A Characterization of Hilbert Spaces in Terms of Multipliers between Spaces of Vector-Valued Analytic Functions

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0. Introduction

Given a complex Banach space $(X, \|\cdot\|)$, we shall denote by $H^1(X)$ the space of X-valued Bochner integrable functions on the circle $\mathbb{T} = \{|z| = 1\}$ whose negative Fourier coefficients vanish; that is,

$$H^1(X) = \{f \in L^1(\mathbb{T}, X) : \hat{f}(n) = 0 \text{ for } n < 0\}.$$

We write

$$\|f\|_{1,X} = \int_0^{2\pi} \|f(e^{it})\| \frac{dt}{2\pi}$$

for the norm in $H^1(X)$.

We shall also denote by $\mathrm{BMOA}(X)$ the space of vector-valued BMO functions on the circle with analytic extension to the unit disk D; that is, $f \in L^1(\mathbb{T}, X)$ with $\hat{f}(n) = 0$ for $n < 0$ such that

$$\|f\|_{*,X} = \sup_I \left(\frac{1}{|I|} \int_I \|f(e^{it}) - f_I\|^2 \frac{dt}{2\pi} \right)^{1/2} < \infty,$$

where the supremum is taken over all intervals $i \in \mathbb{T}$, $|I|$ stands for the normalized Lebesgue measure of I, and

$$f_I = \frac{1}{|I|} \int_I f(e^{it}) \frac{dt}{2\pi}.$$

The norm in the space is given by

$$\|f\|_{\mathrm{BMO}(X)} = \left\| \int_0^{2\pi} f(e^{it}) \frac{dt}{2\pi} \right\| + \|f\|_{*,X}.$$

Finally, we shall use $\mathrm{Bloch}(X)$ to denote the space of X-valued analytic functions on D, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$, such that $\sup_{|z|<1} (1-|z|)\|f'(z)\| < \infty$. To avoid constant functions have zero norm we consider

$$\|f\|_{\mathrm{Bloch}(X)} = \|f(0)\| + \sup_{|z|<1} (1-|z|)\|f'(z)\|.$$

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Now, given two complex Banach spaces X, Y and denoting by $B(X,Y)$ the space of bounded operators from X into Y (or simply $B(X)$ when $X=Y$), we can formulate the following definition, which is the natural analog of the scalar-valued notion of a convolution multiplier.

Given $F\in \text{Bloch}(B(X,Y))$, say $F(z)=\sum_{n=0}^{\infty} T_n z^n$, and given $f\in H^1(X)$, say $f(z)=\sum_{n=0}^{\infty} x_n z^n$, we shall define

$$F*f(z)=\sum_{n=0}^{\infty} T_n(x_n)z^n=\int_0^{2\pi} F(ze^{it})(f(e^{-it}))\frac{dt}{2\pi}.$$

Let us write $(H^1(X), \text{BMOA}(Y))$ for the space of functions $F\colon D\to B(X,Y)$ such that $F*f\in \text{BMOA}(Y)$ for any $f\in H^1(X)$. The norm on it is induced by the norm as subspace of $B(H^1(X), \text{BMOA}(Y))$.

It was proved in [7] that the space of multipliers from H^1 into BMOA can be identified with the space of Bloch functions; that is,

$$(H^1, \text{BMOA}) = \text{Bloch}. \tag{0.1}$$

It is not hard to see that the vector-valued formulation does not hold for general Banach spaces. The aim of this note is to show that the vector-valued extension for $X=Y$ holds only for Hilbert spaces. We shall prove the following theorem.

THEOREM. *Let X be a complex Banach space. Then* $(H^1(X), \text{BMOA}(X)) = \text{Bloch}(B(X))$ *if and only if X is isomorphic to a Hilbert space.*

Throughout the paper, all Banach spaces are assumed to be vector spaces on the complex field and C will stand for a constant that may vary from line to line.

1. Preliminary Results

Let us recall some known facts on vector-valued analytic functions that we shall need for the proof.

First of all, recall the characterization of BMO functions in terms of Carleson measures (see [4, Thm. 3.4]) that we shall use later on. This characterization is still valid for functions taking values in Hilbert spaces (since it simply relies on Plancherel's theorem). Given a Hilbert space X and an analytic function $f\colon \mathbb{D}\to X$, we have

$$\|f\|_{*,X}\approx \sup_{z\in D}\left(\int_0^1\int_0^{2\pi}\frac{(1-s)(1-|z|^2)\|f'(se^{it})\|^2}{|1-\bar{z}se^{it}|^2}\frac{dt}{2\pi}\,ds\right)^{1/2}. \tag{1.1}$$

Another fact to be used is that Kintchine's inequalities hold for BMO functions; actually, this can be achieved using Paley's inequality (see [3]) and duality. That is,

$$\left(\sum_{k=0}^{\infty}|\alpha_k|^2\right)^{1/2}\approx\left\|\sum_{k=0}^{\infty}\alpha_k z^{2^k}\right\|_{\text{BMOA}}. \tag{1.2}$$

The following remarks regard vector-valued Bloch functions. Given $(T_n)\subset B(X,Y)$ and $F(z)=\sum_{n=0}^{\infty}T_nz^n$, it clearly follows from the definition that $F\in \text{Bloch}(B(X,Y))$ if and only if, for any $x\in X$ and $y^*\in Y^*$, the functions $F_{x,y^*}(z)=\sum_{n=0}^{\infty}\langle T_n(x),y^*\rangle z^n\in \text{Bloch}$. Moreover,

$$\|F\|_{\text{Bloch}(B(X,Y))}=\sup_{\|x\|\leq 1,\|y^*\|\leq 1}\|F_{x,y^*}\|_{\text{Bloch}}. \tag{1.3}$$

According to this, it follows from the scalar-valued case (see [1; 2]) that

$$F(z)=\sum_{n=0}^{\infty}T_nz^{2^n}\quad\text{if and only if}\quad \sup_{n\in\mathbb{N}}\|T_n\|<\infty. \tag{1.4}$$

Let us now recall a basic inequality, due to Hardy and Littlewood (see [5, Lemma HL1]), which played an important role in the proof of (0.1) and whose vector-valued extension we shall use.

There exists a constant $C>0$ such that for any $f\in H^1$ one has

$$\left(\int_0^1(1-r)M_1^2(f',r)\,dr\right)^{1/2}\leq C\|f\|_1, \tag{1.5}$$

where $M_1(f',r)=\int_0^{2\pi}|f'(re^{it})|(dt/2\pi)$.

Using the notation $M_{1,X}(f',r)=\int_0^{2\pi}\|f'(re^{it})\|(dt/2\pi)$ when dealing with functions in $H^1(X)$, we have the following vector-valued extension.

LEMMA 1.1. *Let X be a Hilbert space. Then there exists a constant $C>0$ such that*

$$\left(\int_0^1(1-r)M_{1,X}^2(f',r)\,dr\right)^{1/2}\leq C\|f\|_{1,X}$$

for any $f\in H^1(X)$.

Proof. Assume that $X=l^2$ (for general Hilbert spaces, this would follow from the previous case and the fact that X is finitely representable in l^2).

Given $f\in H^1(l^2)$ we can write

$$f=(f_n),\quad \text{where } f_n\in H^1 \text{ and } \left(\sum_{n=1}^{\infty}|f_n(e^{i\theta})|^2\right)^{1/2}\in L^1(\mathbb{T}).$$

Denoting by r_n the Rademacher functions in $[0,1]$, we define

$$F(z)=\sum_{n=1}^{\infty}f_n(z)r_n,\qquad F_t(z)=\sum_{n=1}^{\infty}f_n(z)r_n(t).$$

It follows from Fubini's theorem and Kintchine's inequalities that

$$\|F\|_{1,L^1}\approx\|f\|_{1,l^2},\qquad M_{1,L^1}(F',r)\approx M_{1,l^2}(f',r).$$

Therefore, setting $\alpha_k=1-2^{-k}$,

$$\begin{aligned}\int_0^1 (1-r)M_{1,l^2}^2(f',r)\,dr &\approx \int_0^1 (1-r)M_{1,L^1}^2(F',r)\,dr\\ &= \sum_{k=0}^{\infty}\int_{\alpha_{k+1}}^{\alpha_k}(1-r)M_{1,L^1}^2(F',r)\,dr\\ &\le \sum_{k=0}^{\infty} 2^{-2k}M_{1,L^1}^2(F',\alpha_k)\\ &\le \sum_{k=0}^{\infty}\|2^{-k}M_1(F_t',\alpha_k)\|_{L^1([0,1])}^2.\end{aligned}$$

With this estimate, together with the well-known fact that (due to the cotype-2 condition on L^1; see [7]) that

$$\left(\sum_{k=0}^{\infty}\|\phi_k\|_{L^1([0,1])}^2\right)^{1/2} \le C\left\|\left(\sum_{k=0}^{\infty}(|\phi_k(t)|)^2\right)^{1/2}\right\|_{L^1([0,1])}$$

and applying the scalar inequality (1.5), we can write

$$\begin{aligned}\left(\int_0^1 (1-r)M_{1,l^2}^2(f',r)\,dr\right)^{1/2} &\le \int_0^1\left(\sum_{k=0}^{\infty}2^{-2k}M_1^2(F_t',\alpha_k)\right)^{1/2}dt\\ &\le C\int_0^1\left(\int_0^{2\pi}(1-r)M_1^2(F_t',r)\,dr\right)^{1/2}dt\\ &\le C\int_0^1\int_0^{2\pi}|F_t(e^{i\theta})|\frac{d\theta}{2\pi}\,dt\\ &= C\|F\|_{1,L^1}\approx\|f\|_{1,l^2}.\end{aligned}$$

□

We finish this section by recalling the notions of type and cotype (where we replace the Rademacher functions by lacunary sequences). The reader is referred to [9; 10] for information on these properties.

A Banach space has cotype 2 (respectively type 2) if there exists a constant $C>0$ such that, for all $N\in\mathbb{N}$ and for all $x_1,x_2,\dots,x_N\in X$, one has

$$\left(\sum_{k=1}^{N}\|x_k\|^2\right)^{1/2}\le C\left\|\sum_{k=1}^{N}x_ke^{2^kit}\right\|_{1,X}$$

or, respectively,

$$\left\|\sum_{k=1}^{N}x_ke^{2^kit}\right\|_{1,X}\le C\left(\sum_{k=1}^{N}\|x_k\|^2\right)^{1/2}.$$

Finally, we recall Kwapien's [6] characterization of Hilbert spaces: X is isomorphic to a Hilbert space if and only if X has type and cotype 2.

2. Proof of the Theorem

LEMMA 2.1. *Let X, Y be two complex Banach spaces. Then*

$$(H^1(X),\mathrm{BMOA}(Y))\subset \mathrm{Bloch}(B(X,Y)).$$

Proof. Given $F \in (H^1(X), \mathrm{BMOA}(Y))$ and given $x \in X$ and $y^* \in Y^*$, one clearly has $\langle F(z)(x), y^* \rangle \in (H^1, \mathrm{BMOA}) = \mathrm{Bloch}$. Moreover,

$$\|\langle F(z)(x), y^* \rangle\|_{\mathrm{Bloch}} \le \|F\|_{(H^1(X), \mathrm{BMOA}(Y))} \|x\| \|y^*\|.$$

Hence, (1.3) shows that $F \in \mathrm{Bloch}(B(X, Y))$. □

Proof of the theorem. From Kwapien's result, we shall begin by showing that $(H^1(X), \mathrm{BMOA}(X)) = \mathrm{Bloch}(B(X))$ implies cotype 2 and type 2 on X.

Let us take $x_1, x_2, \ldots, x_N \in X$. Then choose $x_n^* \in X^*$ so that $\langle x_n^*, x_n \rangle = \|x_n\|$ and $\|x_n^*\| = 1$. Then, using (1.2), we have

$$\left(\sum_{k=1}^N \|x_k\|^2 \right)^{1/2} = \left(\sum_{k=1}^N |\langle x_n^*, x_k \rangle|^2 \right)^{1/2} \approx \left\| \sum_{k=1}^N \langle x_k^*, x_k \rangle z^{2^k} \right\|_{\mathrm{BMOA}}.$$

Now let us fix $x \in X$ with $\|x\| = 1$, and consider $F(z) = \sum_{n=1}^N T_n z^{2^n}$ where T_n are the operators in $B(X)$ defined by $T_n(y) = \langle x_n^*, y \rangle x$. From (1.2) we have $F \in \mathrm{Bloch}(B(X))$ and $\|F\|_{\mathrm{Bloch}(B(X))} = 1$. Therefore

$$\left(\sum_{k=1}^N \|x_k\|^2 \right)^{1/2} \le C \left\| \sum_{k=1}^N T_k(x_k) z^{2^k} \right\|_{\mathrm{BMOA}(X)} \le C \left\| \sum_{k=1}^N x_k z^{2^k} \right\|_{1,X}.$$

This shows that X has cotype 2.

Now, given $x_1, x_2, \ldots, x_N \in X$, we fix $x \in X$ and $x^* \in X^*$ with $\|x\| = 1$ and $\langle x^*, x \rangle = 1$. Define $F(z) = \sum_{n=1}^N T_n z^{2^n}$ where T_n are the operators in $B(X)$ defined by $T_n(y) = \langle x^*, y \rangle (x_n / \|x_n\|)$. From (1.2) we have $F \in \mathrm{Bloch}(B(X))$ and $\|F\|_{\mathrm{Bloch}(B(X))} = 1$.

Observe that

$$\sum_{k=1}^N x_k z^{2^k} = \sum_{k=1}^N T_k(\|x_k\| x) z^{2^k} = F * f,$$

where $f(z) = \sum_{k=1}^N \|x_k\| x z^{2^k}$. Then, since $\mathrm{BMOA}(X) \subset H^1(X)$, we have

$$\begin{aligned}\left\| \sum_{k=1}^N x_k z^{2^k} \right\|_{1,X} &\le \left\| \sum_{k=1}^N x_k z^{2^k} \right\|_{\mathrm{BMOA}(X)} \\ &\le C \left\| \sum_{k=1}^N \|x_k\| x z^{2^k} \right\|_{1,X} \\ &\le C \left\| \sum_{k=1}^N \|x_k\| z^{2^k} \right\|_1 \le C \left(\sum_{k=1}^N \|x_k\|^2 \right)^{1/2}.\end{aligned}$$

This shows that X has type 2.

Conversely, let us assume that X is a Hilbert space. From Lemma 2.1, we need only prove

$$\mathrm{Bloch}(B(X)) \subset (H^1(X), \mathrm{BMOA}(X)).$$

Take $F(z) = \sum_{n=0}^\infty T_n z^n \in \mathrm{Bloch}(B(X))$ and $f(z) = \sum_{n=0}^\infty x_n z^n \in H^1(X)$. Now we observe that

$$
\begin{aligned}
z(F*f)'(z^2) &= \sum_{n=1}^{\infty} nT_n(x_n)z^{2n-1} \\
&= \int_0^{2\pi} F'(ze^{it})(f(ze^{-it}))e^{it}\frac{dt}{2\pi} \\
&= 2\int_0^1\int_0^{2\pi}\left(\sum_{n=1}^{\infty} nT_n z^{n-1}r^{n-1}e^{i(n-1)t}\right)\left(\sum_{n=1}^{\infty} nx_n r^{n-1}e^{-i(n-1)t}\right)\frac{dt}{2\pi}r\,dr \\
&= 2\int_0^1\int_0^{2\pi} F'(zre^{it})(f'(re^{-it}))e^{it}\frac{dt}{2\pi}r\,dr.
\end{aligned}
$$

Therefore, since $F\in \mathrm{Bloch}(B(X))$, we have

$$
\begin{aligned}
\|z(F*f)'(z^2)\| &\le C\int_0^1 \frac{1}{(1-s|z|)}M_{1,X}(f',s|z|)\,ds \\
&\le C\left(\int_0^1 \frac{ds}{(1-s|z|)^2}\right)^{1/2}\left(\int_0^{|z|} M_{1,X}^2(f',s)\,ds\right)^{1/2} \\
&\le \frac{C}{(1-|z|)^{1/2}}\left(\int_0^{|z|} M_{1,X}^2(f',s)\,ds\right)^{1/2}.
\end{aligned}
$$

Finally, using (1.1), we obtain

$$
\begin{aligned}
\|F*f\|_{*,X}^2 &\approx \sup_{z\in D}\int_0^1\int_0^{2\pi}\frac{(1-s)(1-|z|^2)\|(F*f)'(se^{it})\|^2}{|1-\bar{z}se^{it}|^2}\frac{dt}{2\pi}\,ds \\
&= 2\sup_{z\in D}\int_0^1\int_0^{2\pi}\frac{(1-r^2)(1-|z|^2)r\|(F*f)'(r^2e^{2it})\|^2}{|1-\bar{z}r^2e^{2it}|^2}\frac{dt}{2\pi}\,dr \\
&\le C\int_0^1\int_0^{2\pi}\frac{(1-|z|^2)}{|1-\bar{z}r^2e^{2it}|^2}\left(\int_0^r M_{1,X}^2(f',s)\,ds\right)\frac{dt}{2\pi}\,dr \\
&\le C\int_0^1\int_0^r M_{1,X}^2(f',s)\,ds\,dr = C\int_0^1(1-s)M_{1,X}^2(f_1,s)\,ds.
\end{aligned}
$$

Of course,

$$
\left\|\int_0^{2\pi} F*f(e^{it})\frac{dt}{2\pi}\right\| = \|T_0(x_0)\| \le \|T_0\|\|x_0\| \le \|F\|_{\mathrm{Bloch}(B(X))}\|f\|_{1,X}.
$$

Therefore, combining both estimates and using Lemma 1.1. the proof is finished. □

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