a filtration

$$H^n(X)^\circ = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_r = 0$$

such that, for each i, there exist E and F such that

$$K_i = \ker(\phi_* q^* : K_{i-1} \to H^n(F \times \mathbb{P}^1)).$$

By taking disjoint unions of all these E and F, we obtain an E and F as in the statement of theorem such that

$$\phi_* q^* : H^n(X)^\circ \to H^n(F \times \mathbb{P}^1)$$

is injective.

Step 4: To complete the proof of the theorem, it is enough to show that

$$\pi_* : H^n(F \times \mathbb{P}^1) \to H^{n-2}(F)$$

is injective when restricted to the image of $H^n(X)^\circ$ in $H^n(F \times \mathbb{P}^1)$. By the Künneth theorem, $H^n(F \times \mathbb{P}^1) \cong H^n(F) \oplus H^{n-2}(F) \cup v$, where v is the Poincaré dual of a section of $\pi : F \times \mathbb{P}^1 \to F$. By construction, $\pi : F \times \mathbb{P}^1 \to F$ has a section s such that $s(F)$ is mapped into a multiple of D in X. Since $H^n(X)^\circ$ has cup product zero with the cohomology class of D, it follows that the image of $H^n(X)^\circ$ in $H^n(F\times\mathbb{P}^1)$ has cup product zero with v. If $x+y\cup v\in H^n(F)\oplus H^{n-2}(F)\cup v$ is in the image of $H^n(X)^\circ$ and $\pi_*(x+y\cup v)=y=0$, then $x\cup v=0$ and thus $x=0$. This means that π_* maps the image of $H^n(X)^\circ$ injectively into $H^{n-2}(F)$. Thus the cohomological Abel–Jacobi mapping

$$(\pi\circ\phi)_*q^*\colon H^n(X)^\circ\to H^{n-2}(F)$$

is injective, completing the proof of Theorem 1.1. □

Notice that nothing in Steps 1–3 depended on the fact we were only considering the primitive cohomology of X in dimension n. Thus, this argument actually shows that there exist E and F as in the statement of Theorem 1.1 such that

$$\phi_*q^*\colon H^*(X,\mathbb{C})\to H^*(F\times\mathbb{P}^1,\mathbb{C})$$

is injective.

There are certain situations in which the parameter space F of Theorem 1.1 can be taken to be equidimensional. Suppose that X is a hyperplane section of a smooth projective variety Y, and that the covering family of rational curves for X deforms generically with X in a Lefschetz pencil $\mathfrak{X}\to T=\mathbb{P}^1$ of hyperplane sections of Y containing X. The construction of Theorem 1.1 can then be carried out over T to yield a diagram

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{\phi} & \mathfrak{F}\times\mathbb{P}^1 & \xrightarrow{\pi} & \mathfrak{F} \\ {\scriptstyle q}\downarrow & & & & \downarrow \\ \mathfrak{X} & & \longrightarrow & & T \end{array}$$

such that, for all t in some open subset $U\subseteq T$, the fibers $\mathfrak{X}_t,\mathcal{E}_t,\mathfrak{F}_t$ of $\mathfrak{X},\mathcal{E},\mathfrak{F}$ over t satisfy the requirements of Theorem 1.1. In particular, it is possible to choose $\mathfrak{F}$ so that the fibers of $\mathfrak{F}$ are equidimensional and, for generic $t\in T$, the Abel–Jacobi mapping

$$H^n(\mathfrak{X}_t,\mathbb{C})^\circ\to H^{n-2}(\mathfrak{F}_t,\mathbb{C})$$

is nonzero. For each $t\in U$, let K_t denote the kernel of this map. Then K_t is a proper subspace of $H^n(\mathfrak{X}_t,\mathbb{C})^\circ$ that is invariant under the action of monodromy. If $H^n(\mathfrak{X}_t,\mathbb{C})^\circ$ is generated by the vanishing cycles of $H^n(\mathfrak{X}_t,\mathbb{C})$ or, equivalently, if $H^n(Y)^\circ=0$, then by classical Lefschetz theory $H^n(\mathfrak{X}_t,\mathbb{C})^\circ$ will be a simple module under the monodromy action. For details, see [10]. Thus $K_t=0$ and the Abel–Jacobi mapping is injective for generic $t\in T$.

3. Applications

In this section we will show that when X is threefold, Theorem 1.1 also gives a surjectivity result for the Abel–Jacobi mapping of Griffiths into the intermediate Jacobian of X. Let X be a smooth projective variety of dimension n. The $(n-d)$th intermediate Jacobian of X is the complex torus

$$J^{n-d}(X) = \frac{(H^{2d+1,0}(X,\mathbb{C})\oplus\cdots\oplus H^{d+1,d}(X,\mathbb{C}))^*}{H_{2d+1}(X,\mathbb{Z})}.$$

Denote by $J^{n-d}(X)^\circ$ the subtorus of $J^{n-d}(X)$ generated by the primitive cohomology of X. When $d=0$, $J^n(X)$ is the Albanese variety of X, denoted $\mathrm{Alb}(X)$. Let $\mathrm{Ch}_d^h(X)$ be the Chow group of rational equivalence classes of algebraic cycles of dimension d on X that are homologous to 0. The Abel-Jacobi mapping

$$\mathrm{AJ}_X\colon \mathrm{Ch}_d^h(X) \to J^{n-d}(X)$$

is then defined as follows. If Z is an algebraic cycle of dimension d that is homologous to zero, then there is a topological $(2d+1)$-chain Γ on X such that $\partial\Gamma = Z$. Integration over Γ then defines $\mathrm{AJ}_X(Z) \in J^{n-d}(X)$, which is independent of the choice of Γ and the choice of Z in its rational equivalence class.

Let F be a smooth projective variety parameterizing a family of curves on a threefold X. Let $E = \{(C,x) \in F\times X : x \in C\}$, and let $p\colon E\to F$ and $q\colon E\to X$ be the natural projections. Then the dual of the cohomological Abel-Jacobi mapping

$$p_*q^*\colon H^3(X,\mathbb{C}) \to H^1(F,\mathbb{C})$$

induces a homomorphism $\Phi\colon \mathrm{Alb}(F) \to J^2(X)$. Note that it is not necessary to assume F is connected. This map is also often called the Abel-Jacobi mapping, and is related to the Abel-Jacobi mappings AJ_X and AJ_F by the commutative diagram

$$\begin{array}{ccc} \mathrm{Ch}_0^h(F) & \xrightarrow{q_*p^*} & \mathrm{Ch}_1^h(X) \\ {\scriptstyle \mathrm{AJ}_F}\downarrow & & \downarrow{\scriptstyle \mathrm{AJ}_X} \\ \mathrm{Alb}(F) & \xrightarrow{\Phi} & J^2(X). \end{array}$$

The following is an immediate reformulation of Theorem 1.1.

THEOREM 3.1. *Let X be a smooth projective threefold that is covered by rational curves. Then there are families of rational curves on X parameterized by a smooth projective variety F such that $J^2(X)^\circ$ is contained in the image of*

$$\Phi\colon \mathrm{Alb}(F) \to J^2(X).$$

Furthermore, F can be chosen so that each component of F parameterizes either the original covering family of rational curves or degenerations of these curves.

COROLLARY 3.2. *Let X be a smooth projective threefold that is covered by rational curves, and assume $H^3(X,\mathbb{C})^\circ \neq 0$. Then the group of one-dimen-*

sional algebraic cycles on X that are algebraically equivalent to zero modulo those that are rationally equivalent to zero is nontrivial even modulo torsion.

COROLLARY 3.3. *Let X be a smooth projective threefold that is covered by rational curves. Then $J^2(X)^\circ$ is contained in the image of*

$$\mathrm{AJ}_X\colon \mathrm{Ch}_1^h(X) \to J^2(X).$$

Proof. Since $\mathrm{AJ}_F\colon \mathrm{Ch}_0^h(F) \to \mathrm{Alb}(F)$ is surjective, this follows from Theorem 3.1 and the preceding commutative diagram. □

If X is a Fano threefold, then it is a theorem of Mori and Kollár [6] that X can be covered by rational curves C satisfying the inequality $(-K_X \cdot C) \leq 4$. In this case, it follows from Kodaira vanishing that $H^3(X, \mathbb{C}) = H^3(X, \mathbb{C})^\circ$ so that $J^2(X) = J^2(X)^\circ$. Combining this with Theorem 3.1 gives our next result.

THEOREM 3.4. *Let X be a smooth Fano threefold, and let F be a smooth variety that parameterizes all rational curves C on X with $(-K_X \cdot C) \leq 4$. Then $\Phi\colon \mathrm{Alb}(F) \to J^2(X)$ is surjective.*

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