The Teichmüller Space of a Punctured Surface Represented as a Real Algebraic Space

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Dedicated to Professor Nobuyuki Suita on his 60th birthday

Introduction

The purpose of this paper is to represent the Teichmüller space of a branched or punctured surface as a real algebraic subspace of the Euclidean space. Let F_g^s denote an oriented closed surface of genus g with a set of s distinguished points $P=\{x_1,\dots,x_s\}$, where $s\geq 1$, $2g-2+s>0$. We consider first the Teichmüller space $\mathbf{T}(g;\nu_1,\dots,\nu_s)$, where $2g+s-2-\sum_{i=1}^{s}(1/\nu_i)>0$, of marked classes of complete hyperbolic metrics with singularities of F_g^s such that the angle around x_i is $2\pi/\nu_i$ with $\nu_i\in\mathbf{N}$, $\nu_i\geq 2$, or equivalently the marked classes of Fuchsian groups representing F_g^s with branched points x_i of order ν_i. Let $d=6g-6+2s$ be the dimension of the Teichmüller space. Then a triangulation of F_g^s by $d+s$ curves with endpoints in P induces global real analytic coordinates, the so-called L-length coordinates (which are essentially the hyperbolic lengths of the geodesic curves homotopic to the curves of the triangulation) introduced by Näätänen and Penner [7; 8]. These coordinates allow $\mathbf{T}(g;\nu_1,\dots,\nu_s)$ a real algebraic representation. Following their ideas, we discuss also a decomposition into subsets of the Teichmüller space that is left invariant under the action of the mapping class group MC_g^s.

The Teichmüller space $\mathbf{T}(g;\infty,\dots,\infty)$, with ∞ repeated s times, consists of marked classes of complete finite-area hyperbolic metrics of F_g^s-P, or equivalently of marked classes of Fuchsian groups representing F_g^s with punctures $x_1,\dots,x_s$. A real algebraic representation of $\mathbf{T}(g;\infty,\dots,\infty)$ is obtained as a limit of the sequence $\mathbf{T}(g;\nu,\dots,\nu)$, $\nu\geq 2$, after a suitable normalization, and is characterized by s homogeneous equations defined in $\mathbf{R}_+^{d+s}$.

Both the L-length coordinates employed in this paper for the Teichmüller space and the convergence have a natural geometric interpretation: In the case of branched surfaces, $L^2=\cosh l-1$, where l is the hyperbolic length of a geodesic with endpoints at elliptic fixed points; for punctured surfaces, $L^2=e^{\delta}$, where δ is the distance between certain horocycles based at parabolic fixed points. The convergence means geometrically that a sequence $D(\nu)$ of normalized fundamental domains equipped with circles centered at

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elliptic fixed points of order ν converges, as $\nu \to \infty$, to a fundamental domain D equipped with horocycles based at parabolic fixed points. If the interiors of the circles are deleted, then the hyperbolic lengths converge for each part of the boundary.

Since the coordinates given for $\mathbf{T}(g; \infty, \ldots, \infty)$ differ by a factor of only $\sqrt{2}$ from the λ-lengths employed in [9], our real algebraic model of $\mathbf{T}(g; \infty, \ldots, \infty)$ turns out to be a natural embedding of the Teichmüller space into the decorated Teichmüller space of Penner [9]. As an example, we consider explicitly the Teichmüller space of a twice-punctured torus and the mapping class group MC_1^2.

We would like to thank T. Kuusalo for valuable remarks.

1. Definitions and Preliminaries

1.1. Hyperbolic Trigonometry in Terms of L-Lengths

Let $\mathbf{H} = \{z = x + iy : y > 0\}$ be the hyperbolic plane with metric

$$ds^2 = y^{-2}(dx^2 + dy^2).$$

We fix the following notation for hyperbolic triangles in $\mathbf{H}$: A hyperbolic triangle T will have vertices labeled v_a, v_b, v_c; the sides opposite these vertices will have lengths l_a, l_b, l_c (respectively); and the interior angles at the vertices will be α, β, γ. The following sine and cosine rules [2, Sec. 7.12] will be used frequently:

$$\frac{\sinh l_a}{\sin \alpha} = \frac{\sinh l_b}{\sin \beta} = \frac{\sinh l_c}{\sin \gamma}, \tag{S}$$

$$\cosh l_c = \cosh l_a \cosh l_b - \sinh l_a \sinh l_b \cos \gamma, \tag{CI}$$

$$\cosh l_c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}. \tag{CII}$$

We shall use the *L-lengths* (see [7; 8]) defined by the equation $L^2 = \cosh l - 1$ rather than the hyperbolic length l. In terms of L-lengths, (CI) reads

$$\cos \gamma = \frac{L_a^2 L_b^2 + L_a^2 + L_b^2 - L_c^2}{L_a L_b \sqrt{(L_a^2 + 2)(L_b^2 + 2)}}; \tag{1.1}$$

hence,

$$\sin \gamma = \frac{\sqrt{2L_a^2 L_b^2 L_c^2 + F(L_a, L_b, L_c)}}{L_a L_b \sqrt{(L_a^2 + 2)(L_b^2 + 2)}}, \tag{1.2}$$

where

$$F(L_a, L_b, L_c) = (L_a + L_b + L_c)(L_a + L_b - L_c)(L_b + L_c - L_a)(L_c + L_a - L_b). \tag{1.3}$$

We then have, for the angle sum of T,

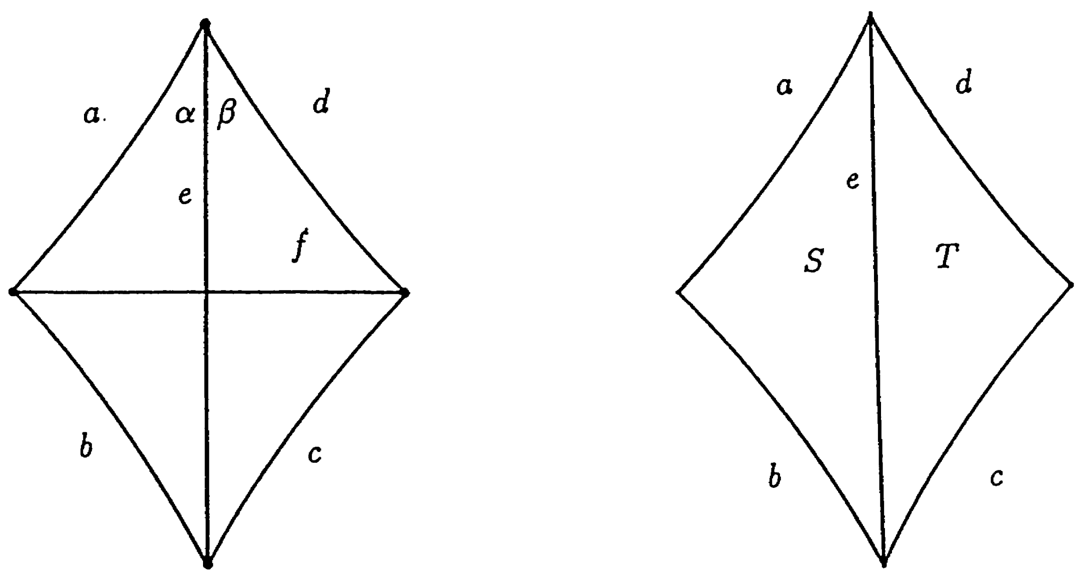
$$\sin(\alpha+\beta+\gamma)=\frac{(L_a^2+L_b^2+L_c^2+4)\sqrt{2L_a^2L_b^2L_c^2+F(L_a,L_b,L_c)}}{(L_a^2+2)(L_b^2+2)(L_c^2+2)} \tag{1.4}$$

and

$$\cos(\alpha+\beta+\gamma)=\frac{L_a^2L_b^2L_c^2-(L_a^4+L_b^4+L_c^4)-4(L_a^2+L_b^2+L_c^2)-8}{(L_a^2+2)(L_b^2+2)(L_c^2+2)}. \tag{1.5}$$

1.2. Hyperbolic Quadrilaterals

The following lemma refers to Figure 1.1(a), and deduces a relation between the L-lengths of the diagonals.



(a) (b)

Figure 1.1

LEMMA 1.2. *For $i \in \{a,b,c,d,e,f\}$, let L_i denote the L-length of the side i. Then*

$$L_e^2L_f^2=\frac{\sqrt{(2L_a^2L_b^2L_e^2+F(L_a,L_b,L_e))(2L_c^2L_d^2L_e^2+F(L_c,L_d,L_e))}}{L_e^2+2} + \frac{(L_a^2+L_b^2-L_e^2)(L_c^2+L_d^2-L_e^2)}{L_e^2+2}+L_a^2L_c^2+L_b^2L_d^2, \tag{1.6}$$

where F is the function in (1.3).

Proof. The equation $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, expressed in terms of L-lengths, together with the formulas (1.1) and (1.2), yield (1.6). □

The diagonal e divides the quadrilateral into two hyperbolic triangles S and T; see Figure 1.1(b). By the Ptolemy equation (see [7; 8]), the circumscribing circles of S and T coincide if and only if

$$L_e L_f = L_a L_c + L_b L_d.$$

Using (1.6), it can be shown that this equality is equivalent to

$$L_a L_b (L_c^2 + L_d^2 - L_e^2) + L_c L_d (L_a^2 + L_b^2 - L_e^2) = 0. \tag{1.7}$$

The triangle T has a vertex outside the circumscribing circle of S if and only if

$$L_a L_b (L_c^2 + L_d^2 - L_e^2) + L_c L_d (L_a^2 + L_b^2 - L_e^2) > 0. \tag{1.8}$$

Since this inequality is symmetric for S and T, (1.8) holds if and only if S has a vertex outside the circumscribing circle of T.

DEFINITION. The equality (1.7) is called the *face equality* on e and (1.8) the *face inequality* on e.

1.3. Triples of Horocycles

Let $h_a \subset \mathbf{H}$ be a horocycle based at v_a. Take a hyperbolic isometry γ so that $h_a = \gamma\{x+i : x \in \mathbf{R}\}$. For $\delta \in \mathbf{R}$, define $h_{a,\delta} = \gamma\{x + ie^{-\delta} : x \in \mathbf{R}\}$. Then $h_{a,\delta}$ is a horocycle based at v_a; it expands to $\partial\mathbf{H}$ as $\delta \to +\infty$ and shrinks to v_a as $\delta \to -\infty$. Let h_b be a horocycle based at v_b distinct from v_a. Denote by $\delta(h_a, h_b)$ the signed hyperbolic distance along the geodesic from v_a to v_b between h_a and h_b, taken with positive sign if $h_a \cap h_b = \emptyset$ and negative sign if $h_a \cap h_b \neq \emptyset$.

REMARK. Let $L = e^{\delta(h_a, h_b)/2}$. Then $\lambda = \sqrt{2}L$ is the λ-length employed by Penner in [9].

Consider triples of horocycles whose base points are mutually distinct. We say that two such triples (h_a, h_b, h_c) and (h_d, h_e, h_f) are *congruent* if there exists an isometry sending h_a, h_b, h_c to h_d, h_e, h_f, respectively, and denote the equivalence class by $[(h_a, h_b, h_c)]$.

LEMMA 1.3. *The set* $\mathbf{R}_+^3 = \{(L_a, L_b, L_c); L_a, L_b, L_c > 0\}$ *parameterizes the classes of triples of horocycles.*

Proof. We show that (L_a, L_b, L_c) determines $[(h_a, h_b, h_c)]$ uniquely. Let δ_i be such that $L_i = e^{\delta_i/2}$, $i = a, b, c$. We may assume that $h_a = \{x + i : x \in \mathbf{R}\}$, that h_b is the horocycle based at 0 going through $ie^{-\delta_c}$, and that h_c is based in $\mathbf{R}_+$. Since h_c is tangent to both h_{a,δ_b} and h_{b,δ_a}, h_c is determined uniquely. □

1.4. The Teichmüller Space of a Branched or Punctured Surface

Let F_g^s be a smooth oriented closed surface of genus g with a subset $P = \{x_1, \dots, x_s\}$ of distinguished points, and let $2g - 2 + s > 0$. We assign to each point x_i of P an integer $\nu_i \geq 2$, or ∞ in case P consists of punctures. The fundamental group G of $F_g^s - P$ has the canonical representation

$$\left(a_1, b_1, \dots, a_g, b_g, e_1, \dots, e_s : \left(\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}\right) e_1 \cdots e_s = 1\right).$$

Let $G(\nu_1, \ldots, \nu_s)$ be the group obtained by adding the relations $e_i^{\nu_i} = 1$, $i = 1, \ldots, s$, to the group G. If, in particular, $\nu = \nu_1 = \cdots = \nu_s$ as in Section 4, then we denote $G_\nu = G(\nu, \ldots, \nu)$.

Let Γ be a Fuchsian group acting on $\mathbf{H}$ such that the quotient surface $\mathbf{H}/\Gamma$ is a closed surface of genus g and the covering map $\pi: \mathbf{H} \to \mathbf{H}/\Gamma$ is branched over s points with branching orders $\nu_1, \ldots, \nu_s$. An orientation-preserving homeomorphism f of F_g^s onto $\mathbf{H}/\Gamma$ that sends each point x_i of P to a branched point of order ν_i induces an isomorphism ϕ of the group $G(\nu_1, \ldots, \nu_s)$ onto Γ. This isomorphism ϕ is called a *marking* of Γ. A *marked group* $\Gamma_m = (\Gamma, \phi)$ defines a marked hyperbolic structure on F_g^s, which is the pull-back by f of the hyperbolic metric (with singularities at the distinguished points) on $\mathbf{H}/\Gamma$. Two marked groups $(\Gamma_1)_m = (\Gamma_1, \phi_1)$ and $(\Gamma_2)_m = (\Gamma_2, \phi_2)$ are said to be *equivalent* if $\phi_2 \circ \phi_1^{-1}$ is a conjugation by a conformal isometry of $\mathbf{H}$. The Teichmüller space $\mathbf{T}(g; \nu_1, \ldots, \nu_s)$ is the space of equivalence classes $[\Gamma_m]$ of marked Fuchsian groups. We choose the group $\Gamma_m = (\Gamma, \phi)$ such that

$$\phi(a_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad 0 < \lambda < 1,$$

and $\phi(b_1)$ has the fixed points p and q with $pq = -1$. Then the entries of $\phi(a_1), \phi(b_1), \ldots, \phi(e_s) \in \mathrm{SL}(2, \mathbf{R})$ (we assume that their traces are nonnegative) give coordinates for the point $[\Gamma_m] \in \mathbf{T}(g; \nu_1, \ldots, \nu_s)$. It is known that $\mathbf{T}(g; \nu_1, \ldots, \nu_s)$ is homeomorphic to $\mathbf{R}^d$, $d = 6g - 6 + 2s$ [10, Chap. 3].

2. Parameterization of the Teichmüller Space of a Branched Surface

2.1

We consider, on $F_g^s - P$, curves c with the following properties:

(i) c is simple and not null-homotopic relative to the boundary in $F_g^s - P$;
(ii) c connects two not necessarily distinct points x_i and x_j of P.

In this paper, homotopy deformations of curves as above are done in $F_g^s - P$. We denote by $[c]$ the homotopy class of c relative to its endpoints. Let $\Delta = (c_1, \ldots, c_p)$ be a system of disjoint curves on $F_g^s - P$ satisfying (i) and (ii) and let $[c_i] \neq [c_j]$ if $i \neq j$.

Definition. Δ is a *cell decomposition* of F_g^s if every component of $F_g^s - \Delta$ is simply connected. Let $d = 6g - 6 + 2s$. A cell decomposition can contain at most $d + s$ curves, and contains this maximal number of curves only when Δ is a *triangulation*. If $\Delta = (c_1, \ldots, c_{d+s})$ is a triangulation, a *triangle* in Δ is a component of $F_g^s - \Delta$. There are $Q = 4g - 4 + 2s$ triangles T in Δ. It is possible that some triangles are as depicted in Figure 2.1.

In the rest of this section and in Section 3, we assume that $\nu_1, \ldots, \nu_s$ are finite for the signature $(g; \nu_1, \ldots, \nu_s)$. With a triangulation $\Delta = (c_1, \ldots, c_{d+s})$ we shall give a global coordinate system of $\mathbf{T}(g; \nu_1, \ldots, \nu_s)$. Take a marked group

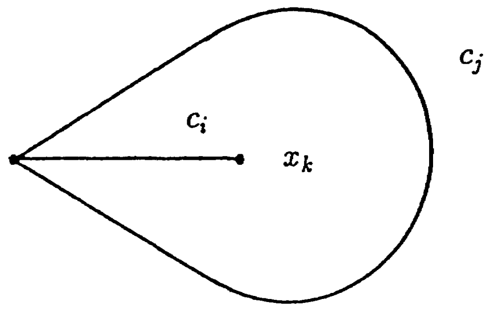


Figure 2.1

Γ_m; it defines a marked hyperbolic structure with singularities on F_g^s. Then $[c_i]$ contains a unique geodesic curve with respect to this hyperbolic structure. We denote by l_i the hyperbolic length of this geodesic curve. (An exception is the case where c_i and c_j are as depicted in Figure 2.1 and the point x_k has branching order 2; for this case, we put $l_j = 2l_i$.) Let L_i denote the L-length, $L_i^2 = \cosh(l_i) - 1$, $i = 1, \dots, d+s$.

PROPOSITION 2.1. *The mapping* $\mathbf{T}(g; \nu_1, \dots, \nu_s) \to \mathbf{R}_+^{d+s}$ *that sends* $[\Gamma_m]$ *to* $(L_1, \dots, L_{d+s})$ *is injective and real analytic.*

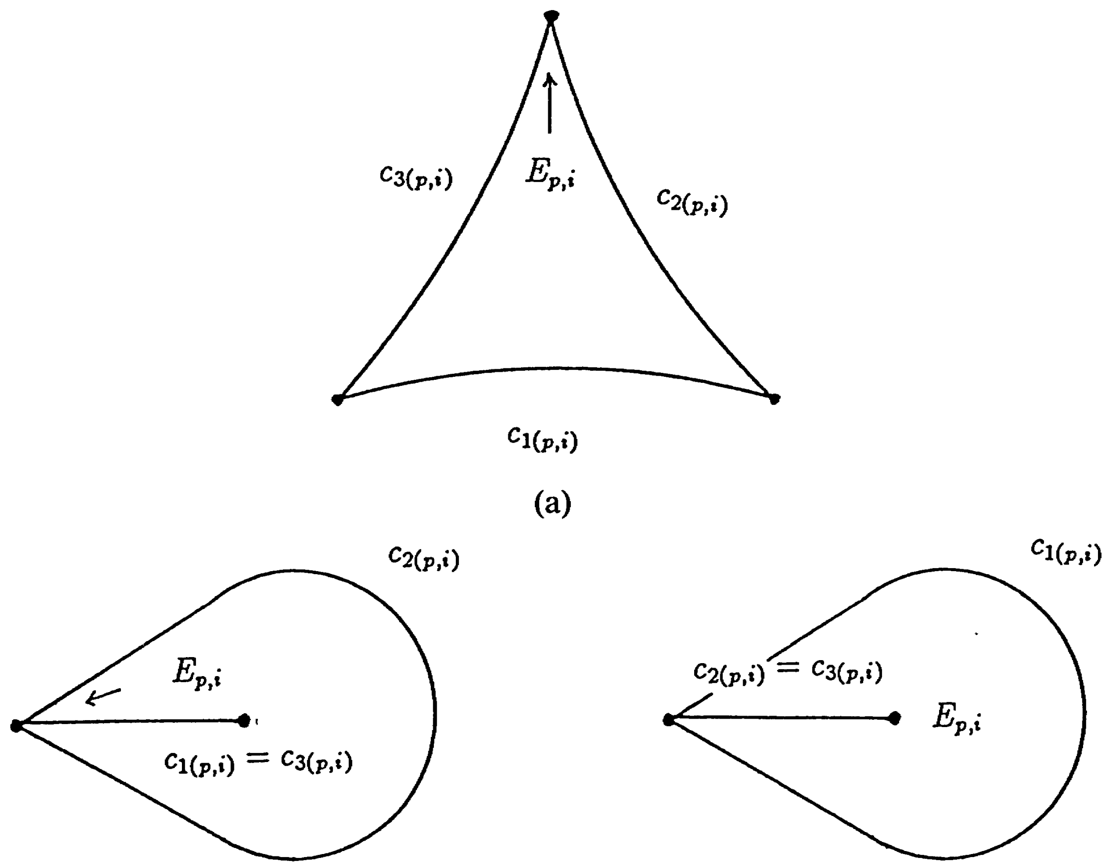
Proof. The mapping is real analytic, since the fixed points of elliptic transformations of Γ are real analytic functions on $\mathbf{T}(g; \nu_1, \dots, \nu_s)$ and the distance between two points depends real analytically on these points.

In order to prove the injectivity, assume that two marked groups $\Gamma_{1,m}$ and $\Gamma_{2,m}$ determine the same point $(L_1, \dots, L_{d+s})$. Let $c_{i,k}$ denote the geodesic curve in $[c_i]$ with respect to $\Gamma_{k,m}$, $k = 1, 2$. If three curves c_i, c_j, c_k are the sides of a triangle T in Δ, then by assumption $(c_{i,1}, c_{j,1}, c_{k,1})$ and $(c_{i,2}, c_{j,2}, c_{k,2})$ bound congruent geodesic triangles T_1 and T_2 (respectively) and there is a conformal isometry $h_T: T_2 \to T_1$. Let $h = h_T$ on T_2 for all triangles T in Δ. Then h extends uniquely to a conformal mapping of $\mathbf{H}/\Gamma_2$ onto $\mathbf{H}/\Gamma_1$. Since h preserves each homotopy class of Δ, it is easy to see that h is homotopic to the identity map. Thus $\Gamma_{1,m} = \Gamma_{2,m}$. ☐

A component of a small neighborhood of a vertex of a triangle defines an *end* of the triangle; see Figure 2.2. Let $E_{1,i}, \dots, E_{p(i),i}$ be the complete list of the ends of the triangles in Δ to which $x_i \in P$ belongs. If a marked group Γ_m is given, choose curves in Δ to be geodesic with respect to Γ_m. Denote by $\theta(p, i)$ the angle of the end $E_{p,i}$. Then we have

$$\sum_{p=1}^{p(i)} \theta(p, i) = \frac{2\pi}{\nu_i}, \quad i = 1, \dots, s. \tag{2.1}$$

Using formulas (1.1) and (1.2), the trigonometric equations induced by (2.1) give s algebraic equations for $L_1, \dots, L_{d+s}$, defining the Teichmüller space as a real algebraic subspace of $\mathbf{R}_+^{d+s}$.



(a)

(b)

Figure 2.2

3. A Model of the Teichmüller Space

3.1. Triangle Inequality

Let l_a, l_b, l_c be any positive numbers. Then the triangle inequality $l_c < l_a + l_b$ is described in terms of the L-lengths L_a, L_b, L_c by

$$L_c^2 < L_a^2 L_b^2 + L_a^2 + L_b^2 + L_a L_b \sqrt{(L_a^2+2)(L_b^2+2)}\,. \tag{3.1}$$

We assume that $L_c \geq \max(L_a, L_b)$. Let $t(L_a, L_b, L_c)$ be the least nonnegative value of t such that the triangle inequality (3.1) holds for tL_a, tL_b, and tL_c; that is,

$$L_c^2 < t^2 L_a^2 L_b^2 + L_a^2 + L_b^2 + L_a L_b \sqrt{(t^2 L_a^2+2)(t^2 L_b^2+2)}\,.$$

This least value is

$$t(L_a, L_b, L_c) = \begin{cases} 0 & \text{if } L_a + L_b > L_c, \\ \tau(L_a, L_b, L_c) & \text{otherwise,} \end{cases}$$

where

$$\tau(L_a, L_b, L_c) = \sqrt{-F(L_a, L_b, L_c)/2L_a^2 L_b^2 L_c^2}$$

with the function F in (1.3).

LEMMA 3.1.1. *Let $0 < L_a, L_b, L_c$. Consider a hyperbolic triangle with L-lengths of sides tL_a, tL_b, tL_c and opposite angles $\alpha(t), \beta(t), \gamma(t)$, where $t > t(L_a, L_b, L_c)$. Then the angle sum $\sum(t) = \alpha(t) + \beta(t) + \gamma(t)$ is a strictly decreasing function of t. $\sum(t)$ converges to π as $t \to t(L_a, L_b, L_c)$ and converges to 0 as $t \to +\infty$.*

Proof. By (1.5),

$$\cos \sum(t) + 1 = \frac{2L_a^2 L_b^2 L_c^2 (1 - t^{-2}\tau(L_a, L_b, L_c)^2)}{(L_a^2 + t^{-2}2)(L_b^2 + t^{-2}2)(L_c^2 + t^{-2}2)};$$

hence $\cos \sum(t)$ is monotone increasing and tends to -1 as $t \to t(L_a, L_b, L_c)$ and to 1 as $t \to \infty$. ❐

The following lemma is a variant of Lemma 3.1.1.

LEMMA 3.1.2. *Fix $\nu \in \mathbf{N}$, $\nu > 2$. Let $L > 0$ and consider a hyperbolic right-angled triangle with an angle π/ν. Suppose that the side opposite the angle π/ν has length $l(t)$ such that $t^2 L^2 = \cosh(2l(t)) - 1$. Then the third angle $\delta(t)$ of the triangle is strictly decreasing for $t > 0$, and converges to $\pi(2^{-1} - \nu^{-1})$ as $t \to 0$ and to 0 as $t \to +\infty$.*

Proof. The lemma is an immediate consequence of the equation $\cot \delta(t) = \tan(\pi/\nu)\sqrt{1 + t^2 L^2/2}$, which is obtained from (S) and (CI). ❐

3.2. *Special Triangulation*

We proceed with a special triangulation $\Delta_0 = (c_1, \ldots, c_{d+s})$ of F_g^s, where c_i separates c_{d+1+i} from all other curves and c_{d+1+i} connects x_1 and x_{i+1}, $i = 1, \ldots, s-1$. Then in Δ_0 there are $s-1$ triangles $T_1, \ldots, T_{s-1}$, as in Figure 2.1. By Section 2, the Teichmüller space $\mathbf{T}(g; \nu_1, \ldots, \nu_s)$ can be represented as a subspace of $\mathbf{R}_+^{d+s}$ determined by Δ_0. Next we shall characterize this subspace. Choose a marked group $\Gamma_m = (\Gamma, \phi)$. Let $(L_1, \ldots, L_{d+s})$ be the L-length coordinates of Γ_m. The geodesic triangle corresponding to T_i, $i = 1, \ldots, s-1$, is isosceles and can be divided into two congruent right-angled triangles. One of the angles of the right-angled triangle is π/ν_{i+1}. Let $\delta_i < \pi/2$ denote the other angle. Using (S), we obtain

$$L_i = \sin\left(\frac{\pi}{\nu_{i+1}}\right) L_{d+1+i}\sqrt{2(L_{d+1+i}^2 + 2)}, \quad i = 1, \ldots, s-1. \tag{3.2}$$

If $\alpha_q, \beta_q, \gamma_q$ denote the angles of T_q, $q = s, \ldots, Q$, then by (2.1) we have

$$\sum_{q=s}^{Q} (\alpha_q + \beta_q + \gamma_q) + \sum_{i=1}^{s-1} 2\delta_i = \frac{2\pi}{\nu_1}. \tag{3.3}$$

By (3.2) we can omit $L_{d+2}, \ldots, L_{d+s}$ from our list of parameters. For $L_1, \ldots, L_{d+1}$, a single relation—namely, the one derived from (3.3)—remains.

PROPOSITION 3.2. *The image of* $\mathbf{T}(g; \nu_1, \ldots, \nu_s)$ *under the mapping in Proposition 2.1 meets each ray from the origin exactly once in*

$$\mathbf{R}_+^{d+1} = \{(L_1, \ldots, L_{d+1}) : L_i > 0\}.$$

Proof. Let $(L_1, \ldots, L_{d+1}) \in \mathbf{R}_+^{d+1}$. Choose t_0 to be the minimum of all t for which the triangle inequalities (3.1) hold among all triples

$$(tL_{1(q)}, tL_{2(q)}, tL_{3(q)}), \quad q = s, \ldots, Q,$$

where the subscripts arise from the coordinates corresponding to the sides of T_q.

Let $\alpha_q(t), \beta_q(t), \gamma_q(t)$ denote the angles of the geodesic triangle with L-lengths of sides $tL_{1(q)}, tL_{2(q)}, tL_{3(q)}$. Next consider the triangle with angles $\pi/2$, π/ν_{i+1}, and $\delta_i(t)$, and assume that the side opposite π/ν_{i+1} has hyperbolic length l defined by $t^2L_i^2 = \cosh(2l) - 1$. Then we need to show that there exists a unique $t > t_0$ for which the angle sum satisfies

$$\sum_{q=s}^{Q} (\alpha_q(t) + \beta_q(t) + \gamma_q(t)) + \sum_{i=1}^{s-1} 2\delta_i(t) = \frac{2\pi}{\nu_1}. \tag{3.4}$$

By Lemmas 3.1.1 and 3.1.2, the left-hand side in (3.4) is strictly decreasing for $t > t_0$; its value is greater than π at $t = t_0$ and tends to 0 as $t \to +\infty$. □

4. Action of the Mapping Class Group on the Teichmüller Space of a Branched Surface

4.1

In this section we consider transformations between Teichmüller spaces related to different triangulations and the mapping class group.

4.1.1. A triangulation Δ of F_g^s defines a representation of the Teichmüller space $\mathbf{T}(g; \nu, \ldots, \nu)$, $\nu \geq 2$, as a real algebraic subspace $\mathbf{T}_\nu(\Delta) \subset \mathbf{R}_+^{d+s}$. When another triangulation Δ' is given, let $\tilde{R}_{\Delta,\Delta'} : \mathbf{T}_\nu(\Delta) \to \mathbf{T}_\nu(\Delta')$ denote the mapping that gives the correspondence between points representing the same class of marked groups.

We first consider a special case. Suppose that $a, b, c, d, e \in \Delta$ are such that (a, b, e) and (c, d, e) define triangles in Δ, as in Figure 4.1. The operation on Δ indicated in Figure 4.1 defines a new triangulation Δ', which is said to arise from Δ by an *elementary move*. Let L_i be the L-length of the geodesic in $[i]$, $i \in \{a, b, c, d, e, f\}$, with respect to a marked group Γ_m. In this case the explicit expression of $\tilde{R}_{\Delta,\Delta'}$ can be obtained by using (1.6).

A sequence of triangulations $(\Delta_j)_{j=1}^{t}$ is called a *chain* if Δ_{j+1} arises from Δ_j by an elementary move. For a general triangulation Δ', there exists a chain $(\Delta_j)_{j=1}^{t}$ such that $\Delta_1 = \Delta$ and $\Delta' = \Delta_t$; see [9, Prop. 7.1]. It is therefore possible to obtain an explicit (but in general very complicated) expression of the transformation $\tilde{R}_{\Delta,\Delta'}$ by using (1.6) successively:

$$\tilde{R}_{\Delta,\Delta'} = \tilde{R}_{\Delta_{t-1},\Delta'} \circ \cdots \circ \tilde{R}_{\Delta,\Delta_2}.$$

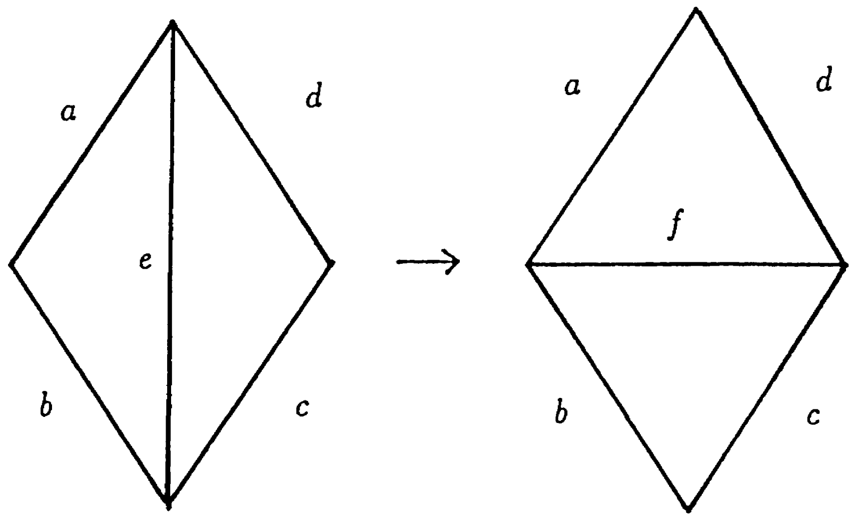


Figure 4.1 Elementary move

4.1.2. The *mapping class group* MC_g^s is the group of homotopy classes of mappings in $\mathrm{Homeo}_+(F_g^s)$ relative to P, where $\mathrm{Homeo}_+(F_g^s)$ is the group of orientation-preserving homeomorphisms of F_g^s that preserve the set $P=\{x_1,\dots,x_s\}$ of distinguished points. A mapping $h\in\mathrm{Homeo}_+(F_g^s)$ induces an isomorphism $h_\sharp\colon G_\nu\to G_\nu$. For $\Gamma_m=(\Gamma,\phi)$, where $[\Gamma_m]\in\mathbf{T}(g;\nu,\dots,\nu)$, we let $h_*\Gamma_m=(\Gamma,\phi h_\sharp^{-1})$. Since $h_\sharp$ is independent of the choice of h from its homotopy class, the change of marking $[\Gamma_m]\to[h_*\Gamma_m]$ induces an action of MC_g^s on $\mathbf{T}(g;\nu,\dots,\nu)$.

Fix a triangulation Δ and identify $\mathbf{T}(g;\nu,\dots,\nu)$ with its model $\mathbf{T}_\nu(\Delta)$. We next study the action of $h_*\in MC_g^s$ on $\mathbf{T}_\nu(\Delta)$. Take a marked group Γ_m. For $c\in\Delta$, let $l(c,\Gamma_m)$ and $L(c,\Gamma_m)$ denote the hyperbolic length and the L-length of the geodesic in $[c]$ with respect to Γ_m. Then there is a natural relation $l(h(c),h_*\Gamma_m)=l(c,\Gamma_m)$, and hence

$$L(h(c),h_*\Gamma_m)=L(c,\Gamma_m). \tag{4.1}$$

The action of $h_*\in MC_g^s$ on $\mathbf{T}(g;\nu,\dots,\nu)$ is expressed by using the coordinates of $\mathbf{T}_\nu(\Delta)$ as

$$h_*(L(c_i,\Gamma_m))_{i=1}^{d+s}=(L(c_i,h_*\Gamma_m))_{i=1}^{d+s}. \tag{4.2}$$

By (4.1), $L(c_i,h_*\Gamma_m)=L(h^{-1}(c_i),\Gamma_m)$. Hence the action of h_* coincides with the mapping $\tilde R_{\Delta,h^{-1}\Delta}$ followed by an auxiliary permutation p of coordinates. Since $\mathbf{T}_\nu(\Delta)$ and $\mathbf{T}_\nu(h^{-1}(\Delta))$ are identical subspaces of $\mathbf{R}_+^{d+s}$, we can think of $p\circ\tilde R_{\Delta,h^{-1}\Delta}$ as a self-mapping of $\mathbf{T}_\nu(\Delta)$. Then we have a representation of the mapping class group MC_g^s in the group of self-mappings of $\mathbf{T}_\nu(\Delta)$ that sends h_* to $p\circ\tilde R_{\Delta,h^{-1}\Delta}$.

4.2. Decomposition of the Teichmüller Space $\mathbf{T}(g;\nu,\dots,\nu)$ *into Subsets*

4.2.1. Let Δ be a cell decomposition. Choose a triangulation $\Delta'\supseteq\Delta$. Whenever a marked group Γ_m of $\mathbf{T}(g;\nu,\dots,\nu)$ is given, take the curves c in Δ' to be geodesic with respect to the hyperbolic structure defined by Γ_m. Fix a curve $e \in \Delta'$. Let S and T be the triangles in Δ' which abut on e from different sides. We identify $S \cup T$ with the hyperbolic quadrilateral as depicted in Figure 1.1(b). (This is visualized in the universal cover $\mathbf{H}$ if e is as in Figure 2.1.) We say that Γ_m satisfies the *face inequality* (resp. the *face equality*) on e if the face inequality (resp. the face equality) holds on e; see Section 1.2.

DEFINITION. Γ_m satisfies *face relations* on $\Delta \operatorname{rel} \Delta'$ if face inequalities hold on each $e \in \Delta$ and face equalities hold on each $e \in \Delta' - \Delta$.

Define a subset of $\mathbf{T}(g; \nu, \ldots, \nu)$ by

$$C_\nu(\Delta) = \{[\Gamma_m] \colon \Gamma_m \text{ satisfies face relations for } \Delta \operatorname{rel} \Delta'\}.$$

It will be shown in Proposition 4.2.2 that $C_\nu(\Delta)$ does not depend on the particular triangulation Δ' containing Δ.

4.2.2. Let Γ be a Fuchsian group with signature $(g; \nu, \ldots, \nu)$. Let $\mathcal{E}(\Gamma)$ denote the set of elliptic fixed points of Γ in $\mathbf{H}$. For each $z_0 \in \mathcal{E}(\Gamma)$, we define

$$P(z_0) = \{z \in \mathbf{H}; \rho(z, z_0) \leq \rho(z, z') \text{ for all } z' \in \mathcal{E} \setminus \{z_0\}\},$$

where $\rho(\cdot, \cdot)$ is the hyperbolic distance. Since $\mathcal{E}(\Gamma)$ is discrete in $\mathbf{H}$, $P(z_0)$ is a locally finite hyperbolic polygon. By Selberg's lemma [10, Thm. 15.14], Γ has a subgroup $\tilde{\Gamma}$ of finite index that contains no elliptic transformations. The closure of the Dirichlet polygon $D(z_0, \tilde{\Gamma})$ centered at z_0 is included in $\mathbf{H}$, because $\tilde{\Gamma}$ is purely hyperbolic and the hyperbolic area of $D(z_0, \tilde{\Gamma})$ is finite. Since $\tilde{\Gamma} z_0 \subset \Gamma z_0 \subset \mathcal{E}(\Gamma)$, we have $P(z_0) \subset D(z_0, \tilde{\Gamma})$. Therefore $P(z_0)$ has finitely many sides and all its vertices lie in $\mathbf{H}$. The collection $P(z_0)$, $z_0 \in \mathcal{E}(\Gamma)$, defines a Γ-invariant tessellation of $\mathbf{H}$. Next we construct a dual tessellation. Let $\mathcal{V}(\Gamma)$ denote the set of vertices of all $P(z_0)$, $z_0 \in \mathcal{E}(\Gamma)$. For each $v \in \mathcal{V}(\Gamma)$, enumerate polygons around v as $P(z_0), \ldots, P(z_n)$. We form the hyperbolic convex hull $D(v)$ of $z_0, \ldots, z_n$. Because $z_0, \ldots, z_n$ are equidistant from v, $D(v)$ inscribes in a hyperbolic circle. The collection $D(v)$, $v \in \mathcal{V}(\Gamma)$, defines a dual tessellation, denoted by $\operatorname{Tess}(\Gamma)$. Under the canonical projection $\pi \colon \mathbf{H} \to \mathbf{H}/\Gamma$, the edges of $\operatorname{Tess}(\Gamma)$ are sent to geodesic curves in $\mathbf{H}/\Gamma$ connecting branched points. Denote by $\tilde{\Delta}(\Gamma)$ the system of these geodesic curves.

For a cell decomposition Δ, let $\Delta(\Gamma_m)$ be the system of geodesic curves in Δ with respect to Γ_m. Then we have the following proposition.

PROPOSITION 4.2.2. *For a cell decomposition* Δ *of* F_g^s,

$$C_\nu(\Delta) = \{[\Gamma_m]; \Delta(\Gamma_m) = \tilde{\Delta}(\Gamma)\}.$$

Proof. Assume $\Delta(\Gamma_m) = \tilde{\Delta}(\Gamma)$ for a marked group Γ_m. Let Δ' be any triangulation containing Δ. We may assume that curves in Δ' are geodesic with respect to Γ_m. Then Δ' induces a triangulation of $\mathbf{H}$, which is a subdivision of $\operatorname{Tess}(\Gamma)$.

Let $e \in \Delta'$. We continue using the notation of Figure 1.1(b). We may identify e with an edge of Tess(Γ). If $e \in \Delta$, then S and T are contained in distinct polygons $D(v)$ and $D(v')$, respectively. It follows that T has a vertex outside the circumscribing circle of $D(v)$, hence also outside of S, and the face inequality holds on e. If $e \in \Delta' - \Delta$, then S and T are contained in the same polygon $D(v)$ and their circumscribing circles are identical. Hence the face equality holds on e. The other direction can be proved by using a similar argument. □

4.2.3. Proposition 4.2.2 implies that Γ_m defines a unique cell decomposition Δ for which $[\Gamma_m] \in C_\nu(\Delta)$, which proves the first claim of the following theorem.

THEOREM 4.2.3. *$\mathbf{T}(g; \nu, \ldots, \nu)$ is a disjoint union of the subsets $C_\nu(\Delta)$,*

$$\mathbf{T}(g; \nu, \ldots, \nu) = \bigcup C_\nu(\Delta),$$

where Δ runs over all cell decompositions. Possibly some $C_\nu(\Delta)$ are empty. This subset decomposition is MC_g^s-invariant.

Proof. The last claim is a consequence of (4.1), since $h_* C_\nu(\Delta) = C_\nu(h(\Delta))$ for $h_* \in MC_g^s$. □

5. Convergence of Teichmüller Spaces of Branched Surfaces to That of a Punctured Surface

5.1. Sequence of Teichmüller Spaces

We consider the Teichmüller spaces $\mathbf{T}(g; \nu, \ldots, \nu)$, $\nu \geq 2$, with ν repeated s times. We fix a triangulation Δ of F_g^s and denote by $\mathbf{T}_\nu(\Delta) \subset \mathbf{R}_+^{d+s}$ the model of the Teichmüller space obtained in Proposition 2.1. Define a positive number $\rho(\nu)$ by $\cosh \rho(\nu) = (2 \sin \pi/\nu)^{-1}$. Then, by Matelski's result [5, Lemma 4.2], for $\nu \geq 7$ all coordinates of a point $(L_1, \ldots, L_{d+s})$ of $\mathbf{T}_\nu(\Delta)$ satisfy $L_i^2 \geq \cosh 2\rho(\nu) - 1 = (2 \sin^2 \pi/\nu)^{-1} - 2$. Thus $L_i \to \infty$ as $\nu \to \infty$. Fix a constant $\alpha > 0$ and consider the homothety

$$h_{\nu,\alpha}(L_1, \ldots, L_{d+s}) = (\xi_\nu L_1, \ldots, \xi_\nu L_{d+s}), \quad \xi_\nu = \frac{\sqrt{2} \sin \pi/\nu}{\alpha}. \tag{5.1}$$

Let $(L_1, \ldots, L_{d+s}) \in \mathbf{T}_\nu(\Delta)$. By (2.1), we have

$$\sin\left(\sum_{p=1}^{p(i)} \theta(p, i)\right) = \sin\left(\frac{2\pi}{\nu}\right), \quad i = 1, \ldots, s. \tag{5.2}$$

We develop the left-hand side of (5.2) in $\sin\theta(p,i)$ and $\cos\theta(p,i)$, and rewrite it in terms of L-lengths by using (1.1) and (1.2). Then we substitute $\xi_\nu^{-1} L_i$ for L_i to obtain the equations defining $h_{\nu,\alpha}\mathbf{T}_\nu(\Delta)$. When ν tends to ∞, all angles appearing in (5.2) tend to 0. We divide both sides by $\sin(2\pi/\nu)$ and let $\nu \to \infty$. Since the terms having at least two sine factors vanish and cosine factors tend to 1, we obtain a subspace of $\mathbf{R}_+^{d+s}$, which we denote by $\mathbf{T}_{\infty,\alpha}(\Delta)$, defined by

$$\sum_{p=1}^{p(i)} \frac{L_{1(p,i)}}{(L_{2(p,i)}L_{3(p,i)})} = \alpha, \quad i = 1, \ldots, s, \tag{5.3}$$

where $1(p, i), 2(p, i), 3(p, i) \in \{1, \ldots, d+s\}$ are determined as depicted in Figure 2.2.

5.2. Algebraic Convergence

5.2.1. We consider the space $\mathbf{T}_{\infty,\alpha}(\Delta) \subset \mathbf{R}_+^{d+s}$ obtained in Section 5.1. Let $(L_1, \ldots, L_{d+s}) \in \mathbf{T}_{\infty,\alpha}(\Delta)$. Take a triangle T in Δ with sides c_i, c_j, and c_k. By Lemma 1.3, (L_i, L_j, L_k) determine a triple of horocycles $(h_{i,T}, h_{j,T}, h_{k,T})$ and then the base points of the horocycles determine a hyperbolic triangle $\tilde{T}$. Taking care that the horocycles fit nicely, replace triangles T in Δ by $\tilde{T}$. This forms $F_g^s - P$ with a marked hyperbolic structure on it. Let Γ_m denote the corresponding marked group. The above horocycles are divided into s classes corresponding to the punctures. The equations in (5.3) mean that the length of each horocycle on F_g^s equals α (see [9, Prop. 2.8] and also (5.6) below). By this observation we have the following result.

THEOREM 5.2.1. *The subspace $\mathbf{T}_{\infty,\alpha}(\Delta)$ represents the Teichmüller space $\mathbf{T}(g; \infty, \ldots, \infty)$ of the closed surface of genus g with s punctures.*

In the preceding argument the coordinates $(L_1, \ldots, L_{d+s})$ of $\mathbf{T}_{\infty,\alpha}(\Delta)$ differ by a factor of $\sqrt{2}$ from Penner's λ-length coordinates. Hence $\mathbf{T}_{\infty,\alpha}(\Delta)$ can be viewed as a natural embedding of the Teichmüller space $\mathbf{T}(g; \infty, \ldots, \infty)$ into the decorated Teichmüller space.

PROPOSITION 5.2.2. *Let $(L_1, \ldots, L_{d+s}) \in \mathbf{T}_{\infty,\alpha}(\Delta)$. Choose*

$$(L_1(\nu), \ldots, L_{d+s}(\nu)) \in \mathbf{T}_\nu(\Delta)$$

so that

$$h_{\nu,\alpha}(L_1(\nu), \ldots, L_{d+s}(\nu)) \to (L_1, \ldots, L_{d+s}), \quad \nu \to \infty. \tag{5.4}$$

Let $[\Gamma(\nu)_m]$ be the point of $\mathbf{T}(g; \nu, \ldots, \nu)$ defined by $(L_1(\nu), \ldots, L_{d+s}(\nu))$. Then suitably normalized marked groups $\Gamma(\nu)_m$ converge to a marked group $\Gamma_m = (\Gamma, \phi)$ determined by $(L_1, \ldots, L_{d+s})$ algebraically in the sense of Jørgensen [3].

Proof. Let $r = 2g+s-1$. Relabel the curves in Δ so that F_g^s reduces to a plane $(2r)$-gon D when it is cut along $c_1, \ldots, c_r$ (for a detailed description of this, see [4, para. 18]). Topologically we may pass from the polygon D to the surface F_g^s by identifying the sides of D in pairs.

Now we consider $\Gamma_m(\nu) = (\Gamma_\nu, \phi_\nu)$ and replace curves c_i in Δ by the geodesic curves $c_{i,\nu}$ in $[c_i]$ with respect to $\Gamma_m(\nu)$. As in the case just illustrated, construct a geodesic $(2r)$-gon D_ν by cutting $\mathbf{H}/\Gamma_\nu$ along $c_{1,\nu}, \ldots, c_{r,\nu}$. We choose from Δ a triangle T_1 that has c_1 as a side, and normalize Γ_ν by conjugation so that:

(A) $c_{1,\nu}$ is an interval on the imaginary axis and the imaginary unit i is its middle point; and
(B) T_1 lies in the right-hand side of the imaginary axis.

Draw hyperbolic circles $C_{1,\nu}, \ldots, C_{2r,\nu}$ of radius $\rho(\nu)+\rho$ centered at vertices of D_ν, where $\alpha = e^\rho$. For each curve $c_{k,\nu}$, $k = 1, \ldots, d+s$, find two circles $C_{i,\nu}$ and $C_{j,\nu}$ centered at its endpoints. For ν large enough, $\rho < \rho(\nu)$. Then the signed hyperbolic distance $\delta_k(\nu)$ along $c_{k,\nu}$ between $C_{i,\nu}$ and $C_{j,\nu}$ is defined by $\delta_k(\nu) = l_k(\nu) - 2(\rho(\nu)+\rho)$ with $L_k(\nu)^2 = \cosh l_k(\nu) - 1$, $i = 1, \ldots, d+s$. A simple calculation shows that

$$e^{\delta_k(\nu)} \to L_k^2 \quad \text{as } \nu \to \infty. \tag{5.5}$$

For each end $E_{p,i}$, $p = 1, \ldots, p(i)$, take the circle $C_{j,\nu}$ which has its center at the vertex in $E_{p,i}$. If the angle at $E_{p,i}$ is $\theta = \theta(p,i)$, then

$$\beta_{p,i}(\nu) = \theta \sinh(\rho(\nu)+\rho)$$

is the hyperbolic distance along $C_{j,\nu}$ between the sides of $E_{p,i}$. Because $\theta < \pi/\nu$, $\sin\theta/\theta \to 1$ as $\nu \to \infty$. Hence by using (1.2) we have

$$\beta_{p,i}(\nu) \to L_{1(p,i)}/(L_{2(p,i)}L_{3(p,i)}) \quad \text{as } \nu \to \infty. \tag{5.6}$$

By (5.5) and (5.6), $\delta_1(\nu), \ldots, \delta_{d+s}(\nu), \beta_{1,1}(\nu), \ldots, \beta_{p(s),s}(\nu)$ are bounded uniformly. By Condition (A), i is the middle point of $c_{1,\nu}$. Then, although all vertices of D_ν move far from i, the distances from i to the circles $C_{1,\nu}, \ldots, C_{2r,\nu}$ are bounded by $\sum_{i=1}^{d+s}|\delta_i(\nu)| + \sum_{i=1}^{s}\sum_{p=1}^{p(i)}\beta_{p,i}(\nu)$. Therefore, by choosing a subsequence if necessary, $C_{i,\nu}$ converges as $\nu \to \infty$ to a horocycle h_i, $i = 1, \ldots, 2r$, and these horocycles have distinct base points. Thus the limit D_∞ of D_ν is still a geodesic $(2r)$-gon and by (5.5) the side pairing that fits the horocycles of D_∞ determines Γ_m, the same marked group as constructed in 5.2.1 from $(L_1, \ldots, L_{d+s})$. Let g_i be the homotopy class on $F_g^s - P$ represented by a loop that cuts $c_1 \cup \cdots \cup c_r$ once and only once in c_i. Then $\gamma_{i,\nu} = \phi'_\nu(g_i)$, $i = 1, \ldots, r$, are the side-pairing transformations of D_ν, where ϕ'_ν is the composite of the projection $G \to G_\nu$ and the marking $\phi_\nu\colon G_\nu \to \Gamma_\nu$. As $\nu \to \infty$, $\gamma_{i,\nu}$ converges to the side-pairing transformation $\gamma_i \in \Gamma$ of D_∞; see Figure 5.1. Hence we have a convergence in $\mathrm{PSL}(2,\mathbf{R})^r$ on the generator system

$$(\gamma_{1,\nu}, \ldots, \gamma_{r,\nu}) \to (\gamma_1, \ldots, \gamma_r),$$

which is the desired algebraic convergence $\Gamma_m(\nu) \to \Gamma_m$. □

5.3

Let $\mathbf{T}_{\infty,\alpha}(\Delta)$ and $\mathbf{T}_{\infty,\alpha}(\Delta')$ be associated with different triangulations Δ and Δ', respectively. Then the natural transformation $R_{\Delta,\Delta'}$ from $\mathbf{T}_{\infty,\alpha}(\Delta)$ to $\mathbf{T}_{\infty,\alpha}(\Delta')$ is the limit of $h_{\alpha,\nu}\tilde{R}_{\Delta,\Delta'}h_{\alpha,\nu}^{-1}$ as $\nu \to \infty$, where $\tilde{R}_{\Delta,\Delta'}$ is the transformation in 4.1.1. To study $\tilde{R}_{\Delta,\Delta'}$, we first assume that Δ' is obtained from Δ by an elementary move on $e \in \Delta$ and that a, b, c, d, e, f are as depicted in Figure 4.1. By substituting $\xi_\nu^{-1}L_i$ for L_i in (1.6) for $i = a, \ldots, f$ and letting $\nu \to \infty$, we obtain the Ptolemy equation of [7; 8]:

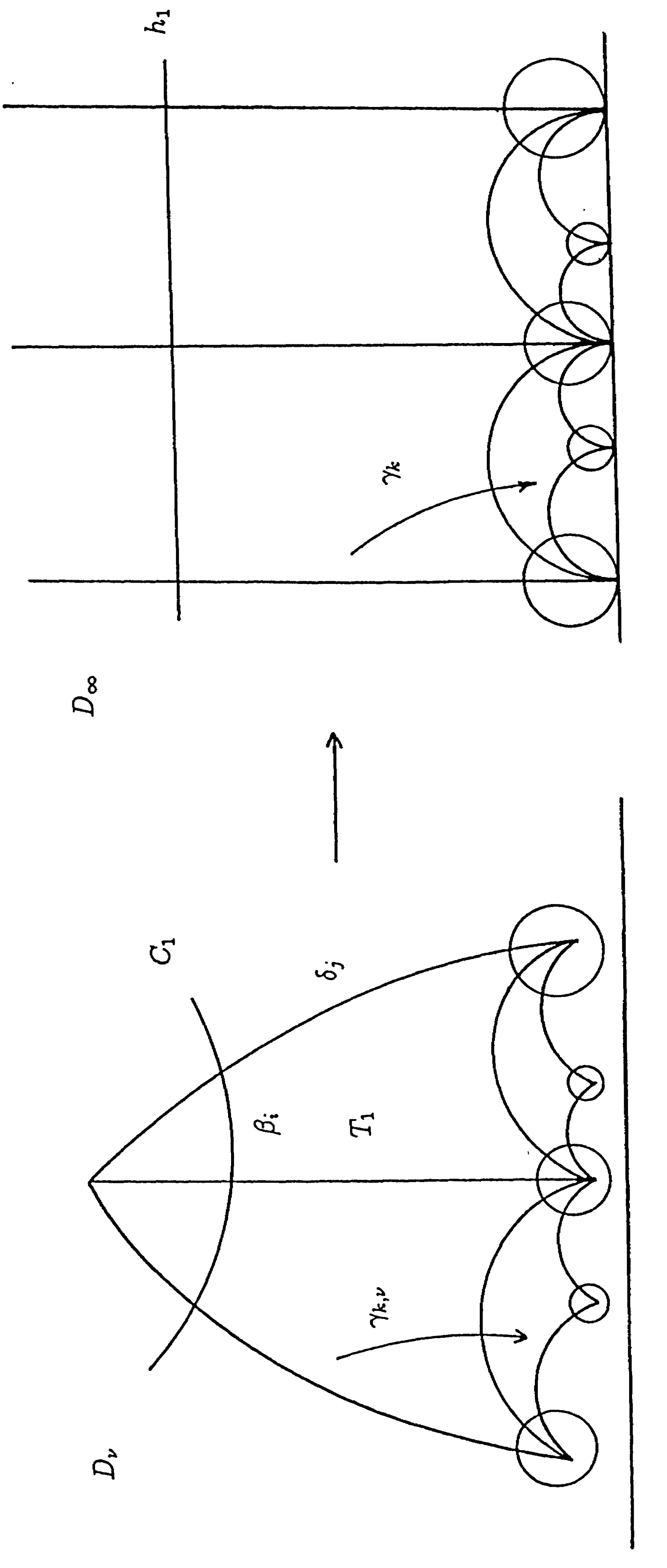


Figure 5.1

$$L_e L_f = L_a L_c + L_b L_d. \tag{5.8}$$

Hence, in this case $R_{\Delta, \Delta'}$ is expressed by (5.8). For a general triangulation Δ', choose a chain $(\Delta_j)_{j=1}^{t}$ from Δ to Δ'. Then

$$R_{\Delta, \Delta'} = R_{\Delta_{t-1}, \Delta'} \circ \cdots \circ R_{\Delta, \Delta_2},$$

and $R_{\Delta, \Delta'}$ is a homogeneous integral rational map of degree 1, as pointed out by Penner (see [9, Sec. 7]). The mapping class group MC_g^s acts on $\mathbf{T}_{\infty, \alpha}(\Delta)$ in the same manner as described in Section 4.1.

Because the face conditions (1.7) and (1.8) are homogeneous, they are preserved under homothety. Thus, if $C_\nu(\Delta)$ is the set defined in 4.2.1 for a cell decomposition Δ, then the limit $C_{\infty, \alpha}(\Delta)$ of $C_\nu(\Delta)$ in the sense of Proposition 5.2.2 consists of all $[\Gamma_m] \in \mathbf{T}(g; \infty, \dots, \infty)$ that satisfy the face relations for Δ. By Theorem 5.5 of [9], we have our next result.

THEOREM 5.3. *$\mathbf{T}(g; \infty, \dots, \infty)$ is a disjoint union of the subsets $C_{\infty, \alpha}(\Delta)$,*

$$\mathbf{T}(g; \infty, \dots, \infty) = \bigcup C_{\infty, \alpha}(\Delta), \tag{5.9}$$

where Δ runs over all cell decompositions. The subset decomposition (5.9) *is MC_g^s-invariant.*

In the statement of the theorem, possibly some sets $C_{\infty, \alpha}(\Delta)$ are empty.

6. Teichmüller Space of a Twice-Punctured Torus

We identify the topological surface F_1^2 with $\mathbf{C}/M$, where $M = \{a + ib : a, b \in \mathbf{Z}\}$ is a lattice. The distinguished points are identified with the congruence classes of the origin and of $(1+i)/2$.

Let Δ_1 be the triangulation of F_1^2 whose lift to $\mathbf{C}$ is depicted in Figure 6.1. To simplify the notation, we write j for L_j (the L-coordinate at $j \in \{a, b, c, d, e, f\}$). Then two equations of type (5.3), with α fixed, determine the Teichmüller space $\mathbf{T}(1; \infty, \infty)$. These equations yield $e = f$. Hence $\mathbf{T}(1; \infty, \infty)$ is represented as a subspace of the (a, b, c, d, e)-space $\mathbf{R}_+^5$ by the equation

$$\frac{d}{ae} + \frac{c}{be} + \frac{e}{bc} + \frac{b}{ce} + \frac{a}{de} + \frac{e}{ad} = \alpha. \tag{6.1}$$

Let $\omega_1, \omega_2, \omega_3$ be the orientation-preserving homeomorphisms whose lifts $\tilde{\omega}_i$, $i = 1, 2, 3$, with respect to the covering $\mathbf{C} \to \mathbf{C}/M$ are:

$$\tilde{\omega}_1(x+iy) = \begin{cases} x + i(y + 2x - [x]) & \text{if } 0 \le x - [x] \le 1/2, \\ x + i(y + [x] + 1) & \text{if } 1/2 \le x - [x] \le 1, \end{cases}$$

$$\tilde{\omega}_2(x+iy) = -y + ix,$$

$$\tilde{\omega}_3(x+iy) = \begin{cases} x + i(y + [x]) & \text{if } 0 \le x - [x] \le 1/2, \\ x + i(y + 2x - [x] - 1) & \text{if } 1/2 \le x - [x] \le 1, \end{cases}$$

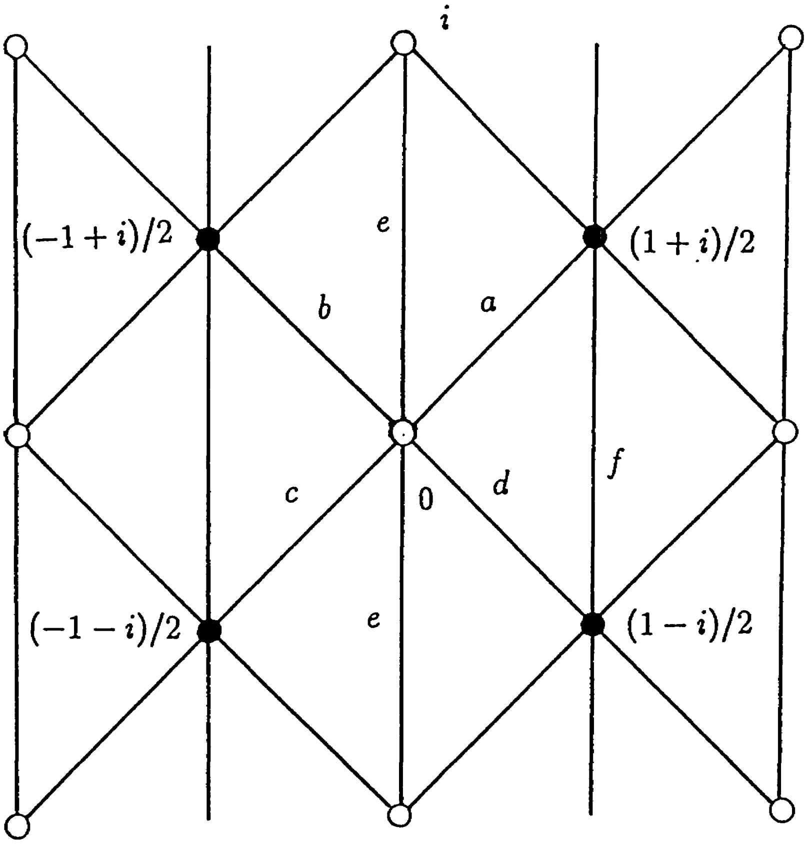


Figure 6.1 Δ_1

where $[x]$ denotes the greatest integer not exceeding x; see Figure 6.2. Each ω_i, $i = 1, 2, 3$, induces a transformation ω_{i*} on the space $\mathbf{T}(1; \infty, \infty)$ represented by (6.1). The images of a point (a, b, c, d, e) under respective transformations are:

$$\omega_{1*}: (d, b, c, (e^2 + d^2)/a, e); \qquad \omega_{2*}: (d, a, b, c, (ac + bd)/e);$$
$$\omega_{3*}: (a, (b^2 + e^2)/c, b, d, e).$$

Let G denote the group generated by ω_{1*}, ω_{2*}, and ω_{3*}. The transformation ω_{2*} has order 4,

$$\omega_{2*}^4 = \text{the identity transformation,} \tag{6.2}$$

and fixes $(4\sqrt{2}/\alpha, 4\sqrt{2}/\alpha, 4\sqrt{2}/\alpha, 4\sqrt{2}/\alpha, 8/\alpha)$. Then $\omega_{2*}^2: (c, d, a, b, e)$ has order 2, preserves $C_{\infty,\alpha}(\Delta_1)$, and fixes every point in

$$\{(a, b, a, b, e); 2(a^2 + b^2 + e^2) = \alpha abe\}.$$

We also have

$$\omega_{2*}^2 \omega_{1*} \omega_{2*}^2 = \omega_{3*}, \qquad \omega_{1*}\omega_{3*} = \omega_{3*}\omega_{1*}; \tag{6.3}$$

$$\omega_{1*}\omega_{2*}\omega_{1*} = \omega_{2*}^{-1}\omega_{1*}^{-1}\omega_{2*}^{-1}, \qquad \omega_{3*}\omega_{2*}\omega_{3*} = \omega_{2*}^{-1}\omega_{3*}^{-1}\omega_{2*}^{-1}. \tag{6.4}$$

We consider the decomposition (5.9) of $\mathbf{T}(1; \infty, \infty)$. For simplicity we write $C(\Delta)$ instead of $C_{\infty,\alpha}(\Delta)$. Let $\overline{C(\Delta)}$ denote the closure of $C(\Delta)$ in $\mathbf{T}(1; \infty, \infty)$.

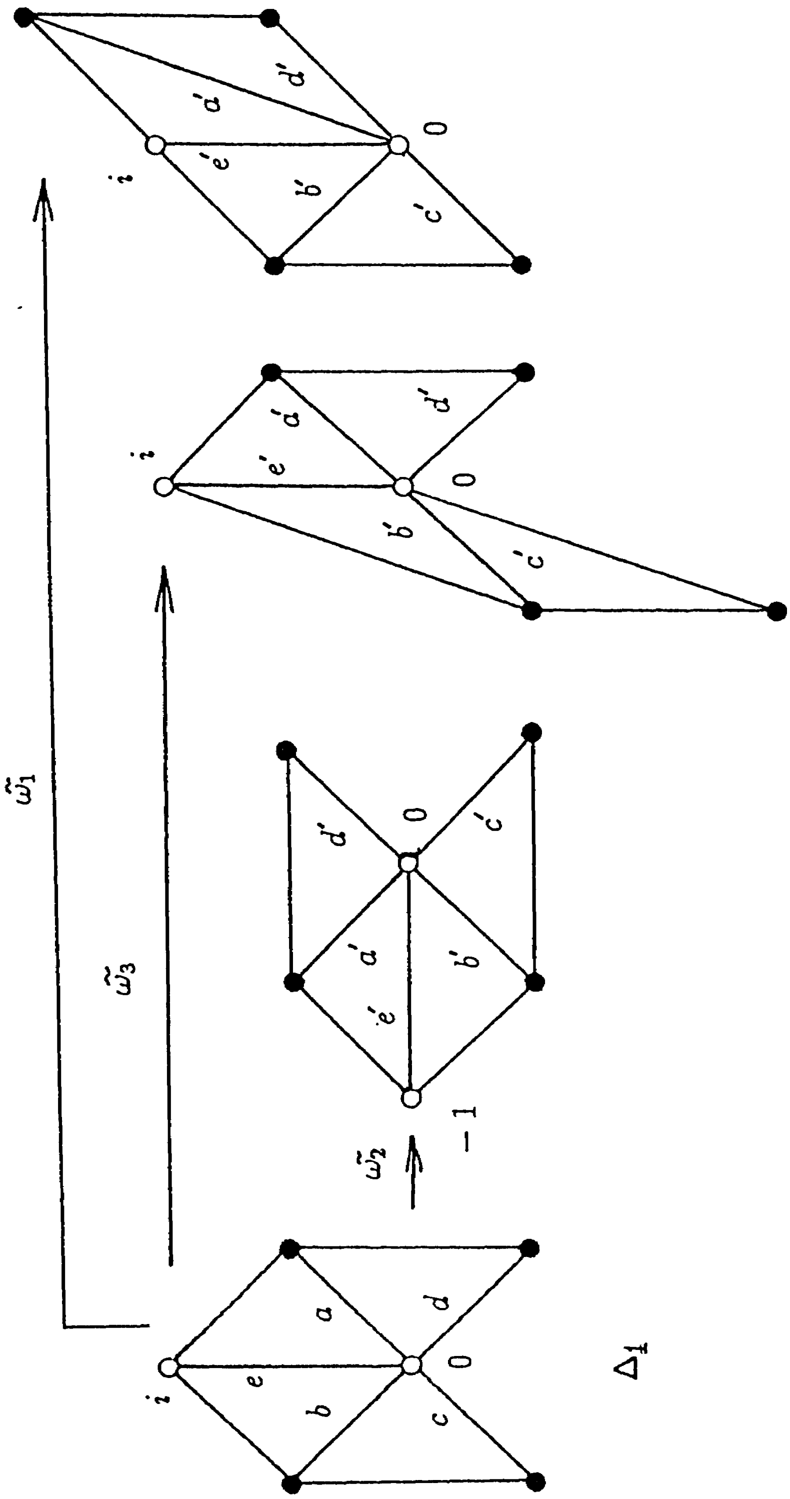

$\tilde{\omega}_1$
$\tilde{\omega}_2$
$\tilde{\omega}_3$
Δ_1
i
0
-1
a
b
c
d
e
a'
b'
c'
d'
e'

Figure 6.2

If Δ_1 is the triangulation of Figure 6.1, then $C(\Delta_1)$ is defined by the face inequalities $d^2+e^2>a^2$, $a^2+e^2>d^2$, $c^2+e^2>b^2$, $b^2+e^2>c^2$, and

$$ad(b^2+c^2-e^2)+bc(a^2+d^2-e^2)>0.$$

LEMMA 6.1. *$\mathbf{T}(1;\infty,\infty)=\bigcup\{\omega_*\overline{C(\Delta_1)}: \omega_*\in G\}$, where $G=\langle\omega_{1*},\omega_{2*},\omega_{3*}\rangle$.*

Proof. We denote the coordinate functions by $a(x),\ldots,e(x)$ and define $m(x)=a+b+c+d$ for $x=(a,b,c,d,e)$. If the lemma is false, there exists a point $x_0=(a_0,b_0,c_0,d_0,e_0)\in\mathbf{T}(1;\infty,\infty)$ such that its G-orbit $G(x_0)$ does not meet $\overline{C(\Delta_1)}$. We choose a point $x=(a,b,c,d,e)\in G(x_0)$. Let M be a number such that $m(x)<M$ and $e<\sqrt{2}M$. Then we can find x' in $G(x_0)$ such that $m(x')<m(x)<M$ and $e(x')<\sqrt{2}M$: Since x does not belong to $\overline{C(\Delta_1)}$, some weak face inequality fails to hold on Δ_1. If it fails to hold on a (i.e., if $e^2+d^2<a^2$), let $x'=\omega_{1*}(x)$. Then $d(x')=(e^2+d^2)/a<a$, so $m(x')<m(x)$ and $e(x')=e<\sqrt{2}M$. Likewise, if the weak face inequality does not hold on b (resp. c,d), then we can choose $x'=\omega_{3*}^{-1}(x)$ (resp. $\omega_{3*}(x)$, $\omega_{1*}^{-1}(x)$) as x'. If the weak face inequality fails to hold on e, then

$$ad(b^2+c^2-e^2)+bc(a^2+d^2-e^2)<0 \Leftrightarrow (ac+bd)(ab+cd)<(ad+bc)e^2.$$

This inequality is the face inequality for $\omega_{2*}(x)$ on e. Then for $\omega_{2*}(x)$ the weak face inequality fails to hold either on a, b, c, or d by our assumption. As before, we can find some $\omega_*\in\{\omega_{1*}^{\pm1},\omega_{3*}^{\pm1}\}$ so that $x''=\omega_*\omega_{2*}(x)$ satisfies $m(x'')<m(\omega_{2*}(x))=m(x)$. If $e(x'')<\sqrt{2}M$, let $x'=x''$. If $e(x'')\geq\sqrt{2}M$, let $x'=\omega_{2*}(x'')$. Then $m(x')=m(x'')$ and

$$e(x')=\frac{a(x'')c(x'')+b(x'')d(x'')}{e(x'')}<\frac{2M^2}{e(x'')}\leq\sqrt{2}M.$$

We apply this argument successively to produce points x_n, $n=1,2,\ldots$, such that $M>m(x_0)>\cdots>m(x_{n-1})>m(x_n)$ and $e_n<\sqrt{2}M$. Since the x_n are uniformly bounded, $\{x_n\}_{n=1}^{\infty}$ contains a converging sequence. This contradicts the discontinuity of the action of the mapping class group, and hence of G, on the Teichmüller space [1, pp. 64–80]. □

The hyperelliptic involution J of F_g^s lifts to the mapping $z\to -z+(1+i)/2$ with respect to the covering $\mathbf{C}\to\mathbf{C}/M$. Then G is a subgroup of

$$MC_{1*}^2=MC_1^2/\langle[J]\rangle,$$

since the class of J generates the isotropy subgroup [6, Sec. 2.3.8].

PROPOSITION 6.2. *MC_{1*}^2 coincides with G.*

Proof. Let $\omega\in MC_1^2$. By Theorem 5.3 and Lemma 6.1, there exists an $\eta_*\in G$ such that $\eta_*^{-1}\omega_*C(\Delta_1)=C(\Delta_1)$. Then $\eta^{-1}\omega$ preserves the triangulation Δ_1. Composing with J if necessary, we may assume that $\eta^{-1}\omega$ fixes each distinguished point of P. Then either $\eta^{-1}\omega$ fixes all homotopy classes in Δ_1 or it sends a,b,c,d,e to c,d,a,b,e, respectively. Therefore $\eta^{-1}\omega$ is homotopic either to the identity map or to ω_2^2. Hence $\omega_*=\eta_*$ or $\omega_*=\eta_*\omega_{2*}^2$. □

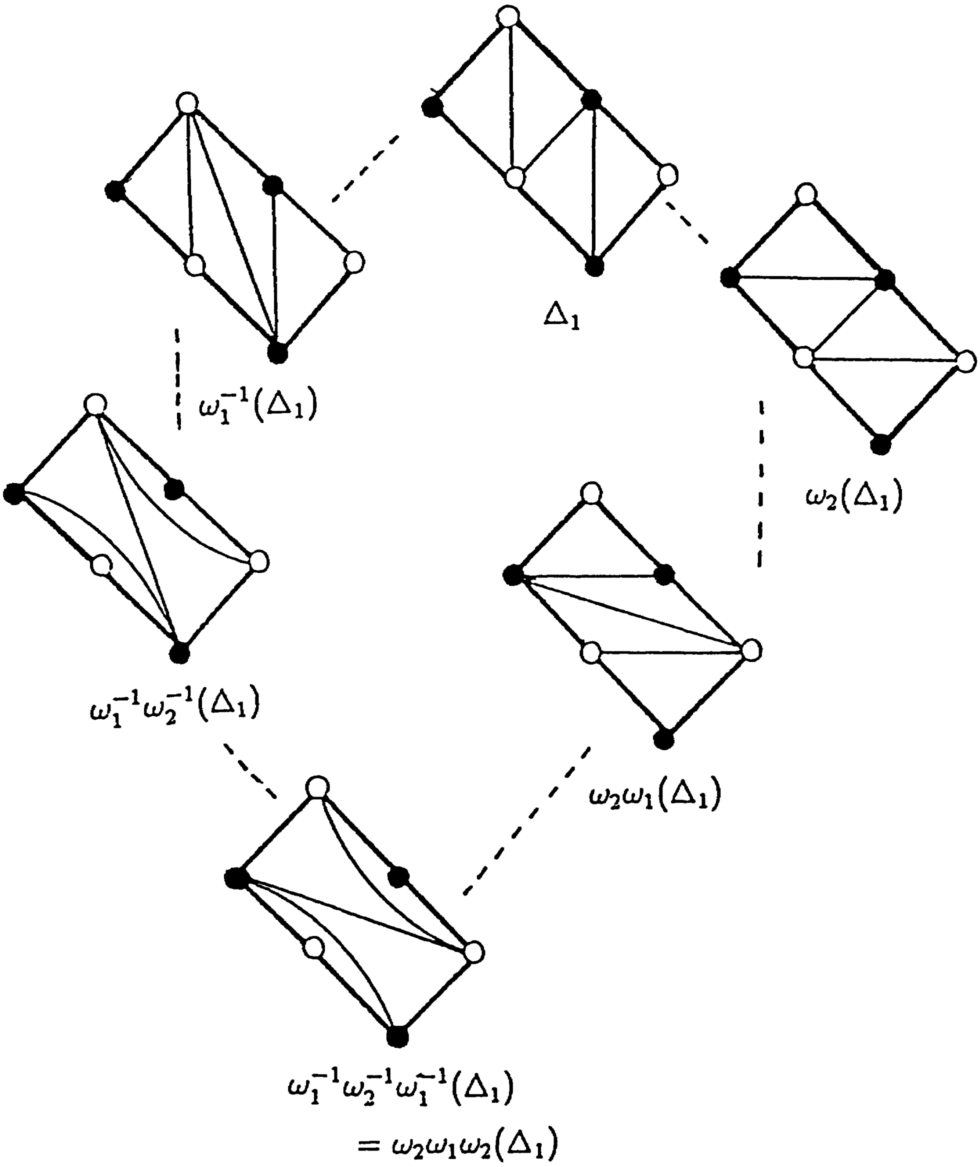


Figure 6.3(a) Triangulations containing $\Delta_{a,e}$

Let $\Delta_{i,j} = \Delta_1 - \{i, j\}$ for distinct $i, j \in \{a, b, c, d, e\}$. Then $C(\Delta_{i,j})$ gives a codimension-2 face of $C(\Delta_1)$, provided that $\Delta_{i,j}$ is a cell decomposition. One can verify that only $\Delta_{a,d}$ and $\Delta_{b,c}$ fail to be cell decompositions. In order to obtain relations in the group MC^2_{1*}, we find all sets $\omega_* \overline{C(\Delta_1)}$, $\omega_* \in MC^2_{1*}$, around a codimension-2 face. If $\omega_* \overline{C(\Delta_1)} = \overline{C(\omega\Delta_1)}$ has a codimension-2 face $\Delta_{i,j}$, then $\omega(\Delta_1)$ contains $\Delta_{i,j}$. In Figure 6.3(a) all triangulations that contain $\Delta_{a,e}$ (and in (b) all that contain $\Delta_{c,d}$) are described. Two triangulations are connected by a broken line if one arises from the other by an elementary move. Note that $\omega_2\omega_1$ preserves $\Delta_{a,e}$. We derive the first relation in (6.4) and the commutativity of ω_{1*} and ω_{3*}. Relations obtained from other codimension-2 faces are consequences of (6.3) and (6.4). Hence the relations (6.2), (6.3), and (6.4) give a representation of the group $G = MC^2_{1*}$.

We finish by considering the special case $\alpha = 6$. In this case, (6.1) can be written as the Diophantine equation

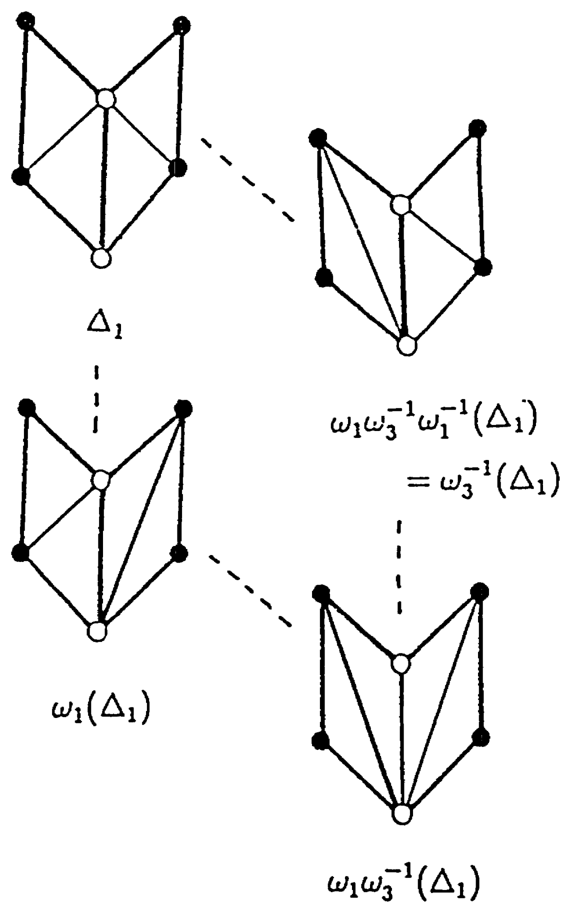


Figure 6.3(b) Triangulations containing $\Delta_{c,d}$

$$ad(b^2+c^2+e^2)+bc(a^2+d^2+e^2) = 6abcde. \qquad (6.5)$$

Equation (6.5), when restricted to the subspace $\{a=c,\ b=d\}$, reduces to the classical Markov equation $a^2+b^2+e^2 = 3abe$. The transformations generated by $\omega_{1*}\omega_{3*}$ and ω_{2*} send $(a,b,e) = (1,1,1)$ to Markov numbers. Hence equation (6.5) can be viewed as a generalization of the Markov equation, and we may expect that (6.5) admits an analogy of the positive integer solutions of the Markov equation. We actually obtain the following.

Proposition 6.3. *Any transformation in MC^2_{1*} sends the solution* $(1,1,1,1,1)$ *to an integer solution of* (6.5).

The rest of this paper is devoted to a proof of this proposition. First, we define

$$F(a,b,c,d,e) = \frac{b^2+c^2+e^2}{bc} \quad \text{and} \quad G(a,b,c,d,e) = \frac{a^2+d^2+e^2}{ad}.$$

Then (6.5) reads $F(a,b,c,d,e)+G(a,b,c,d,e) = 6e$. If (a,b,c,d,e) is a solution of (6.5), then the quadratic equation

$$x^2-\left(6e-\frac{b^2+c^2+e^2}{bc}\right)dx+d^2+e^2 = 0$$

has roots a and $(d^2+e^2)/a$, and

$$a+(d^2+e^2)/a = G(a,b,c,d,e)d.$$

Therefore

$$\omega_{1*}(a,b,c,d,e) = (d,b,c,-a+G(a,b,c,d,e)d,e);$$

likewise for $x=(a,\ b,\ c,\ d,\ e)$, $\omega_{1*}^{-1}(x) = (-d+G(x)a,\ b,\ c,\ a,\ e)$, $\omega_{3*}(x) = (a,-c+F(x)b,b,d,e)$, and $\omega_{3*}^{-1}(x) = (a,c,-b+F(x)c,d,e)$. The functions F (and hence also G) are automorphic with respect to $\{\omega_{1*},\omega_{3*}\}$; that is,

$$F(\omega_{1*}(a,b,c,d,e)) = F(\omega_{3*}(a,b,c,d,e)) = F(a,b,c,d,e).$$

Furthermore,

$$F(\omega_{2*}^2(a,b,c,d,e)) = G(a,b,c,d,e)$$

because $\omega_{2*}^2(a,b,c,d,e) = (c,d,a,b,e)$. Note that $x=(1,1,1,1,1)$ satisfies the following condition:

(*) $\omega_{i*}^{\pm 1}(x)$ and $\omega_{j*}^{\pm 1}\omega_{i*}^{\pm 1}(x)$ are also integer solutions of (6.5), and $F(x)$, $F(\omega_{i*}^{\pm 1}(x))$, and $F(\omega_{j*}^{\pm 1}\omega_{i*}^{\pm 1}(x))$ are integers for $i,j\in\{1,2,3\}$.

We can show that $\omega_*(x)$ is an integer solution of (6.5) for any $\omega_*\in MC_{1*}^2$ if x satisfies the condition (*). However, at present the authors do not know of any integer solutions satisfying condition (*) except for $\omega_*(1,1,1,1,1)$, $\omega_*\in MC_{1*}^2$. To simplify the notation, we denote by L the set of all positive integer solutions $x=(a,b,c,d,e)$ of (6.5) such that $F(x)$ and hence $G(x)$ are integers. The next lemma is immediate from the results mentioned above.

LEMMA 6.4. *For any $x\in L$, $\omega_{1*}^{\pm 1}(x)$, $\omega_{3*}^{\pm 1}(x)$, and $\omega_{2*}^2(x)$ belong to L.*

The following lemma is needed for our argument by induction.

LEMMA 6.5. *For any solution $x=(a,b,c,d,e)$ of* (6.5),

$$F(\omega_{2*}\omega_{1*}^2(x)) = (6e-F(x))F(\omega_{2*}\omega_{1*}(x)) - F(\omega_{2*}(x)),$$
$$G(\omega_{2*}\omega_{1*}^2(x)) = (6e-F(x))G(\omega_{2*}\omega_{1*}(x)) - G(\omega_{2*}(x)).$$

These equations imply that if x, $\omega_{2*}(x)$, and $\omega_{2*}\omega_{1*}(x)\in L$ then $\omega_{2*}\omega_{1*}^2(x)$ and $\omega_{2*}\omega_{1*}^{-1}(x)\in L$, and if x, $\omega_{2*}(x)$, and $\omega_{2*}\omega_{3*}(x)\in L$ then $\omega_{2*}\omega_{3*}^2(x)$ and $\omega_{2*}\omega_{3*}^{-1}(x)\in L$.

Proof. If $(a_n,b,c,d_n,e) = \omega_{1*}^n(a,b,c,d,e)$, $n=0,1,2$, then

$$\begin{aligned} F(\omega_{2*}\omega_{1*}^n(x)) &= \frac{(a_n^2+b^2)e^2+(a_nc+bd_n)^2}{a_nb} \\ &= \frac{a_nc}{e^2}\left(\frac{b^2+c^2+e^2}{bc}\right)+\frac{d_nb}{e^2}\left(\frac{a^2+d^2+e^2}{ad}\right)+\frac{2(cd_n-ba_n)}{e^2}. \end{aligned}$$

This equation yields the first equality in the statement of the lemma. The second equality can be obtained in a similar way. □

We define $\Omega = \{\omega_* \in MC^2_{1*}: \omega_*(1,1,1,1,1) \in L\}$. Proposition 6.3 follows if we prove that Ω coincides with MC^2_{1*}. (We remark that this is true if $(1,1,1,1,1)$ is replaced by any positive integer solution of (6.5) satisfying the condition (*).) Now take an arbitrary element ω_* of MC^2_{1*} and represent it as a word

$$\omega_* = \sigma_1\sigma_2\cdots\sigma_{n+1},$$

where $\sigma_i \in \{\omega_{1*}^{\pm 1}, \omega_{2*}^{\pm 1}, \omega_{3*}^{\pm 1}\}$, $i = 1, \ldots, n+1$, $n \geq 2$. Assume as an induction hypothesis that any element in MC^2_{1*}, represented as a word of length $\leq n$, belongs to Ω. We consider the following cases.

Case 1: $\sigma_1 = \omega_{1*}^{\pm 1}$ *or* $\sigma_1 = \omega_{3*}^{\pm 1}$. The function F is automorphic with respect to $\{\omega_{1*}, \omega_{3*}\}$. Hence, by the induction hypothesis, ω_* belongs to Ω.

Case 2: $\sigma_1 = \omega_{2*}^{-1}$. By (6.2), $\omega_{2*}^{-1} = \omega_{2*}^2\omega_{2*}$. From Lemma 6.4 we can include this case in the next one.

Case 3: $\sigma_1 = \omega_{2*}$. By Lemma 6.4, Lemma 6.5, (6.3), and the induction hypothesis, we need only consider the following subcases:

(a) $\sigma_1\sigma_2\sigma_3 = \omega_{2*}\omega_{1*}\omega_{2*}$;
(b) $\sigma_1\sigma_2\sigma_3 = \omega_{2*}\omega_{1*}\omega_{2*}^{-1}$;
(c) $\sigma_1\sigma_2\sigma_3 = \omega_{2*}\omega_{1*}\omega_{3*} = (\omega_{2*}\omega_{1*}\omega_{2*})(\omega_{2*}\omega_{1*}\omega_{2*}^2)$;
(d) $\sigma_1\sigma_2\sigma_3 = \omega_{2*}\omega_{3*}\omega_{2*}$;
(e) $\sigma_1\sigma_2\sigma_3 = \omega_{2*}\omega_{3*}\omega_{2*}^{-1}$.

Here we have used (6.3). Let $\omega_*' = \sigma_4\cdots\sigma_{n+1}$.

Case (a). By using (6.4) we can write $\omega_* = \omega_{1*}^{-1}\omega_{2*}^{-1}\omega_{1*}^{-1}\omega_*'$, which is a case considered above. The same argument proves case (d).

Case (b). By Lemma 6.5, $\omega_* = \omega_{2*}\omega_{1*}\omega_{2*}^{-1}\omega_*'$ belongs to Ω if both ω_*' and $\omega_{2*}\omega_{1*}^{-1}\omega_{2*}^{-1}\omega_*'$ do. Here $\omega_{2*}\omega_{1*}^{-1}\omega_{2*}^{-1} = \omega_{2*}^2(\omega_{1*}\omega_{2*}\omega_{1*})$ by (6.4). By the induction hypothesis, $\omega_*', \omega_{2*}\omega_{1*}\omega_*' \in \Omega$. Since (by Lemma 6.4) $\omega_{2*}^2\omega_{1*}$ preserves L, $\omega_{2*}^2\omega_{1*}\omega_{2*}\omega_{1*}\omega_*'$ belongs to Ω. Hence $\omega_* \in \Omega$. The same argument shows case (e).

Case (c). By (6.4), $\omega_* = \omega_{1*}^{-1}\omega_{2*}^{-1}\omega_{1*}^{-1}\omega_{2*}\omega_{1*}\omega_{2*}^2\omega_*'$. By Lemma 6.4, $\omega_* \in \Omega$ if $\omega_{2*}^{-1}\omega_{1*}^{-1}\omega_{2*}\omega_{1*}\omega_{2*}^2\omega_*' \in \Omega$. By (6.2) and Lemmas 6.4 and 6.5, the last word belongs to Ω if $\omega_{1*}\omega_{2*}^2\omega_*'$ and $\omega_{2*}^{-1}(\omega_1\omega_{2*}\omega_{1*}\omega_{2*})\omega_{2*}\omega_*' = \omega_{2*}^{-2}\omega_{1*}^{-1}\omega_{2*}\omega_*'$ do. By the induction hypothesis and Lemma 6.5, the last claim is true. □

REMARK. The transformations defined by

$$\omega_{4*}(a,b,c,d,e) = (d,b,c,a,e) \quad \text{and} \quad \omega_{5*}(a,b,c,d,e) = (a,c,b,d,e)$$

preserve the integer solutions of (6.5), but the group generated by these and MC^2_{1*} does not preserve the integer solutions. For example,

$$\omega_{2*}\omega_{4*}\omega_{2*}\omega_{1*}^2(1,1,1,1,1) = (5,1,2,1,11/7).$$

Hence MC^2_{1*} does not preserve the integer solutions of (6.5). This fact causes the difficulty in finding all integer solutions of (6.5).

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