Motions of Trivial Links, and Ribbon Knots

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Introduction

A motion of a link consists of an isotopy of the link through its ambient space that ultimately returns the link to itself. By reducing this notion to the classical dimension, a classical braid can be considered the trace of a motion of a point set in a plane.

The set of motions of a link naturally forms a group, called a *motion group*. It is not easy to describe the motion group explicitly for a given link; Goldsmith [G1; G2] calculated motion groups for a trivial link and torus links in the 3-sphere. One might conjecture that the motion group of a trivial link of n-spheres in S^{n+2} would have the same structure as that in the classical dimension. In this paper, we give a result (Theorem 2.2) on motions of a trivial link of two components in general dimensions that might support its motion group structure.

Using our result on motions of a trivial link, we can define an invariant of ribbon presentations of knots. A *ribbon presentation* is geometric information defining a knot to be a ribbon, which is introduced and studied in [M2], [M3], [NN], and [Ya]. Specifically in this paper, we treat 1-fusion ribbon presentations—that is, a description $\mathcal{R}$ of a knot as obtained from the trivial link of two n-spheres in S^{n+2} by connecting them with a pipe. Then the centerline of the pipe links two components of the trivial link, and we can naturally assign a word w in two letters by reading off the linking of the centerline and the trivial link. Associated with this word w, we define a certain equivalence class $W(\mathcal{R})$ in two leters, and show that this turns out to be an invariant of 1-fusion ribbon presentations (Theorem 4.1).

A ribbon knot possibly has distinct ribbon presentations. We construct a ribbon knot having arbitrarily many different ribbon presentations of 1-fusion in general dimension (Theorem 4.4), and we use our invariant $W(\mathcal{R})$ for distinguishing those ribbon presentations. The first example of a knot

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having two different ribbon presentations is constructed in [NN]; examples in higher dimensions are given in [Ya]. We show that our invariant can work for distinguishing those examples. Another invariant for a ribbon presentation is shown [M3] to work for examples in [NN] and [Ya], but our invariant is slightly easier to calculate than in [M3]. We also apply our result to study handcuffs (Theorem 5.1).

1. Notation

Throughout this paper we work in the piecewise linear category, but all our results hold in the smooth category as well. We assume that every manifold is orientable, and a fixed orientation is given. Any submanifold is presumed to be locally flat.

By a *pair* (M,N) we mean a manifold M and a submanifold or subcomplex N of M. Given a manifold P, $P\cap(M,N)$ is the pair $(P\cap M, P\cap N)$. If N is also a manifold, we define $\partial(M,N)=(\partial M,\partial N)$.

Let f be a map from M to M, and let $N\subset M$. Then f *fixes* N if the restriction $f|_N$ is the identity map of N. Let f_i be orientation-preserving homeomorphisms from an oriented manifold M onto itself such that f_i fixes N. Then f_1 and f_2 are *similar* rel N if there exists an ambient isotopy $\{g_t\}$ of M connecting f_1 and f_2 such that each g_t fixes N, that is, an isotopy $\{g_t\}$ of M such that:

(i) g_0 is the identity and $g_1\circ f_1=f_2$; and
(ii) each g_t fixes N, $0\le t\le 1$.

Let $L=(S^{n+2},K_1\cup K_2)$ be a link of oriented n-spheres K_1 and K_2 embedded in an oriented $(n+2)$-sphere S^{n+2}. The link L is a *trivial* n-link, or simply *trivial,* if one can find two disjoint $(n+1)$-disks in S^{n+2} each bounding K_1 and K_2.

Let α be an oriented arc in S^{n+2} connecting K_1 and K_2 such that α does not intersect K_1 or K_2 except at the endpoints of the arc. Then we call α an *arc spanning* L, and we denote the set of all arcs spanning L by $\mathbf{A}(L)$. If any confusion does not occur, we may regard an arc as a map from the unit interval $I=[0,1]$ into its ambient space. We remark that an arc spanning L is not necessarily simple.

For two arcs α_1 and α_2 in $\mathbf{A}(L)$, the arcs are *homotopic* rel L, denoted $\alpha_1\simeq\alpha_2$ rel L, if there exists a homotopy $\{h_t\}$ connecting α_1 and α_2 such that each h_t is also an arc spanning L for any t. We define $\mathfrak{A}(L)=\mathbf{A}(L)/(\simeq \text{rel}\, L)$.

Given a link $L=(S^{n+2},K_1\cup K_2)$, let f be an orientation-preserving homeomorphism of S^{n+2} that fixes $K_1\cup K_2$. We then have the well-defined bijective map f^* from $\mathfrak{A}(L)$ onto itself by defining $f^*([\alpha])=[f(\alpha)]$, $\alpha\in\mathbf{A}(L)$, where [ ] denotes an equivalence class in $\mathfrak{A}(L)$.

The following lemma is easy, so we omit its proof.

LEMMA 1.1. *Given a link $L=(S^{n+2},K_1\cup K_2)$, let f_i be an orientation-preserving homeomorphism of S^{n+2} onto itself that fixes $K_1\cup K_2$. If f_1 and f_2 are similar* rel$(K_1\cup K_2)$, *then $f_1^*=f_2^*\colon\mathfrak{A}(L)\to\mathfrak{A}(L)$.*

2. Motions of a Trivial Link

The motion group of a link in a space is, roughly speaking, the set of motions of the link, each of which consists of an isotopy of the link through the total space that ultimately returns the link to itself, under certain equivalence relations among motions [Gl]. In [Gl], the motion group of a trivial link in S^3 is completely determined, as Proposition 2.1 states specifically in the case of two components. The authors conjecture that the proposition holds in the case of trivial links in higher dimensions.

PROPOSITION 2.1 [Gl]. *The motion group of a trivial link of two components in S^3 is generated by the following types of motions*:

(1) *turning a component over*;
(2) *interchanging two components*;
(3) *pulling one component through the other component.*

We then prove (in Section 3) the following theorem.

THEOREM 2.2. *For a trivial link $L=(S^{n+2}, K_1 \cup K_2)$ and any orientation-preserving homeomorphism f of S^{n+2} onto itself that fixes $K_1 \cup K_2$, the map $f^*: \mathfrak{A}(L) \to \mathfrak{A}(L)$ is the identity.*

3. Lemmas and Proof of Theorem 2.2

Throughout this section, $L=(S^{n+2}, K_1 \cup K_2)$ denotes a trivial link of two n-spheres K_1 and K_2 in S^{n+2}. We remark that K_1 and K_2 are given suitable orientations.

Let Δ_1 and Δ_2 be disjoint $(n+1)$-disks in S^{n+2} bounding K_1 and K_2, respectively, and let α be an arc spanning L. We may then assume that α is simple and intersects Δ_1 and Δ_2 transversely except at the endpoints of the arc. We give an orientation to α as running from $\alpha(0) \in K_1$ to $\alpha(1) \in K_2$. Give orientations to Δ_1 and Δ_2 coherent with those of K_1 and K_2. For each intersection of α with Δ_1 or Δ_2, we assign a local intersection number, $+1$ or -1, and a letter $x_i^{\pm 1}$ according to the local intersection number at Δ_i. Reading off this assignment from $\alpha(0)$ to $\alpha(1)$ sequentially, we obtain a word $w(\alpha; \Delta_1, \Delta_2)$ in letters x_1 and x_2, which is an element of the free group $F(x_1, x_2)$ on x_1 and x_2. We may assume that $w(\alpha; \Delta_1, \Delta_2)$ represents a reduced word in the free group $F(x_1, x_2)$, by doing reductions in the free group if necessary. In fact, a cancellation $x_*^{\pm 1} x_*^{\mp 1}$ of consecutive letters in $F(x_1, x_2)$, if it occurs, can be realized by a homotopic deformation of the arc $\operatorname{rel}(K_1 \cup K_2)$.

Deforming α isotopically if necessary, we assume that the disk which α meets first is not Δ_1; that is, there exists s, $0 < s \leq 1$, such that $\alpha((0, s))$ does not meet any disks and $\alpha(s) \in \Delta_2$. We assume also that the last disk the arc meets is not Δ_2.

For a word w in $F(x, y)$, let $w = u_1^{\epsilon_1} \cdots u_n^{\epsilon_n}$ be the reduced representation of the word, where n is the length of the word, each u_i is one of the generators

x, y, and $\epsilon_i \in \{+1, -1\}$. Then we call u_1 and u_n the *initial* and *terminal* letters in w. We define $F^*(x;y)$ to be the set of words of $F(x,y)$ consisting of the trivial word 1, along with all those words whose initial letter is x and whose terminal letter is y. We remark that $F^*(x;y)$ does not form a group, and that $F^*(x;y) \neq F^*(y;x)$.

The word $w(\alpha; \Delta_1, \Delta_2)$ constructed above defines an element of $F^*(x_2; x_1)$. Let $\Delta = \{\Delta_1, \Delta_2\}$, and define the map $\varphi_\Delta \colon \mathcal{A}(L) \to F^*(x_2; x_1)$ by $\varphi_\Delta([\alpha]) = w(\alpha; \Delta)$. Then the following lemma is easy.

LEMMA 3.1. *The map $\varphi_\Delta \colon \mathcal{A}(L) \to F^*(x_2; x_1)$ is well-defined and bijective.*

Using Proposition 2.1, we show the following.

LEMMA 3.2. *Let f be an orientation-preserving homeomorphism of S^3 onto itself that fixes $K_1 \cup K_2$, and let α be an arc spanning L. Then $\varphi_\Delta([\alpha]) = \varphi_\Delta([f(\alpha)])$, where $[\,]$ is an equivalence class in $\mathcal{A}(L)$ and $f(\alpha)$ is given the orientation induced from that of α.*

Proof. Any orientation-preserving homeomorphism of S^3 is isotopic to the identity, and this isotopy gives a motion of the trivial link $K_1 \cup K_2$. This motion is obtained as a composition of fundamental motions of types (i), (ii), and (iii) in Proposition 2.1. Motions of types (i) and (ii) do not cause any changes in the intersections of α and Δ.

Consider S^3 to be the one-point compactification of $R^3 = \{(x_1, x_2, x_3) \mid x_i \in R\}$. Then we may identify K_1 with the x_3 axis. A motion of type (iii)—for example, pulling K_2 through K_1—corresponds to a 2π rotation about the x_3 axis, and this motion does not change the intersection of α and Δ (after a suitable isotopy that keeps $K_1 \cup K_2$ fixed).

The observation above shows that there exists an orientation-preserving homeomorphism g similar to f rel$(K_1 \cup K_2)$ such that $w(\alpha; \Delta) = w(g(\alpha); \Delta)$. By Lemmas 1.1 and 3.1, we have $\varphi_\Delta([\alpha]) = \varphi_\Delta([f(\alpha)])$ for $\alpha \in \mathbf{A}$. □

A manifold pair homeomorphic to

$$(D^n \times I^2, D^n \times \{1/2\} \times \{1/3\} \cup D^n \times \{1/2\} \times \{2/3\})$$

is called a *trivial disk pair,* where I is the unit interval $[0,1]$. An $(n+1)$-sphere S' in S^{n+2} is an *equator* if there exists an $(n+2)$-disk in S^{n+2} bounding S'. Let $L' = (S', K_1' \cup K_2')$ be a trivial $(n-1)$-link. Then L' is *equatorial* for L if: S' is an equator of S^{n+2}; S' separates L into two trivial disk pairs $\beta = (B, b_1 \cup b_2)$ and $\delta = (D, d_1 \cup d_2)$ that are symmetric about S'; and $L' = \partial\beta = \partial\delta$.

Assume $n \geq 2$, and take a base point e in $B - (b_1 \cup b_2)$. Choose an element $[\alpha] \in \mathcal{A}(L)$, and assume thereby that the arc $\alpha \subset B - (b_1 \cup b_2) - e$. We then take a projection $p \colon B - e \to \partial B$ such that $p(\alpha)$ is also an arc spanning L'.

LEMMA 3.3. *The projection p above induces a bijection $p^* \colon \mathcal{A}(L) \to \mathcal{A}(L')$ that is defined by $p^*([\alpha]) = [p(\alpha)]$ for $\alpha \in \mathbf{A}(L)$, and we have $\alpha \simeq p(\alpha)$* rel$(K_1 \cup K_2)$.

Proof. Let $\alpha_i \in \mathbf{A}(L)$, $i=1,2$. Deform α_1 and α_2 to be in $B-(b_1 \cup b_2)-e$, by moving homotopically $\operatorname{rel}(K_1 \cup K_2)$ if necessary. Assume $\alpha_1 \simeq \alpha_2 \operatorname{rel}(K_1 \cup K_2)$. We can then realize this homotopic deformation in B, that is, the trace of homotopy between α_1 and α_2 does not intersect D. Deform this homotopy so that the projection of the homotopy gives a homotopy between $p(\alpha_1)$ and $p(\alpha_2)$ rel $\partial(b_1 \cup b_2)$. Thus the map p^* is well-defined, and it is trivial that p^* is surjective. A homotopy between arcs in ∂B naturally gives a homotopy in S^{n+2}, and hence p^* is injective.

The last part of the lemma is shown from the fact that the projection p gives a homotopy between α and $p(\alpha)$. □

LEMMA 3.4. *Let $L'=(S', K_1' \cup K_2')$ be an equatorial trivial link of the trivial link L. Let f be an orientation-preserving homeomorphism of S^{n+2} fixing $K_1 \cup K_2$ such that $f(S')=S'$. Then the following diagram commutes,*

$$\begin{array}{ccc} \mathfrak{A}(L) & \xrightarrow{p^*} & \mathfrak{A}(L') \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f|_{S'})^*} \\ \mathfrak{A}(L) & \xrightarrow{p^*} & \mathfrak{A}(L'), \end{array}$$

where p is the projection chosen in the above.

Proof. It follows from the assumption $f(S')=S'$ that $(f\,|\,S')^*\colon \mathfrak{A}(L') \to \mathfrak{A}(L')$ is well-defined. Let $\alpha \in \mathbf{A}(L)$. By Lemma 3.3 we have $\alpha \simeq p(\alpha) \operatorname{rel}(K_1 \cup K_2)$, and thus $f(\alpha) \simeq f \circ p(\alpha) \operatorname{rel}(K_1 \cup K_2)$. By Lemma 3.3 again, we have $f(\alpha) \simeq p(f(\alpha)) \operatorname{rel}(K_1 \cup K_2)$. Hence $[f \circ p(\alpha)] = [p \circ f(\alpha)] \in \mathfrak{A}(L)$. Both arcs $f \circ p(\alpha)$ and $p \circ f(\alpha)$ are in S', and both span L'. Therefore, we have $[f \circ p(\alpha)] = [p \circ f(\alpha)] \in \mathfrak{A}(L')$ by Lemma 3.3. □

Take two disjoint $(n+1)$-disks Δ_1 and Δ_2 in S^{n+2} such that $K_i = \partial \Delta_i$, $i=1,2$. Let α be a simple arc in S^{n+2} such that $w(\alpha; \Delta_1, \Delta_2)=1$. Then we call α a *trivial* arc spanning L. It is easy to see that any two trivial arcs spanning L are isotopic, keeping $K_1 \cup K_2$ fixed.

LEMMA 3.5. *Assume $n \geq 2$. Let α be a trivial arc spanning L, and let f be an orientation-preserving homemorphism of S^{n+2} that fixes $K_1 \cup K_2$. Then there exists an orientation-preserving homeomorphism g of S^{n+2} similar to $f \operatorname{rel}(K_1 \cup K_2)$ such that $g(\alpha) \doteq \alpha$.*

Proof. We give only an outline of a proof for the lemma, which is essentially the same as that given in [M1].

Thickening the trivial arc α, we have a trivial ribbon n-knot K^n of 1-fusion, and α is a core of the band of K^n. More explicitly, construct an embedding $b\colon B^n \times I \to S^{n+2}$ such that:

(i) $b(B^n \times I) \cap K_i = b(B^n \times \{i-1\})$, $i=1,2$;

(ii) an orientation of $b(B^n \times I)$ is compatible with those of K_i, $i=1,2$; and

(iii) $b|_{(0 \times I)} = \alpha$, where $0 \in \operatorname{int} B^n$.

From the uniqueness of regular neighborhoods and a simple observation (see e.g. [RS, pp. 67ff]), it follows that $b(B^n \times I)$ is unique up to isotopy of S^{n+2}, keeping $K_1 \cup K_2 \cup \alpha$ fixed. Then $K^n = (K_1 \cup K_2 - b(B^n \times \partial I)) \cup b(\partial B^n \times I)$ is a ribbon n-knot of 1-fusion, which defines a trivial knot. Hence the knot $f(K^n)$ is also trivial, which is a ribbon n-knot of 1-fusion, $f(\alpha)$ being a core of the band of $f(K^n)$. The fundamental group $\pi_1(S^{n+2} - f(K^n))$ is a one-relator group, and the relator can be described by using the linking of $f(\alpha)$ and K_1, K_2. From a result of one-relator groups (see e.g. [MKS, p. 261]), it follows that the centerline of the band of $f(K^n)$ is homotopically trivial, keeping K_1 and K_2 fixed; that is, $f(\alpha)$ is a trivial arc. □

THEOREM 3.6. *Let $L' = (S', K_1' \cup K_2')$ be an equatorial trivial link of L. Let f be an orientation-preserving homeomorphism of S^{n+2} that fixes $K_1 \cup K_2$. Then there exists an orientation-preserving homeomorphism g of S^{n+2} similar to f $\text{rel}(K_1 \cup K_2)$ such that $g(S') = S'$.*

Proof. Take an $(n+2)$-disk B in S^{n+2}, bounding S', and a trivial arc α spanning L whose regular neighborhood in S^{n+2} is B. First, from Lemma 3.5, we may assume that $f(\alpha) = \alpha$. Next, from the uniqueness of a regular neighborhood, we can deform f isotopically, keeping $K_1 \cup \alpha \cup K_2$ fixed, to obtain g such that $g(B) = B$. This g is the required one. □

Proof of Theorem 2.2. We prove the theorem by induction on the dimension n. First, assume that $n = 1$. Let f be an orientation-preserving homeomorphism of S^3 onto itself that fixes $K_1 \cup K_2$, and let α be an arc spanning L. Then it follows from Lemmas 3.1 and 3.2 that $[\alpha] = [f(\alpha)]$.

Assume the theorem holds when $n = m-1$. Then choose a trivial link $L' = (S', K_1' \cup K_2')$ that is equatorial for L. From Lemma 1.1 and Theorem 3.6, we may assume that $f(S') = S'$. From Lemmas 3.3 and 3.4 and the inductive assumption, we have that f^* is the identity for $n = m$. □

4. Ribbon Presentations

A *1-fusion ribbon presentation* $\mathcal{R} = (K^n, b)$ is a pair of an n-knot K^n and an embedding $b \colon B^n \times I \to S^{n+2}$, called a *band,* such that

(1) $b(B^n \times I) \cap K^n = b(\partial B^n \times I)$ and
(2) $(K^n - b(\partial B^n \times I)) \cup b(B^n \times \partial I)$ is a trivial link of two embedded n-spheres in S^{n+2}.

Two 1-fusion ribbon presentations $\mathcal{R}_i = (K_i^n, b_i)$, $i = 1, 2$, are *equivalent* if there exists an orientation-preserving homeomorphism $f \colon S^{n+2} \to S^{n+2}$ such that $f(K_1^n) = K_2^n$ and $f \circ b_1(B^n \times I) = b_2(B^n \times I)$. (We can define m-fusion ribbon presentation as well, and equivalence among these ribbon presentations, which we call *simple equivalence* in [M2]. We are describing definitions of "ribbon presentation" and "equivalence" that are slightly different from those in [M2] for the sake of avoiding unnecessary complexity in this paper, but there are no essential differences.)

If a 1-fusion ribbon presentation $\mathcal{R} = (K^n, b)$ is given, K^n is obtained from a trivial link of two n-spheres S_x^n and S_y^n in S^{n+2} by connecting them with a *pipe* using the band b. We may assume that $b(B^n \times I) \cap S_x^n = b(B^n \times 0)$ and $b(B^n \times I) \cap S_y^n = b(B^n \times 1)$, and that orientations of S_x^n and S_y^n are coherent with that of K^n. Let $L = (S^{n+2}, S_x^n \cup S_y^n)$ and $\alpha = b(0 \times I)$, where $0 \in \operatorname{int} B^n$. Then L is a trivial link, and we have $[\alpha] \in \mathcal{A}(L)$. Choosing disjoint disks Δ_x and Δ_y spanning S_x^n and S_y^n, respectively, we get a map $\varphi_\Delta: \mathcal{A}(L) \to F^*(y; x)$ defined in the previous section, where $\Delta = \{\Delta_1, \Delta_2\}$.

We now introduce an equivalence relation $\sim$ on $F(x, y)$ that is generated by

(i) inversion in $F(x, y)$,
(ii) interchanging the symbols x and y, and
(iii) replacing both x and y by their inverses.

We define $F_{x,y}$ to be the quotient of $F(x, y)$ by the equivalence relation $\sim$.

We define $W(\mathcal{R})$ to be the equivalence class of $\varphi_\Delta([\alpha])$ in $F_{x,y}$, and call $W(\mathcal{R})$ the *ribbon invariant* of $\mathcal{R}$. In fact, we prove the following result.

THEOREM 4.1. *The class $W(\mathcal{R})$ is invariant under equivalence among the presentations.*

Proof. Take another set Δ' of spanning disks of the trivial link $S_x^n \cup S_y^n$. There exists an orientation-preserving homeomorphism f of S^{n+2} that fixes $S_x^n \cup S_y^n$ and maps Δ to Δ'. Then, from the definition of φ_Δ, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}(L) & \xrightarrow{\varphi_\Delta} & F^*(x_2; x_1) \\ {\scriptstyle f^*}\downarrow & & \| \\ \mathcal{A}(L) & \xrightarrow{\varphi_{\Delta'}} & F^*(x_2; x_1). \end{array}$$

From Theorem 2.2, f^* is the idenity map. Hence $\varphi_\Delta = \varphi_{\Delta'}$; that is, $\varphi_\Delta([\alpha])$ does not depend on the choice of Δ.

Thus the word $w(x, y) = \varphi_\Delta([\alpha])$ depends only on orientations of the arc α and the original knot K^n, and on the labeling of x and y. Reversing the orientation of α changes $w(x, y)$ into $w(x, y)^{-1}$; another choice of the orientation of K^n makes $w(x, y)$ into $w(x^{-1}, y^{-1})$; exchanging the labels x and y deforms $w(x, y)$ into $w(y, x)$. These do not cause any changes for $W(\mathcal{R})$.

From Theorem 2.2 again, two 1-fusion ribbon presentations that are equivalent define the same class $W(\mathcal{R})$. □

REMARKS. (1) In [Ya], Lemma 1.4 is described without proof. Theorem 4.1 gives a supplement for this insufficiency.

(2) In general, $W(\mathcal{R})$ has eight different representatives. That is, if $\varphi_\Delta([\alpha])$ is a word $w(x, y)$, then $w(x, y) \sim w(u, v)^\delta$, where $\{u, v\} = \{x^\epsilon, y^\epsilon\}$, $|\delta| = |\epsilon| = 1$.

COROLLARY 4.2. *The correspondence $\mathcal{R} \mapsto W(\mathcal{R})$ gives a bijection from the set of 1-fusion ribbon presentations of an n-knot onto $F_{x,y}$, if $n > 1$. In the case of $n = 1$, the map is surjective.*

Proof. From Theorem 4.1, the correspondence is a well-defined map.

For any word $w(x,y) \in F^*(y;x)$, we can easily realize an arc spanning a trivial link $S_x^n \cup S_y^n$ in any dimension n, and then construct a ribbon presentation having the ribbon invariant with a representative $w(x,y)$. This gives the surjectivity of the correspondence.

The arc above is uniquely determined up to homotopy, and hence up to isotopy in the case of $n>1$. Using the same argument as in the proof of Lemma 3.5, we can prove that the ribbon presentation is unique up to equivalence. This gives the injectivity if $n>1$. □

We use the notation $\mathcal{R}=(K^n,b)$ and Δ_x, Δ_y introduced earlier in this section. Let $\bar{S}$ be the $(n+3)$-sphere obtained by capping two $(n+3)$-disks D_+ and D_- on $S^{n+2}\times[-2,2]$. That is, $\bar{S}=D_- \cup (S^{n+2}\times[-2,2]) \cup D_+$, where

$$D_\pm \cap (S^{n+2}\times[-2,2]) = \partial D_\pm = S^{n+2}\times\{\pm 2\}.$$

Let $\overline{\Delta_x}=\Delta_x\times[-2,2]$ and $\overline{\Delta_y}=\Delta_y\times[-2,2]$ be embedded $(n+2)$-disks in $\bar{S}$, where S^{n+2} is identified with $S^{n+2}\times\{0\}\subset\bar{S}$. We define an embedding $\bar{b}$: $(B^n\times[-1,1])\times I\to\bar{S}$ by $\bar{b}((x,s),t)=(b(x,t),s)$, where $x\in B^n$, $s\in[-1,1]$, and $t\in I$. Let $\bar{K}$ be the $(n+1)$-knot in $\bar{S}$ obtained from $\partial\overline{\Delta_x}\cup\partial\overline{\Delta_y}$ by the fusion with the band $\bar{b}$. That is, $\bar{K}$ is an $(n+1)$-knot with the 1-fusion ribbon presentation $\bar{\mathcal{R}}=(\bar{K},\bar{b})$. We say that $\bar{\mathcal{R}}$ is *associated* with $\mathcal{R}$. From this construction, we have $W(\mathcal{R})=W(\bar{\mathcal{R}})$.

Using Corollary 4.2, we have the following.

COROLLARY 4.3. *For 1-fusion ribbon presentations of n-knots, $\mathcal{R}_1=\mathcal{R}_2$ if and only if $\overline{\mathcal{R}_1}=\overline{\mathcal{R}_2}$ in the case of $n>1$. In the case of $n=1$, $\overline{\mathcal{R}_1}=\overline{\mathcal{R}_2}$ if $\mathcal{R}_1=\mathcal{R}_2$.*

THEOREM 4.4. *For any positive integers $m>1$ and n, there exists a ribbon n-knot with at least m inequivalent 1-fusion ribbon presentations.*

Proof. Choose m distinct positive integers $p_1,p_2,\dots,p_m$. Take a trivial link of $m+1$ oriented n-spheres, labeled $x_0,x_1,\dots,x_m$, in S^{n+2}. Call the n-sphere labeled x_i the ith sphere. Connect the 0th sphere and ith sphere using a tube, called the ith tube, to get a new n-sphere. Choosing m such tubes as shown in Figure 1, we obtain an n-knot K_n; in the figure, each box in which p_i is

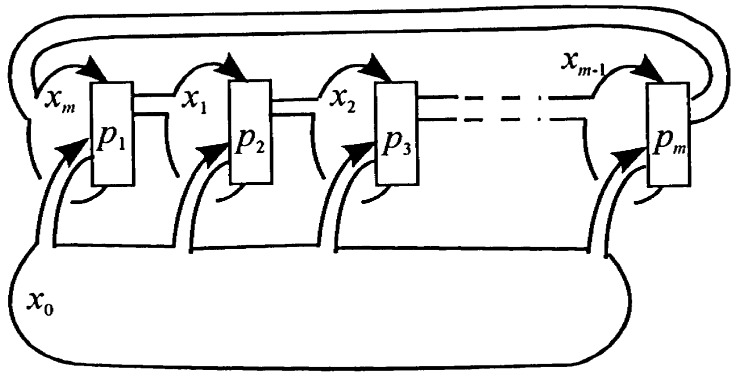


Figure 1

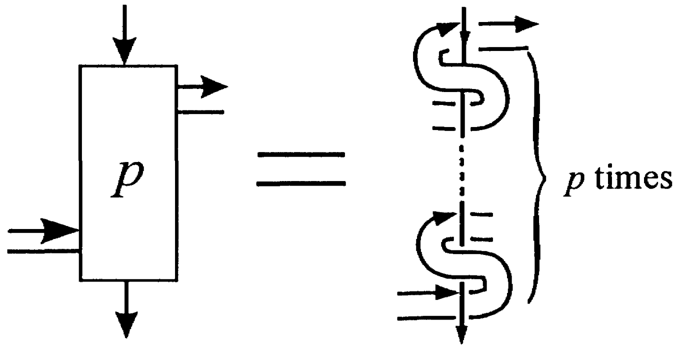


Figure 2

filled means that the ith connecting tube links with the $(i-1)$th sphere p_i times, as shown in Figure 2. In the case of $n=1$, we choose such tubes to have no self-twist.

Consider the band b_i associated with the ith tube of the knot. This gives a 1-fusion ribbon presentation $\mathcal{R}_i = (K^n, b_i)$ so that the fission of the knot along b_i results in a trivial link of two components.

For simplicity of argument, we assume that $n > 1$. Then the knot group $\pi_1(S^{n+2} - K^n)$ is generated by $x_0, x_1, \dots, x_m$ with relations

$$\begin{aligned} x_1 &= x_m^{p_1} x_m^{-p_1} \\ x_2 &= x_1^{p_2} x_0 x_1^{-p_2} \\ x_3 &= x_2^{p_3} x_0 x_2^{-p_3} \\ &\vdots \\ x_m &= x_{m-1}^{p_m} x_0 x_{m-1}^{-p_m}, \end{aligned}$$

where x_i is a meridianal loop of the ith sphere. First, deform K^n by pulling the jth sphere along the jth tube to get a new n-sphere, which is made from a fusion of the 0th and jth spheres. Make this deformation for each j, $j \neq i$, consecutively. Then the resulting knot appears to be two spheres connected by the band b_i, which shows a 1-fusion ribbon presentation $\mathcal{R}_i = (K^n, b_i)$ of K^n. Reading off the linking of the core of b_i and the two spheres, we can describe $W(\mathcal{R}_i)$.

The geometrical deformation above can be interpreted as an algebraic modification of the knot group with the presentation above, which eliminates each generator x_j $(j \neq i)$ by substituting one relation into the others. Applying the modification, we (finally) get a new group presentation with two generators x_0, x_i and one relation $x_i = w_i x_0 w_i^{-1}$, where $w_i x_0 w_i^{-1}$ is a reduced word in the free group $F(x_0, x_i)$, and $w_i \in F^*(x_0; x_i)$. Then the word w_i represents a core of the band b_i, and is a representative of the invariant $W(\mathcal{R}_i)$.

We now demonstrate the validity of the above procedure in the case of $i = 1$. The following can be shown by a simple calculation.

SUBLEMMA. *Let $x_{j-1} = u x_0 u^{-1}$, $x_j = x_{j-1}^p x_0 x_{j-1}^{-p}$, and $j \geq 3$. Assume that $u x_0 u^{-1}$ is a reduced word, and that the initial and terminal letters of u are x_1. Then there exists a reduced word uV such that $x_j = uVx_0(uV)^{-1}$.*

If $m=2$, then a representative of $W(\mathfrak{R}_1)$ is $x_0^{p_1}x_1^{p_2}$, and $x_0^{p_1}x_1^{p_2}$ for $W(\mathfrak{R}_2)$. From Corollary 4.2, two ribbon presentations $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are inequivalent.

Assume $m\geq 3$. First, we have $x_3=ux_0u^{-1}$, $u=x_1^{p_2}x_0^{p_3}x_1^{-p_2}$. Next, applying our sublemma repeatedly, we obtain a reduced word V such that

$$x_m=x_1^{p_2}x_0^{p_3}x_1^{-p_2}Vx_0V^{-1}x_1^{p_2}x_0^{-p_3}x_1^{-p_2}.$$

By substituting this relation into the first relation $x_1=x_m^{p_1}x_0x_m^{-p_1}$ to delete the generator x_m, we obtain a single relation

$$x_1=w(x_0,x_1)x_0w(x_0,x_1)^{-1},$$

where

$$w(x_0,x_1)=x_1^{p_2}x_0^{p_3}x_1^{-p_2}\cdots x_1^{p_2}x_0^{-p_3}x_1^{-p_2}.$$

Taking a conjugation of the above relation by $x_1^{p_2}$, we deform it to derive a new relation

$$x_1=w_1(x_0,x_1)x_0w_1(x_0,x_1)^{-1},$$

where

$$w_1(x_0,x_1)=x_0^{p_3}x_1^{-p_2}\cdots x_1^{p_2}x_0^{-p_3}x_1^{-p_2}$$

is a reduced word in $F^*(x_0;x_1)$. This conjugation can be realized by an isotopic deformation that changes $\mathfrak{R}_1$ into another ribbon presentation equivalent to $\mathfrak{R}_1$. We therefore have a representative $w_1(x_0,x_1)$ of the ribbon invariant $W(\mathfrak{R}_i)$.

The calculation above can be applied to the case of general $\mathfrak{R}_i$, yielding a representative $w_i(x_0,x_i)$ of the ribbon invariant $W(\mathfrak{R}_i)$ such that

$$w_i(x_0,x_1)=x_0^{p_{i+2}}x_i^{-p_{i+1}}\cdots x_i^{p_{i+1}}x_0^{-p_{i+2}}x_i^{-p_{i+1}},$$

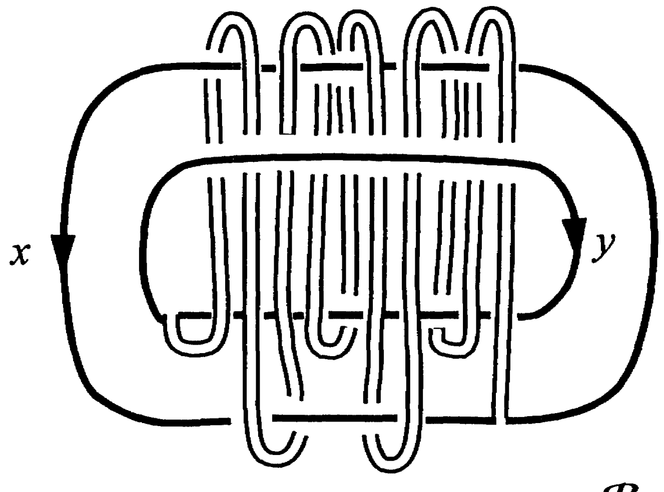
where subscripts for p should be taken as $m+1:=1$, $m+2:=2$. Therefore all $W(\mathfrak{R}_i)$ are shown to be different, and thus all $\mathfrak{R}_i$ are inequivalent.

In the case of $n=1$, given the ribbon presentation $\mathfrak{R}_i=(K^1,b_i)$, we get a ribbon presentation $\overline{\mathfrak{R}_i}=(K^2,\overline{b_i})$ of the ribbon 2-knot that is associated with $\mathfrak{R}_i$. From the result above in the case of $n=2$ together with Corollary 4.3, the theorem in the case of $n=1$ is proved. Hence the 1-fusion ribbon presentations we constructed are inequivalent in any dimension $n\geq 1$. □

REMARK. A suitable choice of $\{p_i\}$ shows that the knot K^n constructed in the proof of Theorem 4.4. cannot be obtained as the spun knot of any classical knot—for example, by calculating an Alexander polynomial.

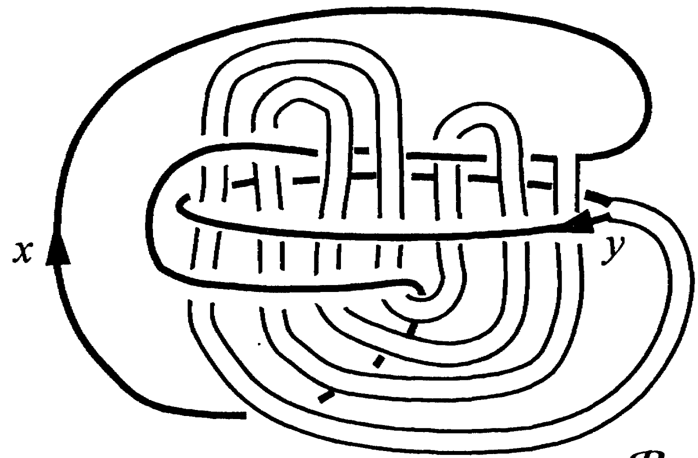
In [M3], a polynomial invariant is studied for distinguishing certain 1-fusion ribbon presentations that is effective for examples in [NN] and [Ya]. Using Theorem 4.1, we can distinguish those ribbon presentations much more easily.

NAKAGAWA–NAKANISHI EXAMPLE. Nakagawa and Nakanishi [NN] construct two 1-fusion ribbon presentations $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of a 1-knot, which are deformed to give Figures 3 and 4. In the figure, a thin part means a band of



$\mathcal{R}_1$

Figure 3



$\mathcal{R}_2$

Figure 4

a ribbon presentation. These two ribbon presentations define the same 1-knot, which is not very easily seen from the figures in [NN] and [M3].

Let

$$w_1(x,y) = yxy^{-1}x^{-1}y^{-1}xyxyx^{-1}y^{-1}x^{-1}yx;$$

$$w_2(x,y) = yxy^{-1}x^{-1}yx^{-1}y^{-1}xyxy^{-1}xyx^{-1}.$$

Then, reading off the linking of the centerline of the band and the two circles in each figure, we can show that $w_i(x,y)$ is a representative for the ribbon invariant $W(\mathcal{R}_i)$. The labels x and y are assigned with orientations as shown in the figures, and the centerlines of the bands are oriented as running from S_x^1 to S_y^1. From Theorem 4.1, two ribbon presentations $\mathcal{R}_1$ and $\mathcal{R}_2$ are inequivalent.

Yasuda Example. In [Ya], Yasuda constructs two 1-fusion ribbon presentations of the spun n-knot of a 2-bridge knot.

Let k be a 2-bridge knot of type (p,q) in S^3, and let $\langle x,y \mid y=wxw^{-1}\rangle$ be the Wirtinger presentation of the knot group that is naturally obtained from the knot diagram of 2-bridge type (p,q) [Sc]. The spun n-knot K^n of k is shown to be a ribbon n-knot of 1-fusion, and the ribbon presentation $\mathcal{R}_{(p,q)}$ is naturally constructed from the 2-bridge knot structure. Then it is verified that the word w is a representative of the ribbon invariant $W(\mathcal{R}_{(p,q)})$ of the spun knot.

Choose a 2-bridge knot k having two different 2-bridge presentations (p,q) and (p',q'), that is, $qq'\equiv 1 \bmod 2p$ and $q'\not\equiv \pm q$; the 5_2-knot provides an example (see Figures 5 and 6). In Figure 5, the left picture shows a 2-bridge presentation of the 5_2-knot, while the right picture shows the ribbon presentation of its spun 2-knot naturally obtained by the left one. Figure 6 shows another 2-bridge presentation of the knot and ribbon presentation. Thus the spun n-knot K^n has two, seemingly different, ribbon presentations $\mathcal{R}$ and $\mathcal{R}'$ that are constructed from the regular projections of 2-bridge types (p,q) and (p',q') respectively. From [Fu], it is shown that $W(\mathcal{R})\neq W(\mathcal{R}')$, and hence these ribbon presentations are distinct. The method stated here to distinguish ribbon presentations is essentially the same as that in [Ya].

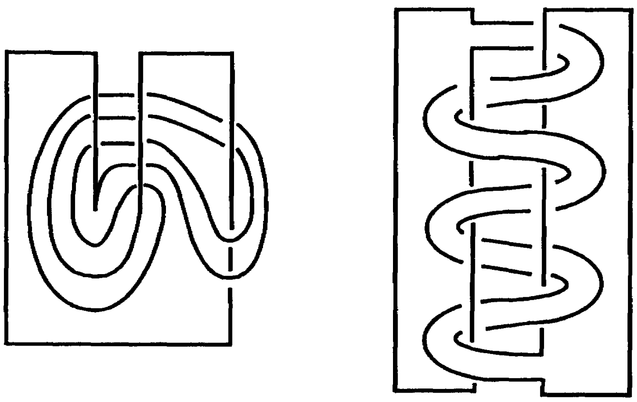


Figure 5

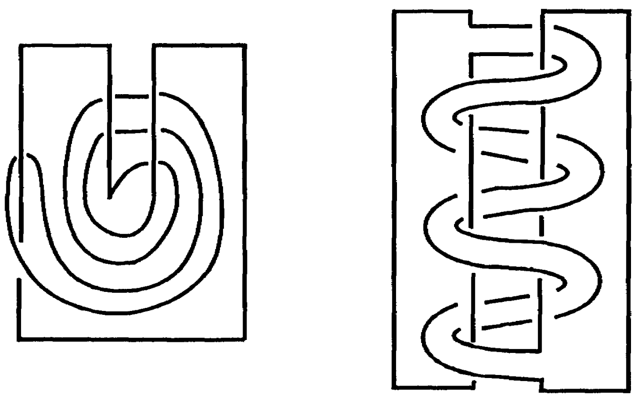


Figure 6

5. Handcuffs

An n-dimensional handcuff $C = K_1 \cup \alpha \cup K_2$ consists of two n-spheres K_1 and K_2 and a connecting simple arc α of K_1 and K_2, embedded in S^{n+2}. We call the two spheres of the handcuff *bases,* and the connecting arc a *string*. A handcuff is *simple* if bases form a trivial link of two n-spheres in S^{n+2}. Two handcuffs are *equivalent* if there exists an orientation-preserving homeomorphism of S^{n+2} that maps one handcuff to the other.

Given a simple n-dimensional handcuff $C = K_1 \cup \alpha \cup K_2$, we get a trivial link $L = (S^{n+2}, K_1 \cup K_2)$ by giving suitable orientations to bases of the handcuff. Introducing an orientation on the string α of the handcuff, and taking a family $\Delta = \{\Delta_1, \Delta_2\}$ of two $(n+1)$-disks that span the bases of C, we can define the word $\varphi_\Delta([\alpha])$ in $F^*(y;x)$ in the same manner as in Section 4.

We next introduce an equivalence relation $\sim'$ on $F(x,y)$ that is generated by

(i) inversion in $F(x,y)$,
(ii) interchanging the symbols x and y, and
(iii) replacing x by x^{-1}, or y by y^{-1}.

We define $F'_{x,y}$ to be the quotient of $F(x,y)$ by the equivalence relation $\sim'$.

Using the word $\varphi_\Delta([\alpha])$ constructed above, we define $\tilde{W}(C)$ to be the equivalence class of $\varphi_\Delta([\alpha])$ in $F'_{x,y}$. We can prove the following theorem by the same argument used in the proof of Theorem 4.1.

THEOREM 5.1.
(1) *$\tilde{W}(C)$ is invariant under equivalence among simple n-dimensional handcuffs.*
(2) *If $\tilde{W}(C) = \tilde{W}(C')$, two simple handcuffs C and C' are equivalent after deforming the string of a handcuff homotopically, that is, allowing the string to have self-intersections during the deformation.*

We remark that a handcuff cannot give the unique orientations to its bases, and that the words $w(u,v)^\delta$ define the same equivalence class $\tilde{W}(C)$ in $F'_{x,y}$, where $\{u,v\} = \{x^{\epsilon_1}, y^{\epsilon_2}\}$, $|\delta| = |\epsilon_1| = |\epsilon_2| = 1$.

In [K1], Kinoshita gives an invariant of an embedded 1-dimensional complex in terms of elementary ideals of the complementary space, and calculates the invariant of the handcuff in Figure 7. Using our invariant, the handcuff is directly shown to be inequivalent to the trivial one.

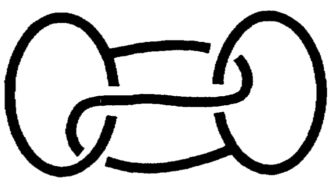


Figure 7

6. Final Remark

As mentioned previously, we posit the following.

CONJECTURE. Motions of a trivial link in higher dimensions are generated by Goldsmith's generators (see Proposition 2.1 or [G1]).

The following question is related to this conjecture.

QUESTION 1. Can a nontrivial link be obtained by gluing two trivial disk pairs along their common boundary?

In the case of classical dimension, the answer to this question is clearly No. Consider a 2-bridge knot, for example. The knot is constructed as two trivial arc pairs, and indeed any classical knot may be obtained by such a gluing with a suitable number of arc components. The answer to Question 1 is affirmative in the case of 2-dimensional embeddings in dimension 4 [Ka, Lemma 1.6]. However, the authors do not know about the other dimensions.

QUESTION 2. Does there exist a ribbon knot having infinitely many 1-fusion ribbon presentations?

Among two or more fusion ribbon presentations, equivalences are discussed in [M2]. In fact, the ribbon presentations we constructed in the proof of Theorem 4.4 are shown to be stably equivalent under the definition in [M2].

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