On the Equivalence of Holomorphic and Plurisubharmonic Phragmén–Lindelöf Principles

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There are several papers in recent years which studied partial differential operators $P(D)$ on classes of infinitely differentiable functions on convex open sets in $\mathbb{R}^N$ or $\mathbb{C}^N$ in terms of Phragmén–Lindelöf type estimates for plurisubharmonic functions on algebraic varieties. In the early work of Hörmander [7] it is shown that the surjectivity of $P(D)$ on the space of all real analytic functions on a convex open set in $\mathbb{R}^N$ is equivalent to Phragmén–Lindelöf conditions on the tangent cone at infinity of the variety $V(P) := \{z \in \mathbb{C}^N \mid P(z) = 0\}$. Later Zampieri [13], Braun, Meise, and Vogt [3; 4], and Braun [1] made similar investigations for classes of ultradifferentiable functions. Kaneko [8] proved that Hartogs problems for partial differential operators $P(D)$ can be characterized by Phragmén–Lindelöf conditions on $V(P)$. To treat the problem of existence of continuous linear right inverses for partial differential operators, Meise, Taylor, and Vogt [9], Momm [11], and Palamodov [12] also used Phragmén–Lindelöf conditions. In most of the aforementioned cases one first derives the Phragmén–Lindelöf conditions for all plurisubharmonic functions $u = \log|f|$, where f is a holomorphic function on $V(P)$. Meise, Taylor, and Vogt [10] proved a general result which shows that the conditions for all plurisubharmonic functions of type $u = \log|f|$, where f is a holomorphic function on $V(P)$, hold if and only if the conditions hold for all plurisubharmonic functions on $V(P)$. The idea is to write the plurisubharmonic function u as an upper envelope of functions $\log|f|$. More precisely, they have shown that for each $0 < \theta < 1$ and for each plurisubharmonic function u on the variety $V(P)$ with $u(z) \le |z|$, and for most of the regular points $\zeta \in V_{\text{reg}}(P)$, there exists a holomorphic function f on $V(P)$ such that

$$\log|f(z)| \le \sup\{u(y) \mid |z-y| \le 1\} + C\log(2+|z|), \quad z \in V(P)$$

and

$$\log|f(\zeta)| \ge \theta u(\zeta) - C\log(2+|\zeta|),$$

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where C is independent of u. In the present paper we show that $\theta = 1$ can be achieved if one modifies the proof of Meise, Taylor, and Vogt [10] appropriately. The motivation for doing this comes from the article by Franken and Meise [5], in which holomorphic Phragmén–Lindelöf conditions APL(K, Q) and APL$'(K, Q)$ (see Definition 9 below) are used to characterize when zero-solutions of $P(D)$ on a given compact and convex set $K \subset \mathbb{R}^N$ can be extended as zero-solutions to a larger compact and convex set $Q \subset \mathbb{R}^N$. The essential difference with [10] is that we use a modified version of their Lemma 3.4, given as Lemma 2 of the present paper. In order to prove our main result, applying Lemma 2 we use the growth estimates of the function u at one more place in our proof. At the end of the paper we show the equivalence of the holomorphic conditions APL(K, Q) and APL$'(K, Q)$ to the plurisubharmonic conditions PL(K, Q) and PL$'(K, Q)$ (see Theorem 10).

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1. LEMMA. *Let h be harmonic in $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and continuous on $\bar{\mathbb{D}}$. Then*

$$|h(z_1) - h(z_2)| \le 256(\max_{|\xi|=1}|h(\xi)|)|z_1 - z_2|, \quad |z_1|, |z_2| \le 1/2.$$

Proof. We use Poisson's integral formula

$$h(z) = \frac{1}{2\pi}\int_{|\xi|=1} \frac{1-|z|^2}{|z-\xi|^2} h(\xi)\, d\sigma(\xi), \quad |z| < 1.$$

For $|z_1|, |z_2| \le 1/2$ and $|\xi| \ge 1$,

$$\begin{aligned}
&\left|\frac{1-|z_1|^2}{|\xi - z_1|} - \frac{1-|z_2|^2}{|\xi - z_2|}\right| \\
&\quad \le \left|\frac{(1-|z_1|^2)|\xi - z_2|^2 - (1-|z_2|^2)|\xi - z_1|^2}{|\xi - z_1|^2|\xi - z_2|^2}\right| \\
&\quad \le 16|(1-|z_1|^2)|\xi - z_2|^2 - (1-|z_1|^2)|\xi - z_1|^2 \\
&\qquad + (1-|z_1|^2)|\xi - z_1|^2 - (1-|z_2|^2)|\xi - z_1|^2| \\
&\quad \le 16((1-|z_1|^2)||\xi - z_2|^2 - |\xi - z_1|^2| + |\xi - z_1|^2||z_1|^2 - |z_2|^2|) \\
&\quad \le 256|z_1 - z_2|.
\end{aligned}$$

This proves Lemma 1. □

2. LEMMA. *Let u be subharmonic on a neighborhood of $z \in \mathbb{C}$, where $|z| \le s$ and $0 < s \le 1$. Suppose also that $(1/2\pi)\int_{|\zeta| \le s} \Delta u(\zeta)\, d\lambda(\zeta) \le 1$. Then there exists a number $C > 0$, independent of u and s, such that for all $0 < r \le 1/2$,*

$$\frac{1}{\pi(sr)^2}\int_{|\zeta| \le sr} e^{-u(\zeta)}\, d\lambda(\zeta) \le C \exp\left[Cr\left(\max_{|\xi|=s}|u(\xi)|\right)\right] e^{-u(0)}.$$

Proof. It is no loss of generality to take $s = 1$. Then we can write $u = h + p$, where h is harmonic in $|z| < 1$ and equal to u on $|z| = 1$; p is the function

$$p(\zeta) = \frac{1}{2\pi} \int_{|z|<1} \log \left| \frac{z-\zeta}{1-\bar{\zeta}z} \right| d\mu(z), \quad |\zeta| < 1,$$

where $d\mu(z) = \Delta u(z)\, d\lambda(z)$. With Lemma 1 we get

$$\begin{aligned} &\frac{1}{\pi r^2} \int_{|\zeta|\le r} e^{-u(\zeta)}\, d\lambda(\zeta) \\ &\quad = \frac{1}{\pi r^2} \int_{|\zeta|\le r} e^{-h(0)+h(0)-h(\zeta)-p(\zeta)}\, d\lambda(\zeta) \\ &\quad \le \frac{1}{\pi r^2} \int_{|\zeta|\le r} e^{-p(\zeta)+p(0)}\, d\lambda(\zeta) \exp\left[512r\left(\sup_{|\xi|=1} |u(\xi)|\right)\right] e^{-u(0)}. \end{aligned}$$

Therefore it suffices to show that there exists a number $C \ge 1$ such that, for all $0 < r \le 1/2$,

$$\frac{1}{\pi r^2} \int_{|\zeta|\le r} e^{-p(\zeta)+p(0)}\, d\lambda(\zeta) \le C. \tag{1}$$

To show this claim (1) set $a := (1/2\pi)\int_{|z|\le 1} d\mu(z) \le 1$. We get the estimate:

$$\begin{aligned} \frac{1}{\pi r^2} \int_{|\zeta|\le r} e^{-p(\zeta)+p(0)}\, d\lambda(\zeta) &\le \frac{1}{\pi r^2} \int_{|\zeta|\le r} \int_{|z|<1} \left| \frac{1-\bar{\zeta}z}{\zeta - z} \right|^a |z|^a \frac{d\mu(z)}{2\pi a}\, d\lambda(\zeta) \\ &= \int_{|z|<1} \left[\frac{|z|^a}{\pi r^2} \int_{|\zeta|\le r} \left| \frac{1-\bar{\zeta}z}{\zeta - z} \right|^a d\lambda(\zeta) \right] \frac{d\mu(z)}{2\pi a}. \end{aligned}$$

By the proof of Meise, Taylor, and Vogt [10, 3.4], the following two estimates hold:

$$\begin{aligned} \frac{|z|^a}{\pi r^2} \int_{|\zeta|\le r} \left| \frac{1-\bar{\zeta}z}{\zeta - z} \right|^a d\lambda(\zeta) &\le \frac{|z|^a}{\pi r^2} 2^{a+1} \frac{2\sqrt{3}}{3} \frac{r}{|z|} 2|z|^{1-a} 2r \\ &\le \frac{2^{a+4}\sqrt{3}}{3\pi} \quad \text{for } |z| \ge 2r; \end{aligned}$$

$$\begin{aligned} \frac{|z|^a}{\pi r^2} \int_{|\zeta|\le r} \left| \frac{1-\bar{\zeta}z}{\zeta - z} \right|^a d\lambda(\zeta) &\le \frac{2^{a+1}\pi}{2-a} (3r)^{2-a} \frac{(2r)^a}{\pi r^2} \\ &= \frac{2^{2a+1} 3^{2-a}}{2-a} \quad \text{for } |z| \le 2r. \end{aligned}$$

Thus we have established the claim (1). If we choose $C \ge 512$, the proof is complete. □

3. PROPOSITION. *There exist constants $\Delta, C > 0$, depending only on the dimension n, such that for all plurisubharmonic functions ψ on $|\zeta - z| \le \delta$ with*

$$\frac{1}{|B(z,\delta)|} \int_{B(z,\delta)} \psi(\zeta)\, d\lambda(\zeta) \le \psi(z) + \Delta,$$

one also has

$$\frac{1}{|B(z,r\delta/2)|}\int_{B(z,r\delta/2)} e^{-2\psi(\zeta)}\,d\lambda(\zeta) \le C\exp\left[Cr\left(\sup_{|\xi-z|=\delta/2}|\psi(\xi)|\right)\right]e^{-2\psi(z)}$$

for all $0<r\le 1/2$.

Proof. The proof of Proposition 3 is word-for-word the same as the proof of Meise, Taylor, and Vogt [10, 3.3], except that in the last inequality one uses Lemma 2. □

Let Ω be an open subset of $\mathbb{C}^N$ and let ψ be a plurisubharmonic function in Ω. For $z\in\mathbb{C}^N$ and $\epsilon>0$ we set $B(z,\epsilon):=\{\xi\in\mathbb{C}^N\,|\,|z-\xi|<\epsilon\}$. Then $|B(z,\epsilon)|=\pi^n\epsilon^{2n}/n!$. If $z_0\in\Omega$ and $0<\epsilon<\operatorname{dist}(z_0,\mathbb{C}^N\backslash\Omega)$, then let

$$A(z_0,\epsilon)=\epsilon\left\{\frac{1}{|B(z_0,\epsilon)|}\int_{B(z_0,\epsilon)} e^{-2\psi(\zeta)}\,d\lambda(\zeta)\right\}^{-1/2}.$$

In the proof of the main result we use the following proposition, which can be proved using the standard arguments in Hörmander [6, Chap. 4].

4. PROPOSITION. *Let* $\Omega\subset\mathbb{C}^N$ *be pseudoconvex. Then there exists a constant* $C>0$ *such that, for each* $z_0\in\Omega$ *and* $0<\epsilon\le\min(\operatorname{dist}(z_0,\mathbb{C}^N\backslash\Omega),\frac{1}{2}(1+|z_0|))$, *there exists a function* $f\in A(\Omega)$ *with*

(i) $$f(z_0)=A(z_0,\epsilon).$$

Moreover, f satisfies the following estimates:

(ii) $$\int_\Omega \frac{|f(z)|^2e^{-2\psi(z)}}{(1+|z|^2)^{3n+1}}\,d\lambda(z)\le C^2$$

and

(iii) $$|f(z)|\le C\epsilon^{-n}(1+|z|)^{3n+1}\exp[\tilde{\psi}(z,\epsilon)],$$

where $\tilde{\psi}(z,\epsilon):=\max\{\psi(z+\zeta)\,|\,|\zeta|\le\epsilon\}$.

We recall some notation from Meise, Taylor, and Vogt [10].

5. SPECIAL COORDINATES. In order to formulate the main result of this paper, we introduce special coordinates in $\mathbb{C}^N$. In the sequel we denote by V a pure k-dimensional algebraic variety in $\mathbb{C}^N$. Let $P_1,\dots,P_l$ be a set of generators of the ideal $I(V)$ in $\mathbb{C}[z_1,\dots,z_N]$ associated with the variety V. We define $|P(z)|^2=|P_1(z)|^2+\cdots+|P_l(z)|^2$, $z\in\mathbb{C}^N$. After a real linear change of coordinates, we can assume that, for the coordinates $z=(s,w)\in\mathbb{C}^{N-k}\times\mathbb{C}^k$, the following holds:

$$V\subset\{z=(s,w)\in\mathbb{C}^N\,|\,|s|\le\tilde{C}(1+|w|)\}$$

for some constant $\tilde{C}>0$. Moreover, there exists a polynomial $D(w)$ in the w coordinate so that the projection map onto the w coordinate is an m-sheeted covering over the set $\{w\in\mathbb{C}^k\,|\,D(w)\neq 0\}$; that is, we have

$$V = \{(s_j(w), w) \mid 1 \le j \le m\}.$$

The functions $s_j(w)$ are all distinct when $D(w) \neq 0$. After a further real linear change of coordinates we can assume that, for $w \in \mathbb{C}^k$ with $D(w) \neq 0$, the polynomial D has the form

$$D(w) = \prod_{j \neq k} (\pi_1(s_j(w)) - \pi_1(s_k(w))),$$

where π_1 denotes the first coordinate function of π. For $\delta, C > 0$ we set

$$S_0 := S_0(\delta, C) := \{z = (s, w) \in \mathbb{C}^N \mid |D(w)| < \delta(1+|w|)^{-C}\}. \tag{2}$$

Moreover, for $A_1, B_1 > 0$ we define the set

$$\Omega := \Omega(A_1, B_1) := \{z = (s, w) \in \mathbb{C}^N \mid \varphi_{A_1, B_1}(z) < 0, D(w) \neq 0\}, \tag{3}$$

where the function φ_{A_1, B_1} is defined by

$$\varphi_{A_1, B_1}(z) := \log|P(z)| + A_1\left\{\log\left(\frac{1}{|D(w)|}\right) + B_1 \log(2+|z|)\right\}.$$

6. Definition. Let V be an analytic variety. A function $u: V \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* if u is plurisubharmonic in the regular points V_{reg} of V and locally bounded on V. In order that u be upper semicontinuous on the singular points V_{sing} of V, we set

$$u(\zeta) = \limsup_{V_{\text{reg}} \ni z \to \zeta} u(z), \quad \zeta \in V_{\text{sing}}.$$

From Meise, Taylor, and Vogt [10] we recall the following proposition.

7. Proposition. *There exist constants $\delta, C, A_1, B_1 > 0$ such that the following six conditions are satisfied when S_0 and Ω are defined as in* (2) *and* (3).

(i) *Ω is pseudoconvex and $\Omega \supset V \cap \{z = (s, w) \in \mathbb{C}^N \mid D(w) \neq 0\}$.*

(ii) *For each $z = (s, w)$ in Ω there exists a unique number $j \in \mathbb{N}$ with $1 \le j \le m$ such that $|s - s_j(w)| < \min_{k \neq j} |s - s_k(w)|$.*

(iii) *The map $\rho: \Omega \to \Omega \cap V$ given by $\rho(s, w) = (s_j(w), w)$, where j is as in* (ii), *is a holomorphic retract of Ω into V.*

(iv) *If u is plurisubharmonic on V, then for all $z \in V \cap S_0$,*

$$u(z) \le \max\{u(\zeta) \mid \zeta \in V \setminus S_0, |\zeta - z| \le 1\}.$$

(v) *There are numbers $\epsilon_1, C_1 > 0$ such that for all $z = (s_j(w), w) \in V \setminus S_0$ and $\epsilon(z) = \epsilon_1(1+|w|)^{-C_1}$:*

(a) *$z(\tau) = (s_j(w+\tau), w+\tau), w+\tau) \in \Omega$ for all $|\tau| \le 8\epsilon(z)$;*

(b) *$B(z(\tau), 8\epsilon(z)) \subset \Omega$ for all $|\tau| \le 8\epsilon(z)$; and*

(c) *$|z(\tau) - z| \le 1$ for all $|\tau| \le 8\epsilon(z)$.*

(vi) *There exist constants $C_2, C_3 > 0$ such that:*

(a) *$\varphi(w) = C_2\{\log|D(w)| + C_3 \log(2+|w|)\} \ge 0$ whenever $z \in V \setminus S_0$, $(s, w) \in B(z(\tau)\, \epsilon(z))$, and $z(\tau)$ is as in part* (v); *and*

(b) *if* $f \in A(\Omega)$ *and* $\int_\Omega |f|^2 e^{-2(\varphi+u)}\,d\lambda < \infty$ *for some function* u, *locally bounded on* $\bar{\Omega}$, *then there exists an entire function* F *on* $\mathbb{C}^N$ *such that* $F|_V = f$ *and* $F = 0$ *on* $V \cap \{z \in \mathbb{C}^N \mid D(w) = 0\}$.

We introduce some more notation. For a point $z = (s_j(w), w) \in V \setminus S_0$ we define, with $\epsilon(z)$ as in (v) of Proposition 7,

$$B = B(0, \epsilon(z)) = \{\tau \in \mathbb{C}^k \mid |\tau| < \epsilon(z)\}. \tag{4}$$

Now we can formulate the main result of this paper.

8. THEOREM. *There exists a constant* $C_4 > 0$, *depending only on* V *and the choices of the constants in Proposition 7, such that for each plurisubharmonic function on the regular points* V_{reg} *of* V *satisfying* $|u(z)| \le L|z|$, *and for all* $z = (s_j(w), w) \in V \setminus S_0$, *there exists a subset* E *of* B, *where* B *is defined as in* (4), *with*

(i) $|E| \le |B| \max(1, L)(1+|z|)^{-2}$;

furthermore, for all $\tau \in B \setminus E$ *there exists an entire function* f_τ *on* $\mathbb{C}^N$ *such that*

(ii) $\log|f_\tau(z(\tau))| \ge u(z(\tau)) - C_4 \log(2+|z(\tau)|)$ *with* $z(\tau)$ *as in part* (v) *of Proposition 7, and*

(iii) $\log|f_\tau(\zeta)| \le \max\{u(\zeta') \mid \zeta' \in V,\ |\zeta - \zeta'| \le 1\} + C_4 \log(2+|\zeta|)$ *for all* $\zeta \in V$.

Proof. Let $0 < \delta, C, A_1, B_1, C_2, C_3, \Omega \subset \mathbb{C}^N$ with ρ as in Proposition 7. We define, for $z \in \Omega$,

$$\psi(s, w) = u \circ \rho(s, w) + C_2\{\log|D(w)| + C_3 \log(2+|w|)\}.$$

Now let $z = (s_j(w), w) \in V \setminus S_0$ be arbitrarily given. Set $r := \frac{1}{2}(2+|z|)^{-1}$. We use a number $0 < \delta < \epsilon(z)/4$ which will be fixed at the end of the proof. For $\tau \in B$, we denote by $f_\tau \in A(\Omega)$ the holomorphic function given in Proposition 4 which satisfies Proposition 4(i) at $z(\tau)$. Because of Propositions 4(ii) and 7(vi)(b), the function f_τ extends to a holomorphic function on $\mathbb{C}^N$. Let $\Delta > 0$ be the number in Proposition 3. Let E denote the set of all points $\zeta = z(\tau)$ in B not satisfying the estimate

$$\psi_\delta(\zeta) := \frac{1}{|B(\zeta, \delta)|} \int_{B(\zeta, \delta)} \psi(\xi)\, d\lambda(\xi) \le \psi(\zeta) + \Delta.$$

Let $C' > 0$ be the constant in Proposition 3. By Propositions 3 and 4(i),

$$\begin{aligned}
&\log|f_\tau(z(\tau))| \\
&\quad = \log A\left(z(\tau), \frac{r\delta}{2}\right) \\
&\quad \ge \log\left(\frac{r\delta}{2}\right) - \frac{1}{2}\log(C') - \frac{C'r}{2}\left(\sup_{|\xi - z(\tau)| = \delta/2} |\psi(\xi)|\right) + \psi(z(\tau)).
\end{aligned} \tag{5}$$

In order to estimate the supremum of $|\psi|$, let $\xi=(s_\xi, w_\xi)\in\mathbb{C}^N$ with $|\xi-z(\tau)|=\delta/2$ and let $\tau_\xi = w_\xi - w$. Then

$$|\tau_\xi|\le|\tau|+|\tau-\tau_\xi|\le\epsilon(z)+\delta/2\le\epsilon(z)(1+1/8)\le 2\epsilon(z).$$

There exists a number $E_1\ge 1$ such that, with Propositions 7(v)(c) and 7(vi)(a), the following holds:

$$\begin{aligned}|\psi(\xi)| &\le |u\circ\rho(\xi)|+C_2|\{\log|D(w)|+C_3\log(2+|w_\xi|)\}| \\ &= |u(z(\tau_\xi))|+E_1\log(2+|w_\xi|)\le L|z(\tau_\xi)|+E_1|z(\tau_\xi)|+E_1 \\ &\le (L+E_1)(|z|+|z-z(\tau_\xi)|+1)\le(L+E_1)(|z|+2).\end{aligned}$$

This implies with the definition of r that

$$r\cdot \sup_{|\xi-z(\tau)|=\delta/2}|\psi(\xi)|\le\frac{L+E_1}{2}. \tag{6}$$

By (5) and (6) there exists a constant $C''>0$, independent of u and z, such that

$$\log|f_\tau(z(\tau))|\ge\psi(z(\tau))+\log\left(\frac{\delta}{1+|z(\tau)|}\right)-C''. \tag{7}$$

From the proof of Meise, Taylor, and Vogt [10, 5.1], we get

$$|E|\le\max(1,L)|B|(1+|z|)^{-2}$$

whenever

$$\delta=\frac{\epsilon(z)}{C'''(1+|z|)^3}$$

for a sufficiently large number $C'''>0$. This implies (i). With the choice of δ we obtain from (7) that (ii) holds with a sufficiently large number $C_4>0$. By Propositions 4 and 7(iv) and the definition of S_0 we get (iii). This completes the proof of Theorem 8. □

Next we apply Theorem 8 to certain kinds of Phragmén-Lindelöf conditions.

9. Definition. Let $Q, K\subset\mathbb{R}^N$ be compact and convex sets with $K\subset Q$. Moreover, let $V\subset\mathbb{C}^N$ be an algebraic variety. The support function H_K of the set K is defined by

$$H_K(y):=\sup_{x\in K}\langle x,y\rangle,\quad y\in\mathbb{R}^N.$$

(a) We say that V satisfies the Phragmén-Lindelöf condition PL(K, Q) if for each $k\ge 1$ there exist $l\ge 1$ and $C>0$ such that, for all plurisubharmonic functions u on V, the conditions (1) and (2) imply (3), where:

(1) $u(z)\le H_K(\mathrm{Im}(z))+O(\log(1+|z|))$, $z\in V$;
(2) $u(z)\le H_Q(\mathrm{Im}(z))+k\log(1+|z|)$, $z\in V$; and
(3) $u(z)\le H_K(\mathrm{Im}(z))+l\log(1+|z|)+C$, $z\in V$.

The algebraic variety V satisfies APL(K, Q) if the above implications hold for all plurisubharmonic functions $u = \log|f|$, where f is a holomorphic function on V.

(b) We say that the variety satisfies PL$'(K, Q)$ if for each $l \geq 1$ there exist $k \geq 1$ and $C > 0$ such that, for each plurisubharmonic function u on V, the conditions (1′) and (2′) imply (3′), where:

(1′) $u(z) \leq H_K(\mathrm{Im}(z)) - j\log(1+|z|) + O(1)$, $z \in V$, for all $j \geq 1$;
(2′) $u(z) \leq H_Q(\mathrm{Im}(z)) - k\log(1+|z|)$, $z \in V$; and
(3′) $u(z) \leq H_K(\mathrm{Im}(z)) - l\log(1+|z|) + C$, $z \in V$.

V satisfies APL$'(K, Q)$ if the above implications hold for all plurisubharmonic functions $u = \log|f|$, where f is holomorphic on V.

10. THEOREM. *Let V be a pure k-dimensional algebraic variety in $\mathbb{C}^N$ and let $K \subset Q \subset \mathbb{R}^N$, be compact sets with nonempty interior. Then we have*

(a) PL(K, Q) *is equivalent to* APL(K, Q) *and*
(b) PL$'(K, Q)$ *is equivalent to* APL$'(K, Q)$.

Proof. The idea of the proof of Theorem 10 is the same as in [10, 2.3]. To show (a), assume that APL(K, Q) holds. Moreover, let u be a plurisubharmonic function on V satisfying Definitions 9(1) and (2). Because of the estimate in Proposition 7(iv), we need only prove the estimate of Definition 9(3) at points $z = (s_j(w), w) \in V \setminus S_0$. Let $\epsilon(z)$, B and $z(\tau)$ be defined as above. By the subaveraging property for plurisubharmonic functions,

$$u(z) \leq \frac{1}{|B|}\int_B u(z(\tau))\,d\lambda(\tau).$$

We write the integral as a sum of the parts in E and in $B \setminus E$; E is the exceptional set in Theorem 8 satisfying $|E| \leq |B|\max(1, L)(1+|z|)^{-2}$. Since u satisfies the condition of Definition 9(2), we can choose a number $L \geq 1$ with $H_Q(\mathrm{Im}(v)) + k\log(1+|v|) \leq L|v|$ for $v \in \mathbb{C}^N$. Without loss of generality we may assume that $u \geq 0$. Then

$$u(z) \leq \frac{|E|}{|B|}L(1+|z|) + \sup\{u(z(\tau)) \mid \tau \in B \setminus E\}. \tag{8}$$

Because of the estimate for $|E|$, the first term of the right-hand side of (8) does not exceed L^2. For $\tau \in B \setminus E$, let f_τ be the function in Theorem 8. The estimate in Theorem 8(iii) implies that $\log|f_\tau|$ satisfies Definitions 9(1) and (2) with some larger constant $k' \geq 1$. Consequently there exist constants $l' \geq 1$ and $C' > 0$ such that Definition 9(3) holds for $\log|f_\tau|$, where l' and C' are independent of τ and u. By Theorem 8(ii) there exist $l \geq 1$ and $C'' > 0$ such that

$$u(z(\tau)) \leq H_K(\mathrm{Im}(z(\tau))) + l\log(1+|z(\tau)|) + C'', \quad \tau \in B \setminus E. \tag{9}$$

From Propositions 7(v)(c) and 7(iv) and inequalities (8) and (9), we get a constant $C > 0$ such that

$$u(z) \leq H_K(\mathrm{Im}(z)) + l \log(1+|z|) + C, \quad z \in V.$$

For (b), let $l \geq 1$ be arbitrarily given. Let $C_4 > 0$ as in Theorem 8 and set $l' = l + C_4$. Choose constants $k' \geq 1$ and $C' > 0$ such that Definitions 9(1′) and (2′) imply (3′) with the constant l' for all plurisubharmonic functions $u = \log|f|$, where f is a holomorphic function on V. There exists $\epsilon > 0$ with $B^\infty(0,\epsilon) := \{z \in \mathbb{C}^N \mid |z_j| \leq \epsilon, 1 \leq j \leq N\} \subset K$. It is well known that there exists a subharmonic function v and a number $D \geq 1$ such that

$$-D\sqrt{|z|} \leq v(z) \leq \epsilon|\mathrm{Im}(z)| - \sqrt{|z|}, \quad z \in \mathbb{C}$$

(see Braun and Meise [2, Prop. 5]). Obviously there exists a number $D_1 \geq 1$ such that

$$v_1(z) := \sum_{j=1}^{N} v(z_j) \leq H_K(\mathrm{Im}(z)) - k \log(1+|z|) + D_1, \quad z \in \mathbb{C}^N.$$

Now let u be a plurisubharmonic function on V satisfying Definitions 9(1′) and (2′) with the constant $k := k'+1$. We let

$$U(z) := \max(u(z), v_1(z) - D_1), \quad z \in V.$$

By the definition of U there exists a constant $L \geq 1$, independent of u, such that

$$|U(z)| \leq L|z|, \quad z \in \mathbb{C}^N.$$

Because of Proposition 7(iv), we need only prove the estimate of Definition 9(3′) at points $z = (s_j(w), w) \in V \setminus S_0$. Choose B and $z(\tau)$ as above. As in part (a), we get

$$u(z) \leq L^2 + \sup\{u(z(\tau)) \mid \tau \in B \setminus E\}. \tag{10}$$

In order to evaluate the second term of the right-hand side of (10), we again use Theorem 8. For each $\tau \in B \setminus E$, let f_τ be the holomorphic function in Theorem 8 for the plurisubharmonic function U. There exists a constant $E_1 \geq 1$, depending only on Q and k, such that $\log|f_\tau| - E_1$ satisfies the conditions of Definition 9(1′) and (2′) with the constant $k' \geq 1$. By hypothesis, $\log|f_\tau| - E_1$ satisfies Definition 9(3′) with the constants l' and C'. From Proposition 7(v)(c) and Theorem 8(ii) we get a number $E_2 \geq C_4 + E_1 + C'$ such that

$$\begin{aligned} U(z(\tau)) &\leq \log|f_\tau(z(\tau))| + C_4 \log(1+|z|) + C_4 \\ &\leq H_K(\mathrm{Im}(z(\tau))) - (l' - C_4)\log(1+|z(\tau)|) + C_4 + E_1 + C' \\ &\leq H_K(\mathrm{Im}(z)) - l \log(1+|z|) + E_2. \end{aligned}$$

This, together with (10), implies that

$$U(z) \leq H_K(\mathrm{Im}(z)) - l \log(1+|z|) + C,$$

where $C := E_2 + L^2$. Therefore $u \leq U$ satisfies Definition 9(3′) with the constants l and C.

11. REMARK. In Franken and Meise [5], Theorem 10 is used to characterize those linear partial differential operators $P(D)$, with constant coefficients and a compact set $K \subset \mathbb{R}^N$ with nonempty interior, having one of the following properties:

(a) For each C^∞ Whitney jet on K (resp. $f \in \mathfrak{D}'(K)$) satisfying $P(D)f = 0$, there exists a global zero solution F of $P(D)$ in $C^\infty(\mathbb{R}^N)$ (resp. $\mathfrak{D}'(\mathbb{R}^N)$) which extends f; that is, Whitney's extension theorem holds for the zero solutions of $P(D)$ on K.

(b) For each $f \in C^\infty(\mathbb{R}^N)$ (resp. $f \in \mathfrak{D}'(\mathbb{R}^N)$) satisfying $f\,|\,\mathring{K} \equiv 0$, there exists $g \in C^\infty(\mathbb{R}^N)$ (resp. $g \in \mathfrak{D}'(\mathbb{R}^N)$) satisfying $P(D)g = f$ and $g\,|\,\mathring{K} \equiv 0$; that is, the equation $P(D)g = f$ can be solved preserving the lacuna K.

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