A Harmonic Quadrature Formula Characterizing Bi-Infinite Cylinders

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1. Introduction and Results

In the following let $K_r = \{x \in \mathbb{R}^m : |x| < r\}$ be an open ball of radius $r > 0$ centered at the origin. $|\cdot|$ always denotes the Euclidean norm and λ_m the m-dimensional Lebesgue measure. Here m and (later on) n will be natural numbers.

We are concerned with *harmonic quadrature formulas*. The prototype is Gauss's well-known mean value formula:

For every harmonic and integrable function $h: K_r \to \mathbb{R}$, the following mean value property holds:

$$\int_{K_r} h\, d\lambda_m = \lambda_m(K_r) \cdot h(0).$$

For a $(1+n)$-dimensional strip $(-r, r) \times \mathbb{R}^n$, the following quadrature formula is true for harmonic and integrable functions $h: (-r, r) \times \mathbb{R}^n \to \mathbb{R}$:

$$\int_{(-r,r)\times\mathbb{R}^n} h\, d\lambda_{1+n} = \lambda_1(K_r) \cdot \int_{\mathbb{R}^n} h(0, \xi)\, d\lambda_n(\xi)$$

(see [2] or [7]).

Now consider $m \geq 2$. For an $(m+n)$-dimensional bi-infinite cylinder $K_r \times \mathbb{R}^n \subset \mathbb{R}^{m+n}$, we shall prove a similar quadrature formula in Section 2, as follows.

THEOREM 1. *Let $h: K_r \times \mathbb{R}^n \to \mathbb{R}$ be harmonic and integrable on $K_r \times \mathbb{R}^n$. Then*

$$\int_{K_r \times \mathbb{R}^n} h\, d\lambda_{m+n} = \lambda_m(K_r) \cdot \int_{\mathbb{R}^n} h(0, \xi)\, d\lambda_n(\xi).$$

Open balls and open strips can even be characterized by harmonic quadrature. Indeed, Kuran [11] gave a simple proof of the following result:

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Let D be an open subset of $\mathbb{R}^m$ such that $0 \in D$ and $\lambda_m(D) < \infty$. If, for every integrable and harmonic function h on D,

$$\int_D h \, d\lambda_m = \lambda_m(D) \cdot h(0),$$

then D is an open ball centered at 0.

A corresponding result characterizing open strips is due to Armitage and Nelson [2] and, under somewhat stronger assumptions, attributable to the authors [7].

Let D be an open subset of $\mathbb{R}^{1+n}$ such that $\{0\} \times \mathbb{R}^n \subset D$ and D is a subset of some (arbitrarily large) strip. If for every positive integrable harmonic function h on D the equation

$$\int_D h \, d\lambda_{1+n} = \lambda_1(K_r) \cdot \int_{\mathbb{R}^n} h(0, \xi) \, d\lambda_n(\xi)$$

holds true, then

$$D = (-r, r) \times \mathbb{R}^n.$$

Our main result shows that a bi-infinite cylinder also can be characterized by harmonic quadrature. More precisely, we have the following result.

THEOREM 2. *Let D be a regular open subset of $\mathbb{R}^{m+n}$ such that*

$$\{(0, \dots, 0)\} \times \mathbb{R}^n \subset D$$

and D is a subset of some (arbitrarily large) cylinder. If for every positive integrable harmonic function h on D we have

$$\int_D h \, d\lambda_{m+n} = \lambda_m(K_r) \cdot \int_{\mathbb{R}^n} h(0, \xi) \, d\lambda_n(\xi),$$

then

$$D = K_r \times \mathbb{R}^n.$$

Note that a regular open set $D \subset \mathbb{R}^{m+n}$ is defined by $(\bar{D})^0 = D$.

The proof of Theorem 2 will be given in Section 4. It is based on two auxiliary results which will be proved in Section 3.

Throughout the paper we shall use the following notation. Let $m \geq 2$ and $n \geq 1$. A typical point of $\mathbb{R}^{m+n}$ will be denoted by

$$X = (x, \xi) = (x_1, \dots, x_m, \xi_1, \dots, \xi_n).$$

By $B_r(X_0)$ we mean the $(m+n)$-dimensional open ball of radius $r > 0$ centered at $X_0 \in \mathbb{R}^{m+n}$. Open balls in $\mathbb{R}^m$ are denoted by $K_r(x_0)$, where $x_0 \in \mathbb{R}^m$ and $r > 0$. Hence, in particular,

$$K_r = K_r(0).$$

For an $(m+n)$-dimensional bi-infinite cylinder of radius $r > 0$ centered at 0 we also use the notation Z_r; that is, $Z_r = K_r \times \mathbb{R}^n$. The volume of the d-dimensional unit ball is called ω_d. Note that $\omega_d = 2\pi^{d/2}/(d \cdot \Gamma(d/2))$.

For any open set $D \subset \mathbb{R}^{m+n}$, let $H(D)$ be the set of all harmonic functions on D; for $E \subset \mathbb{R}^{m+n}$ we use $C(E)$ for the set of all continuous functions on E. By χ_F we denote the characteristic function of a set $F \subset \mathbb{R}^{m+n}$.

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2. Proof of Theorem 1

In order to prove Theorem 1, we first show a proposition that follows easily from the work of Gardiner [4].

Proposition 3. *Let $h: K_r \times \mathbb{R}^n \to \mathbb{R}$ be harmonic and λ_{m+n}-integrable. Then*

$$M(h;x) = \int_{\mathbb{R}^n} h(x,\xi)\, d\lambda_n(\xi) \text{ is harmonic on } K_r. \tag{2.1}$$

Proof. We have

$$M(|h|;x) = \int_{\mathbb{R}^n} |h(x,\xi)|\, d\lambda_n(\xi) \in [0,\infty],$$

and by Fubini's theorem

$$\int_{K_r} M(|h|;x)\, d\lambda_m(x) = \int_{K_r \times \mathbb{R}^n} |h|\, d\lambda_{m+n} < \infty. \tag{2.2}$$

According to (2.2), the function $M(|h|;x)$ belongs to Gardiner's class $\mathfrak{F}(K_r)$ (see [4, bottom of p. 343]). As $|h|$ is subharmonic on $K_r \times \mathbb{R}^n$, it follows from Theorem 1 of Gardiner [4] that $M(|h|;x)$ is subharmonic on K_r and hence locally bounded. Applying Gardiner's Theorem 1 again, now to h and $-h$, one obtains that $M(h;x)$ and $M(-h;x)$ are subharmonic on K_r; this means that $M(h;x)$ is harmonic there. □

Proof of Theorem 1. The mean value property of harmonic functions applied to $M(h;x)$ leads to

$$M(h;0) = \frac{1}{\lambda_m(K_r)} \int_{K_r} M(h;x)\, d\lambda_m(x). \tag{2.3}$$

Using Fubini's theorem, we obtain

$$\begin{aligned}
\int_{K_r \times \mathbb{R}^n} h(x,\xi)\, d\lambda_{m+n}(x,\xi) &= \int_{K_r}\left(\int_{\mathbb{R}^n} h(x,\xi)\, d\lambda_n(\xi)\right) d\lambda_m(x) && \text{(Fubini)}\\
&= \int_{K_r} M(h;x)\, d\lambda_m(x) && \text{(by (2.1))}\\
&= \lambda_m(K_r) M(h;0) && \text{(by (2.3))}\\
&= \lambda_m(K_r) \int_{\mathbb{R}^n} h(0,\xi)\, d\lambda_n(\xi). && \text{(by (2.1))}
\end{aligned}$$

This completes the proof of Theorem 1. □

3. Auxiliary Results

In this section we prove two auxiliary results which are also of independent interest.

LEMMA 4. *Suppose that* $\emptyset \neq D \subset \mathbb{R}^d$ *is an open set, where* $d \geq 2$. *Assume that the function* $h \in C(\bar{D}) \cap H(D) \cap L^\infty(D)$ *satisfies*

(α) $h = 0$ *on* ∂D,

and assume also that

(β) $\lambda_d(D \cap B_r(0)) = o(r^d)$ *as* $r \to \infty$.

Then $h = 0$ *on* $\bar{D}$.

Proof. Define the function S on $\mathbb{R}^d$ by putting $S = |h|$ on $\bar{D}$ and $S = 0$ on $\mathbb{R}^d \setminus \bar{D}$. Then, by the continuity of h and by (α), S is continuous. For any $X \in \mathbb{R}^d$ there exists an $R = R(X) > 0$ such that

$$S(X) \leq \frac{1}{\lambda_d(B_r(X))} \int_{B_r(X)} S(Y)\, d\lambda_d(Y) \quad \text{for all } r \text{ with } 0 < r < R.$$

Indeed, for $X \in D$ choose $R > 0$ such that $B_R(X) \subset D$, and for $X \notin D$ choose any $R > 0$. Thus S is continuous on $\mathbb{R}^d$ and satisfies the sub-mean value property; hence S is subharmonic.

Since $h \in L^\infty(D)$, we have $S \in L^\infty(D)$. Thus

$$0 \leq S \leq M \quad \text{on } \mathbb{R}^d$$

for some bound $M > 0$.

Hence, if $X \in \mathbb{R}^d$ and $r > 0$, then

$$\begin{aligned} 0 \leq S(X) &\leq \frac{1}{\lambda_d(B_r(X))} \int_{B_r(X)} S\, d\lambda_d \\ &\leq \frac{1}{\omega_d r^d} \int_{B_r(X) \cap D} M\, d\lambda_d \\ &\leq \frac{M}{\omega_d r^d} \cdot \lambda_d(B_{|X|+r}(0) \cap D) \to 0 \end{aligned}$$

for $r \to \infty$ by (β), so that $S = 0$ and hence $h = 0$. ☐

For the rest of this section we assume that $s > 0$, $m \geq 2$, and $n \geq 1$.

THEOREM 5. *Denote by* G *the Green function of* $K_s \times \mathbb{R}^n$ *and by* g *the Green function of* K_s. *Then, for fixed* $x, y \in K_s$ *with* $x \neq y$ *and arbitrary* $\eta \in \mathbb{R}^n$, *we have*

$$\int_{\mathbb{R}^n} G((x,\xi),(y,\eta))\, d\lambda_n(\xi) = c_{m,n} \cdot g(x,y),$$

where

$$c_{m,n} = \begin{cases} \dfrac{d(d-2)\omega_d}{2\pi} & \text{for } m=2 \\ \dfrac{d(d-2)\omega_d}{m(m-2)\omega_m} & \text{for } m \geq 3 \end{cases}$$

with $d = m+n$.

REMARK. For $m=2$ and $n=1$, this result can be found in Lévy [12] with $c_{2,1}=2$.

Proof. Let $Y \in K_s \times \mathbb{R}^n$ be fixed. Since G is the Green function of $K_s \times \mathbb{R}^n$, it has the following properties:

(1) $K_s \times \mathbb{R}^n \ni X \mapsto G(X,Y)$ is harmonic in $(K_s \times \mathbb{R}^n) \setminus \{Y\}$;
(2) $G(X,Y) \to 0$ as $X \to X_0 \in \partial K_s \times \mathbb{R}^n$; and
(3) $G(X,Y) - 1/|X-Y|^{d-2}$ is harmonic for $X = Y$.

It is sufficient to show that

$$h(x,y) = \int_{\mathbb{R}^n} G((x,\xi),(y,\eta))\, d\lambda_n(\xi) \tag{3.1}$$

has the properties (1′), (2′) and (3′):

(1′) $K_s \ni x \mapsto h(x,y)$ is harmonic in $K_s \setminus \{y\}$;
(2′) $h(x,y) \to 0$ as $x \to x_0 \in \partial K_s$; and
(3′) $\dfrac{1}{c_{m,n}} \cdot h(x,y) + \begin{cases} \log|x-y| & \text{for } m=2 \\ -\dfrac{1}{|x-y|^{m-2}} & \text{for } m \geq 3 \end{cases}$ is harmonic for $x = y$.

By a standard argument (dominate G by a Green function of a half-space), the integral (3.1) exists for $x \neq y$ (see Nualtaranee [13]). Note that the integral in (3.1) does not depend on η, since $G((x,\xi),(y,\eta)) = G((x,\xi-\eta),(y,0))$.

Now consider $\Omega = K_s \setminus \{y\}$. By Gardiner [4, Thm. 1] applied to $G(X,Y)$ and $-G(X,Y)$, we see that h and $-h$ are subharmonic in Ω and hence harmonic. (Note that G and $-G$ belong to the class $\mathfrak{F}$ of Gardiner [4] by his sufficiency criterion on the bottom of p. 343). Hence h satisfies (1′).

Let us now prove (2′) for h. Let $Y = (y,\eta) \in K_s \times \mathbb{R}^n$ be fixed and let $x_p \to x_0 \in \partial K_s$ for $p \to \infty$. We have $G((x_p,\xi),(y,\eta)) \to 0$ for $p \to \infty$ by (2). Hence the dominated convergence theorem of Lebesgue gives

$$\lim_{p\to\infty} h(x_p,y) = \lim_{p\to\infty} \int_{\mathbb{R}^n} G((x_p,\xi),(y,\eta))\, d\lambda_n(\xi) = 0$$

if there exists, for sufficiently large p, a λ_n-integrable function F with

$$G((x_p,\xi),(y,\eta)) \leq F(\xi) \quad \text{for all } \xi \in \mathbb{R}^n. \tag{3.2}$$

F will be defined with the aid of the Green function G_H of some half-space H, which is given by

$$G_H(X,Y) = \frac{1}{|X-Y|^{d-2}} - \frac{1}{|X-Y^*|^{d-2}},$$

where Y^* is the mirror image of Y with respect to ∂H.

For the construction of H, first let a half-space H_0 of $\mathbb{R}^m$ be chosen as follows. Take a ball $K_R(y)$ of radius $R > 0$ centered at y such that $K_s \subset K_R(y)$. Take a tangent hyperplane P in $\mathbb{R}^m$ to $K_R(y)$ orthogonal to the line connecting x_0 and y, such that the mirror image y^* of y with respect to P satisfies

$$|x_0 - y^*| < |y - y^*| \tag{3.3}$$

and such that $\partial H_0 = P$ and H_0 contains $K_R(y)$. Finally, put

$$H = H_0 \times \mathbb{R}^n.$$

Then for $Y = (y, \eta)$ we have $Y^* = (y^*, \eta)$. Because of inequality (3.3) we can choose $\sigma > 0$ so small that, for each $x \in K_\sigma(x_0) \cap K_s$ and for a fixed $w_0 \in K_\sigma(y) \subset K_s$, $w_0 \neq y$, we have

$$|w_0 - y| < |x-y| \quad \text{and} \quad |x-y^*| < |w_0 - y^*|.$$

These inequalities extend to

$$|W-Y| < |X-Y| \quad \text{and} \quad |X-Y^*| < |W-Y^*|, \tag{3.4}$$

where $X = (x, \xi)$, $Y = (y, \eta)$, $Y^* = (y^*, \eta)$, and $W = (w_0, \xi)$ for $x \in K_\sigma(x_0) \cap K_s$ with arbitrary $\xi, \eta \in \mathbb{R}^n$. Now (3.4) yields

$$\frac{1}{|X-Y|^{d-2}} - \frac{1}{|X-Y^*|^{d-2}} < \frac{1}{|W-Y|^{d-2}} - \frac{1}{|W-Y^*|^{d-2}};$$

that is, $G_H((x,\xi),(y,\eta)) < G_H((w_0,\xi),(y,\eta))$ for all $x \in K_\sigma(x_0) \cap K_s$, with w_0 as before and for arbitrary $\xi, \eta \in \mathbb{R}^n$.

Since $K_s \times \mathbb{R}^n \subset H = H_0 \times \mathbb{R}^n$, for all $\xi, \eta \in \mathbb{R}^n$ and sufficiently large p we have

$$G((x_p,\xi),(y,\eta)) \le G_H((x_p,\xi),(y,\eta)) \le G_H((w_0,\xi),(y,\eta)).$$

Hence $F(\xi) = G_H((w_0,\xi),(y,\eta))$ satisfies (3.2) since it is a λ_n-integrable function by Nualtaranee [13].

For (3′), we now examine the singularity of $h(x,y)$ for $x = y$. Take again the Green function G_H of a half-space H containing the cylinder $K_s \times \mathbb{R}^n$. Then $G_H - G$ is harmonic in $K_s \times \mathbb{R}^n$. Define

$$h_1(x,y) = \int_{\mathbb{R}^n} G_H((x,\xi),(y,\eta))\, d\lambda_n(\xi).$$

By Gardiner [4, Thm. 1] applied to $\pm(G_H - G)$, we have that

$$\int_{\mathbb{R}^n} (G_H((x,\xi),(y,\eta)) - G((x,\xi),(y,\eta)))\, d\lambda_n(\xi) = h_1(x,y) - h(x,y)$$

is harmonic. Hence h and h_1 have the same singularity at $x=y$. In order to show (3′) it is sufficient to prove that the singularity of $(1/c_{m,n})\cdot h_1(x,y)$ is

$$\begin{cases} -\log|x-y| & \text{for } m=2, \\ 1/|x-y|^{m-2} & \text{for } m\geq 3. \end{cases}$$

Case 1: First let $m\geq 3$. Since $Y^*=(y^*,\eta)$ does not belong to the half-space H, the function

$$x\mapsto \int_{\mathbb{R}^n} \frac{1}{|(x,\xi)-(y^*,\eta)|^{m+n-2}}\,d\lambda_n(\xi)$$

is finite and hence harmonic according to Gardiner [4]. Thus it is sufficient to prove, for all $n\in\mathbb{N}$,

$$\int_{\mathbb{R}^n} \frac{1}{|(x,\xi)-(y,\eta)|^{m+n-2}}\,d\lambda_n(\xi) = c_{m,n}\cdot\frac{1}{|x-y|^{m-2}} \quad \text{for all } m\geq 3. \tag{3.5}$$

We prove (3.5) by induction with respect to n, so let at first $n=1$; we start with $d=m+n$ even. Then

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{|(x,\xi)-(y,\eta)|^{d-2}}\,d\lambda_1(\xi) &= \int_{-\infty}^{\infty} \frac{d\xi}{(\sum_{i=1}^{d-1}(x_i-y_i)^2+(\xi-\eta)^2)^{d/2-1}} \\ &= \int_{-\infty}^{\infty} \frac{d\zeta}{(a^2+\zeta^2)^{d/2-1}},\end{aligned} \tag{3.6}$$

where $a^2=\sum_{i=1}^{d-1}(x_i-y_i)^2$. By Gröbner and Hofreiter [8, p. 14, formula 9], the last integral in (3.6) can be expressed as

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\zeta}{(a^2+\zeta^2)^{d/2-1}} &= 2\cdot\frac{2^{d/2-2}\cdot\Gamma(1/2+d/2-2)\cdot\pi}{2^{d/2-1}\cdot\Gamma(1/2)\cdot(d/2-2)!}\cdot\frac{1}{a^{d-3}} \\ &= \frac{1}{a^{d-3}}\cdot\frac{(d-2)\cdot\pi^{1/2}\cdot(d/2-3/2)\cdot\Gamma(d/2-3/2)}{(d-3)\cdot(d/2-1)!} \\ &= \frac{1}{a^{d-3}}\cdot\frac{d(d-2)\omega_d}{(d-1)(d-3)\omega_{d-1}} \\ &= \frac{c_{d-1,1}}{|x-y|^{d-3}},\end{aligned}$$

where we have used $\omega_d=2\pi^{d/2}/(d\cdot\Gamma(d/2))$. This shows (3.5) for $n=1$ and $d=m+n$ even.

A similar calculation shows that (3.5) is also true for $n=1$ and $d=m+n$ odd. Here we use Gröbner and Hofreiter [8, p. 35, formula 2a for $m=0$]. Note that the symbol $(\mu;\delta;\nu)$ in Gröbner and Hofreiter is defined as

$$(\mu;\delta;\nu)=\frac{\delta^\nu\Gamma(\mu/\delta+\nu)}{\Gamma(\mu/\delta)};$$

see [8, p. 1].

Now assume that (3.5) holds for some $n \geq 1$. Let m be fixed and put $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, $\xi^{(n)} = (\xi_n, \ldots, \xi_1)$, and $\eta^{(n)} = (\eta_n, \ldots, \eta_1)$. Note that $\xi^{(n+1)} = (\xi_{n+1}, \xi^{(n)})$ and $\eta^{(n+1)} = (\eta_{n+1}, \eta^{(n)})$. With this notation we derive

$$\int_{\mathbb{R}^{n+1}} \frac{d\lambda_{n+1}(\xi^{(n+1)})}{|(x, \xi^{(n+1)}) - (y, \eta^{(n+1)})|^{m+(n+1)-2}}$$

$$= \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^n} \frac{d\lambda_n(\xi^{(n)})}{|(x, \xi_{n+1}, \xi^{(n)}) - (y, \eta_{n+1}, \eta^{(n)})|^{(m+1)+n-2}} \right) d\lambda_1(\xi_{n+1}) \quad \text{(Fubini)}$$

$$= c_{m+1,n} \cdot \int_{-\infty}^{\infty} \frac{d\lambda_1(\xi_{n+1})}{|(x, \xi_{n+1}) - (y, \eta_{n+1})|^{m+1-2}} \quad \text{(by (3.5))}$$

$$= c_{m+1,n} \cdot c_{m,1} \cdot \frac{1}{|x-y|^{m-2}}. \quad \text{(by (3.5))}$$

A straightforward calculation shows that

$$c_{m+1,n} \cdot c_{m,1} = c_{m,n+1}.$$

Hence we have proved (3.5) for $(n+1)$. By induction, (3.5) is valid for all $n \in \mathbb{N}$ as long as $m \geq 3$.

Case 2: Now we consider the case $m = 2$ and $n = 1$. Here it is sufficient to show that

$$I = \int_{-\infty}^{\infty} \left(\frac{1}{|(x, \xi) - (y, \eta)|} - \frac{1}{|(x, \xi) - (y^*, \eta)|} \right) d\lambda_1(\xi)$$
$$= -2 \cdot \log|x-y| + 2 \cdot \log|x-y^*|.$$

This follows with

$$a^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 \quad \text{and} \quad (a^*)^2 = (x_1 - y_1^*)^2 + (x_2 - y_2^*)^2$$

from

$$I = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{a^2 + \zeta^2}} - \frac{1}{\sqrt{(a^*)^2 - \zeta^2}} \right) d\zeta$$
$$= \lim_{b \to \infty} 2 \cdot [\log(x + \sqrt{x^2 + a^2}) - \log(x + \sqrt{x^2 + (a^*)^2})]_0^b$$
$$= \lim_{b \to \infty} 2 \cdot \left[\log\left(\frac{b + \sqrt{b^2 + a^2}}{b + \sqrt{b^2 + (a^*)^2}} \right) - \log \frac{a}{a^*} \right]$$
$$= -2 \cdot \log|x-y| + 2 \cdot \log|x-y^*|.$$

Case 3: In the remaining case, $m = 2$ and $n \geq 2$, it is sufficient to show that the singularity of

$$x \mapsto \int_{\mathbb{R}^n} \left(\frac{1}{|(x, \xi) - (y, \eta)|^n} - \frac{1}{|(x, \xi) - (y^*, \eta)|^n} \right) d\lambda_n(\xi)$$

is equal to $-c_{2,n} \cdot \log|x-y|$. This follows by an argument similar to that used in Case 1 and in Case 2. □

Table 1 displays some of the constants $c_{m,n}$ for $2 \le m \le 5$ and $1 \le n \le 4$.

Table 1

	$n=1$	$n=2$	$n=3$	$n=4$	$\cdots$
$m=2$	2	2π	4π	$2\pi^2$	$\cdots$
$m=3$	π	2π	π^2	$\frac{4}{3}\pi^2$	$\cdots$
$m=4$	2	π	$\frac{4}{3}\pi$	$\frac{1}{2}\pi^2$	$\cdots$
$m=5$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{1}{4}\pi^2$	$\frac{4}{15}\pi^2$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

4. Proof of Theorem 2

The proof of Theorem 2 uses several lemmas. Throughout this section, let $d=m+n$ where $m \ge 2$ and $n \ge 1$, and let $Z_r = K_r \times \mathbb{R}^n \subset \mathbb{R}^{m+n}$ for some $r > 0$.

LEMMA 6. *Let $\emptyset \ne D \subset Z_s$ be an open set, and let G be the Green function of Z_s for some $s > 0$. Then the function u defined by*

$$u(Y) = \int_D G(X,Y)\, d\lambda_{m+n}(X) \quad \textit{for } Y \in Z_s$$

has the following properties:

(γ) $u \in C^1(Z_s)$;
(δ) $u \in C^2(D)$; *and*
(ϵ) $\Delta u = d(2-d)\omega_d$ *on* D.

REMARK. The integrability of G is guaranteed by an estimate in Nualtaranee [13, Lemma 1], since G is dominated by the Green function of a half-space H containing Z_s.

Proof. (γ) Let $Y_0 \in Z_s$ and $\delta > 0$ with $B = B_\delta(Y_0) \subset Z_s$. We show that $u \in C^1(B)$. Consider

$$u(Y) = \int_{D \setminus B} G(X,Y)\, d\lambda_{m+n}(X) + \int_{B \cap D} G(X,Y)\, d\lambda_{m+n}(X) \quad \text{for } Y \in B. \tag{4.1}$$

Now we want to apply Helms [10, Thm. 6.6]. To this end we define a measure μ on the Borel field $\mathcal{B}(Z_s)$ by

$$\mu(C) = \int_{D \setminus B} \chi_C(X)\, d\lambda_{m+n}(X) \quad \text{for } C \in \mathcal{B}(Z_s).$$

Then $\mu(B) = 0$, and B is open. Hence, according to Helms [10, Thm. 6.6], the function

$$B \ni Y \mapsto \int_{D \setminus B} G(X,Y)\, d\lambda_{m+n}(X) \text{ is harmonic in } B. \tag{4.2}$$

The second term on the right-hand side of (4.1) can be written as

$$\int_{B \cap D} \left(G(X,Y) - \frac{1}{|X-Y|^{d-2}} \right) d\lambda_{m+n}(X) + \int_{B \cap D} \frac{d\lambda_{m+n}(X)}{|X-Y|^{d-2}}. \tag{4.3}$$

Now, in order to apply Lemma 6.7 of Helms [10], put $U = D \cap B$ and $V = B$. Then, for each fixed $Y \in V$, the function $X \mapsto G(X,Y) - 1/|X-Y|^{d-2}$ is continuous on U (condition (i) of Helms). Furthermore, for each fixed $X \in U$, the function $Y \mapsto G(X,Y) - 1/|X-Y|^{d-2}$ is harmonic in V (condition (ii) of Helms). Finally, since G is dominated by the Green function $G_{\mathbb{R}^d}(X,Y) = 1/|X-Y|^{d-2}$, we obtain

$$G(X,Y) - \frac{1}{|X-Y|^{d-2}} \le 0 \quad \text{on } Z_s \times Z_s.$$

Since, in addition, the integral

$$\int_{B \cap D} \left(G(X,Y) - \frac{1}{|X-Y|^{d-2}} \right) d\lambda_{m+n}(X)$$

is finite for $Y \in V$, we obtain by Helms [10, Thm. 6.7] that

$$B \ni Y \mapsto \int_{B \cap D} \left(G(X,Y) - \frac{1}{|X-Y|^{d-2}} \right) d\lambda_{m+n}(X)$$

is harmonic in B. Applying [5, Lemma 4.1] to

$$B \ni Y \mapsto \int_{B \cap D} \frac{d\lambda_{m+n}(X)}{|X-Y|^{d-2}} \tag{4.4}$$

with $\Omega = B$ and $f = \chi_{B \cap D}$, we obtain that the function defined in (4.4) is in $C^1(B)$. Hence also the function in (4.3) is in $C^1(B)$. Thus, by (4.1) and (4.2), we see that $u \in C^1(B)$.

(δ) and (ϵ) Let $Y_0 \in D$ and $r > 0$ be chosen such that

$$B = B_r(Y_0) \subset D.$$

Then, according to the proof of (γ), it is sufficient to show that the function

$$g(Y) = \int_B \frac{d\lambda_{m+n}(X)}{|X-Y|^{d-2}} \quad (Y \in B)$$

is in $C^2(B)$ with $\Delta g = d(2-d)\omega_d$.

To prove this, we apply Lemma 4.2 of Gilbarg and Trudinger [5] to $f = \chi_B$. This yields $g \in C^2(B)$ and $\Delta g = d(2-d)\omega_d$ in B. □

REMARK. In applying [5, Lemma 4.2], note that Gilbarg and Trudinger consider $(1/d(2-d)\omega_d)(1/|X-Y|^{d-2})$ as a Newtonian kernel, whereas we consider $1/|X-Y|^{d-2}$.

LEMMA 7. *Let $D \subset \mathbb{R}^{m+n}$ be a regular open set containing $\{(0,\dots,0)\} \times \mathbb{R}^n$. Assume that $D \subset Z_{s-1}$ for some $s > 1$, and let*

$$\int_D h(x,\xi)\, d\lambda_{m+n}(x,\xi) = \lambda_m(K_1) \cdot \int_{\mathbb{R}^n} h(0,\xi)\, d\lambda_n(\xi) \tag{4.5}$$

for all positive, integrable and harmonic functions on D. Let G be the Green function of Z_s for some $s > 1$, and let

$$u(Y) = \int_D G(X,Y)\, d\lambda_{m+n}(X) \quad \textit{for } Y \in Z_s.$$

Then, for $Y = (y, \eta) \in Z_s \setminus D$, we have

$$(\zeta) \qquad u(Y) = c_{m,n} \cdot \omega_m \cdot \begin{cases} \log(s/|y|) & \textit{for } m = 2 \\ 1/|y|^{m-2} - 1/s^{m-2} & \textit{for } m \geq 3 \end{cases}$$

and

$$(\eta) \qquad \operatorname{grad} u(Y) = c_{m,n} \cdot \omega_m \cdot \begin{cases} -(1/|y|^2)(y,0) & \textit{for } m = 2, \\ ((2-m)/|y|^m)(y,0) & \textit{for } m \geq 3. \end{cases}$$

Proof. (ζ) Denote by g the Green function of the m-dimensional ball K_s with center 0 and radius $s > 1$. Then, by Theorem 5,

$$\int_{\mathbb{R}^n} G((x,\xi),(y,\eta))\, d\lambda_n(\xi) = c_{m,n} \cdot g(x,y). \tag{4.6}$$

By Helms [10, p. 77],

$$g(0,y) = \begin{cases} \log(s/\sqrt{y_1^2 + y_2^2}) & \text{for } m = 2, \\ 1/|y|^{m-2} - 1/s^{m-2} & \text{for } m \geq 3. \end{cases} \tag{4.7}$$

for $y = (y_1, \dots, y_m) \in K_s \setminus \{(0,\dots,0)\}$.

Let $Y \in Z_s \setminus D$ be fixed. Then $h_Y(X) = G(X,Y)$ is positive, harmonic, and integrable on D. By our assumption we obtain

$$\begin{aligned} u(Y) &= \int_D h_Y(X)\, d\lambda_{m+n}(X) \\ &= \omega_m \int_{\mathbb{R}^n} h_Y(0,\xi)\, d\lambda_n(\xi) && \text{(by (4.5))} \\ &= \omega_m \int_{\mathbb{R}^n} G((0,\xi),(y,\eta))\, d\lambda_n(\xi) \\ &= c_{m,n} \cdot \omega_m \cdot g(0,y) && \text{(by (4.6))} \\ &= c_{m,n} \cdot \omega_m \cdot \begin{cases} \log(s/\sqrt{y_1^2 + y_2^2}) & \text{for } m = 2, \\ 1/|y|^{m-2} - 1/s^{m-2} & \text{for } m \geq 3. \end{cases} && \text{(by (4.7))} \end{aligned}$$

(η) Let at first $Y \in Z_s \setminus \bar{D}$. This set is open, and hence we obtain (η) by differentiating (ζ). By Lemma 6(γ), $u \in C^1(Z_s)$, and so (η) holds also for all points that belong to the closure of $Z_s \backslash \bar{D}$ in Z_s, that is, for all points $y \in Z_s \backslash (\bar{D})^0$. But this means $y \in Z_s \backslash D$, since D is a regular open set. □

LEMMA 8. *With the same assumptions as in Lemma 7, for* $Y = (y_1, \dots, y_m, \eta_1, \dots, \eta_n)$ *we have*

$$(\vartheta) \qquad \frac{\partial u}{\partial \eta_j}(Y) = 0 \quad \text{for } Y \in \bar{D},\ 1 \le j \le n,$$

$$(\kappa) \qquad \operatorname{grad} u(Y) = -\frac{d(d-2)\omega_d}{m}(y, 0) \quad \text{for } Y \in \bar{D}.$$

Proof. We shall show that, for $1 \le j \le n$,

$$\frac{\partial u}{\partial \eta_j} \in C(\bar{D}) \cap H(D), \tag{4.8}$$

$$\frac{\partial u}{\partial \eta_j} = 0 \quad \text{on } \partial D, \tag{4.9}$$

$$\frac{\partial u}{\partial \eta_j} \text{ is bounded on } D. \tag{4.10}$$

Then Lemma 4 implies (ϑ). In order to show (4.8)–(4.10), we define

$$v(Y) = u(Y) + \frac{d(d-2)\omega_d}{2m} \cdot (y_1^2 + \cdots + y_m^2) \quad \text{for } Y \in Z_s. \tag{4.11}$$

Then $v \in C^2(D)$ according to Lemma 6(δ). In addition,

$$\begin{aligned} \Delta v &= \Delta u + d(d-2)\omega_d && \text{(by (4.11))} \\ &= 0 \quad \text{on } D. && \text{(by Lemma 6}(\epsilon)) \end{aligned} \tag{4.12}$$

From (4.11) and (4.12), we obtain that

$$\frac{\partial u}{\partial \eta_j} = \frac{\partial v}{\partial \eta_j} \text{ is harmonic on } D \quad \text{for } 1 \le j \le n. \tag{4.13}$$

Since $\bar{D} \subset Z_s$, we obtain by Lemma 6(γ) that

$$\frac{\partial u}{\partial \eta_j} \in C(\bar{D}) \quad \text{for } 1 \le j \le n. \tag{4.14}$$

Hence (4.8) is satisfied according to (4.13) and (4.14). Equation (4.9) is fulfilled by Lemma 7(η) because $\partial D \subset Z_s \backslash D$. In order to complete the proof of (ϑ) it remains only to show (4.10).

Consider the half-space

$$H = (-s-1, \infty) \times \mathbb{R}^{d-1},$$

and let G_H be the Green function of H. Since $Z_s \subset H$ implies $G \le G_H$, we have

$$G_H(X, Y) - G(X, Y) \ge 0 \quad \text{on } Z_s \times Z_s. \tag{4.15}$$

For $Y \in Z_s$ we split u into two parts:

$$u(Y) = \int_D G(X,Y)\, d\lambda_{m+n}(X) = -w_1(Y) + w_2(Y),$$

with

$$w_1(Y) = \int_D (G_H(X,Y) - G(X,Y))\, d\lambda_{m+n}(X), \tag{4.16}$$

$$w_2(Y) = \int_D G_H(X,Y)\, d\lambda_{m+n}(X). \tag{4.17}$$

In order to prove (4.10), we show that

$$\frac{\partial w_1}{\partial \eta_j} \text{ and } \frac{\partial w_2}{\partial \eta_j} \text{ are bounded in } D \quad \text{for } 1 \le j \le n. \tag{4.18}$$

To this end we use the following result (see Hayman and Kennedy [9, p. 37, Example 1]):

Let $h: B_r(Y_0) \to \mathbb{R}$ be a nonnegative harmonic function; then the partial derivatives satisfy

$$\left| \frac{\partial h}{\partial \eta_j}(Y_0) \right| \le \frac{d}{r} \cdot h(Y_0) \quad \textit{for } 1 \le j \le n. \tag{4.19}$$

The Green function of the half-space H is given by

$$G_H = V_1 - V_2 \tag{4.20}$$

with

$$V_1(X,Y) = \frac{1}{|X-Y|^{d-2}} \tag{4.21}$$

and

$$V_2(X,Y) = \frac{1}{|X-Y^*|^{d-2}}, \tag{4.22}$$

where Y^* is the mirror image of Y with respect to $\partial H = \{(-s-1)\} \times \mathbb{R}^{d-1}$; that is, for $Y = (y_1, \ldots, y_m, \eta_1, \ldots, \eta_n)$ we have

$$Y^* = (-y_1 - 2s - 2, y_2, \ldots, y_m, \eta_1, \ldots, \eta_n).$$

Now we prove (4.18). According to (4.15) and (4.16), we have $w_1 \ge 0$ on Z_s; w_1 is also harmonic in Z_s by Helms [10, Lemma 6.7]. For any $Y_0 \in Z_{s-1}$ and $1 \le j \le n$,

$$\left| \frac{\partial w_1}{\partial \eta_j}(Y_0) \right| \le \frac{d}{s-(s-1)} w_1(Y_0) \qquad \text{(by (4.19))}$$

$$\le d \cdot \int_D G_H(X, Y_0)\, d\lambda_{m+n}(X) \le$$

$$\le d\cdot\int_{[-s+1,s-1]\times\mathbb{R}^{d-1}} G_H(X,Y_0)\,d\lambda_{m+n}(X)$$

$$= d\cdot\int_{[-s+1,s-1]\times\mathbb{R}^{d-1}} G_H(X,(y_1^{(0)},0,\dots,0))\,d\lambda_{m+n}(X) \tag{4.23}$$

by an appropriate change of variables. The function

$$[-s+1,s-1]\ni y_1^{(0)}\mapsto \int_{[-s+1,s-1]\times\mathbb{R}^{d-1}} G_H(X,(y_1^0,0,\dots,0))\,d\lambda_{m+n}(X)$$

is continuous, and hence bounded. Thus (4.23) implies (4.18) for w_1.

Now we consider (4.18) for w_2. Let $Y_0\in Z_{s-1}$. Then, by (4.17) and (4.20) with $B=B_1(y_0)$, we have

$$w_2(Y)=\int_{D\setminus B}(V_1-V_2)(X,Y)\,d\lambda_{m+n}(X)+\int_{B\cap D}(V_1-V_2)(X,Y)\,d\lambda_{m+n}(X).$$

Since the function

$$B\ni Y\mapsto h_1(Y)=\int_{D\setminus B}(V_1-V_2)(X,Y)\,d\lambda_{m+n}(X)$$

is harmonic and nonnegative (by (4.21) and (4.22)), we conclude from (4.19) with $r=1$ for $1\le j\le n$ that

$$\left|\frac{\partial h_1}{\partial\eta_j}(Y_0)\right|\le d\cdot h_1(Y_0)$$

$$\le d\cdot\int_{[-s+1,s-1]\times\mathbb{R}^{d-1}} G_H(X,(y_1^{(0)},0,\dots,0))\,d\lambda_{m+n}(X). \tag{4.24}$$

Similarly $h_2(Y)=\int_{B\cap D}V_2(X,Y)\,d\lambda_{m+n}(X)$ is harmonic and nonnegative on B. Hence, by (4.19), we obtain

$$\left|\frac{\partial h_2}{\partial\eta_j}(Y_0)\right|\le d\cdot h_2(Y_0)\le d\cdot\int_B V_2(X,Y_0)\,d\lambda_{m+n}(X)$$

$$\le d\cdot C\cdot\lambda_d(B)\quad\text{for } 1\le j\le n, \tag{4.25}$$

where C is an upper bound of V_2 on $Z_s\times Z_s$.

Finally, by [5, Lemma 4.1], we obtain

$$\left|\frac{\partial}{\partial\eta_j}\int_{B\cap D}V_1(X,Y)\,d\lambda_{m+n}(X)\right|_{Y=Y_0}$$

$$\le\int_{B\cap D}\left|\frac{\partial}{\partial\eta_j}V_1(X,Y)\right|_{Y=Y_0}d\lambda_{m+n}(X)\qquad\text{(by [5, Lemma 4.1])}$$

$$\le(d-2)\int_{B\cap D}\frac{|\xi_j-\eta_j^{(0)}|}{|X-Y_0|^d}\,d\lambda_{m+n}(X)$$

$$\le(d-2)\int_{B\cap D}\frac{d\lambda_{m+n}(X)}{|X-Y_0|^{d-1}}\le\int_{B_1(0)}\frac{d\lambda_{m+n}(X)}{|X|^{d-1}}<\infty. \tag{4.26}$$

Hence, by (4.24), (4.25), and (4.26), we have (4.18) for w_2, and this proves (ϑ).

Let us now turn to assertion (κ). As in (4.11), set

$$v(Y)=\frac{d(d-2)\omega_d}{2m}\cdot(y_1^2+\cdots+y_m^2)+u(Y)\quad \text{for } Y\in Z_s.$$

By Lemma 7(η), for $1\le i\le m$ and $y\in\partial D$ we have

$$\frac{\partial v}{\partial y_i}(Y)=\frac{d(d-2)\omega_d}{m}\cdot y_i-c_{m,n}\cdot\omega_m\cdot\begin{cases} y_i/|y|^2 & \text{for } m=2,\\ (m-2)y_i/|y|^m & \text{for } m\ge 3.\end{cases}\tag{4.27}$$

For $1\le i,k\le m$ we define

$$g_{ik}(Y)=y_k\frac{\partial v}{\partial y_i}(Y)-y_i\frac{\partial v}{\partial y_k}(Y)\quad \text{on } Z_s.$$

Then, according to Lemma 6(γ), $g_{ik}\in C(Z_s)$ with

$$g_{ik}(Y)=0\quad \text{for } Y\in\partial D.\qquad \text{(by (4.27))}\tag{4.28}$$

With Lemma 6(ϵ) we see that v is harmonic in D, and we have

$$\frac{\partial v}{\partial\eta_j}(Y)=0\quad \text{for } 1\le j\le n \text{ and } Y\in\bar D$$

by Lemma 8(ϑ). Hence we obtain

$$\frac{\partial g_{ik}}{\partial\eta_j}=0\quad \text{on } D\tag{4.29}$$

and, by a straightforward calculation,

$$g_{ik}\text{ is harmonic on } D\quad \text{for } 1\le i,k\le m.\tag{4.30}$$

By the same argument as in [2], we can see that D is connected. For a positive, integrable and harmonic function h on D, consider the functions $h_m=h\chi_{D_0}+mh\chi_{D\setminus D_0}$, $m=1,2$, where D_0 is the connected component of D containing $\{(0,\ldots,0)\}\times\mathbb{R}^n$. Since (4.5) is true for both h_1 and h_2, it follows that $D\setminus D_0=\emptyset$.

In addition, g_{ik} is real analytic on D. Hence we obtain by (4.29) that the g_{ik} do not depend on $\eta_1,\eta_2,\ldots,\eta_n$. Since g_{ik} is continuous on $\overline{Z_{s-1}}$ (which follows from Lemma 6(γ)), this implies that g_{ik} is bounded on D. Hence, from (4.28) and (4.30) we conclude that

$$g_{ik}=0 \text{ on } \bar D\quad \text{for } 1\le i,k\le m,$$

by Lemma 4. Now, using the wedge product (see Avci [3]), we have

$$\begin{aligned}(\operatorname{grad}_y v,0)\wedge(y,0)&=\left(\frac{\partial v}{\partial y_1},\ldots,\frac{\partial v}{\partial y_m},0,\ldots,0\right)\wedge(y_1,\ldots,y_m,0,\ldots,0)\\ &=0\quad \text{on } D.\end{aligned}$$

Hence $(\operatorname{grad}_y v,0)$ and $(y,0)$ are linearly dependent; that is,

$$\left(\frac{\partial v}{\partial y_1},\ldots,\frac{\partial v}{\partial y_m},0,\ldots,0\right)=\beta(y_1,\ldots,y_m)\cdot(y_1,\ldots,y_m,0,\ldots,0)$$
$$\text{on } D\setminus\{(0,\tau):\tau\in\mathbb{R}^n\}.$$

A similar calculation as in Avci [3] shows that

$$\beta(y_1, \dots, y_m) = \gamma\left(\sum_{i=1}^{m} y_i^2\right)^{-m/2} \quad \text{on } D \setminus \{(0, \tau) : \tau \in \mathbb{R}^n\},$$

where γ is a constant (note that D is connected). Thus

$$\operatorname{grad}_y v = \frac{\gamma}{|y|^m} \cdot y.$$

Since $\operatorname{grad}_y v \in C(D)$, this can be true only if $\gamma = 0$ (let $|y| \to 0$).

By the definition of v in (4.11), we see that

$$\operatorname{grad} u(Y) = -\frac{d(d-2)\omega_d}{m} \cdot (y, 0)$$

for $Y \in D$ and, by continuity, for $Y \in \bar{D}$. Hence (κ) is proved. □

Now we are ready to give the proof of Theorem 2.

Proof of Theorem 2. Without loss of generality, let $r = 1$. Let $s > 1$ be chosen such that $D \subset Z_{s-1}$. It is sufficient to show that

$$\partial D \subset \partial Z_1. \tag{4.31}$$

Since $D \subset Z_{s-1}$, (4.31) implies $\bar{D} = \bar{Z}_1$, and since D is a regular open set, $D = Z_1$.

Let $G(X, Y)$ be the Green function of Z_s. We define

$$u(Y) = \int_D G(X, Y)\, d\lambda_{m+n}(X) \quad \text{for } Y \in Z_s.$$

Since $r = 1$, the assumptions of Lemmas 7 and 8 are fulfilled. According to Lemma 8(κ), we have

$$\operatorname{grad} u(Y) = -\frac{d(d-2)\omega_d}{m}(y, 0) \quad \text{for } Y \in \bar{D}. \tag{4.32}$$

Lemma 7(η) yields

$$\operatorname{grad} u(Y) = c_{m,n} \cdot \omega_m \cdot \begin{cases} -(1/(y_1^2 + y_2^2))(y_1, y_2, 0) & \text{for } m = 2 \\ ((2-m)/|y|^m)(y, 0) & \text{for } m \geq 3 \end{cases} \tag{4.33}$$

for $Y \in Z_s \setminus D$.

Now let $m \geq 3$. Then, from (4.32) and (4.33), we have

$$-\frac{d(d-2)\omega_d}{m} = \frac{c_{m,n} \cdot \omega_m \cdot (2-m)}{|y|^m}$$

for any boundary point $Y = (y, \eta) \in \partial D$. This gives

$$|y|^m = \frac{d(d-2) \cdot \omega_d}{m(m-2)\omega_m} \cdot \frac{m(m-2)\omega_m}{d(d-2)\omega_d} = 1;$$

that is, $Y \in \partial Z_1$.

In the case $m = 2$ we have, for all $Y = (y, \eta) \in \partial D$,

$$-\frac{d(d-2)\omega_d}{2} = -\frac{c_{2,d-2} \cdot \omega_2}{|y|^2}.$$

Hence

$$|y|^2 = \frac{d(d-2)\omega_d}{2\pi} \cdot \frac{2\pi}{d(d-2)\omega_d} = 1;$$

that is, $Y \in \partial Z_1$, too. Thus, in either case $D = Z_1$. □

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