Essentially Normal Multiplication Operators on the Dirichlet Space

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1. Introduction

Let U be the open unit disk in the complex plane $\mathbf{C}$. The Dirichlet space D is the Hilbert space of analytic functions $f(z)=\sum_{n=0}^{\infty} a_n z^n$ on U such that

$$f(0)=0 \quad \text{and} \quad \|f\|_D^2=\int_U |f'(z)|^2 \frac{dA}{\pi}=\sum_{n=1}^{\infty} n|a_n|^2<\infty,$$

where dA denotes the usual area measure.

An analytic function φ on U is called a *multiplier* of D if $\varphi D \subset D$. The set of all multipliers of D will be denoted by $M(D)$. Each multiplier generates a bounded multiplication operator M_φ on D defined by $M_\varphi f=\varphi f$ for $f \in D$.

Multiplication operators on D are almost never normal (they are normal only for constant multipliers). In [AS], Axler and Shields asked whether the self-commutator $M_\varphi^* M_\varphi - M_\varphi M_\varphi^*$ is compact for $\varphi \in M(D)$; that is, whether multiplication operators on D are normal in the Calkin algebra. A Hilbert space operator whose self-commutator is compact is called *essentially normal.*

This paper answers negatively the question of Axler and Shields. An example of a multiplication operator that is not essentially normal is given in Section 3. Section 2 contains a description of essentially normal multipliers that is used throughout the rest of the paper.

A few more definitions are in order. The *harmonic* Dirichlet space D_h is the Hilbert space of functions f on the unit circle T for which

$$\|f\|_{D_h}^2=|\hat{f}(0)|^2+\sum_{n=-\infty}^{\infty} |n||\hat{f}(n)|^2<\infty,$$

where $(\hat{f}(n))$ is the sequence of Fourier coefficients of f. It can be shown that

$$\begin{aligned}\|f\|_{D_h}^2&=|\hat{f}(0)|^2+\int_U |\nabla P[f]|^2 \frac{dA}{\pi}\\ &=|\hat{f}(0)|^2+\int_0^{2\pi}\int_0^{2\pi}\left|\frac{f(e^{i\theta})-f(e^{i\xi})}{e^{i\theta}-e^{i\xi}}\right|^2 \frac{d\theta}{2\pi}\frac{d\xi}{2\pi},\end{aligned}$$

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where $P[f]$ denotes the Poisson integral of f. (The first of these equalities follows from an easy computation; for the proof of the second see [Do, pp. 307–311].) Since each function in D can be identified with its boundary values, we may think of D as being a closed subspace of D_h. This allows us to consider the projection map $P: D_h \to D$.

The Bergman space B is the Hilbert space of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on U such that

$$\|f\|_B^2 = \int_U |f|^2 \frac{dA}{\pi} = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty.$$

It is well known that for the Bergman and Hardy spaces the set of all multipliers is equal to $H^\infty(U)$ (the set of all bounded analytic functions on U). Nice characterizations of the multipliers that generate essentially normal multiplication operators have been found for both of these spaces (see [Ax, Prop. 3 & Thm. 7] for the Bergman space case and [Sa, Chaps. 4, 5, & 9] for the Hardy space case). In particular, it can be shown that every multiplication operator by a function in $D+\mathbf{C}$ on the Bergman and Hardy spaces does have a compact self-commutator. It is also known that $M(D) \subsetneqq H^\infty(U) \cap (D+\mathbf{C})$ (see [Ta, Thm. 9]). An easy application of the product rule shows that

$$\varphi \in M(D) \quad \text{if and only if} \quad \varphi \in H^\infty(U) \quad \text{and} \quad \varphi' D \subset B.$$

Hence (using the closed graph theorem) if $\varphi \in M(D)$ then the operator $M_{\varphi'}$: $D \to B$ of multiplication by φ' is bounded. It turns out that the essential normality of M_φ is equivalent to the compactness of $M_{\varphi'}$. This is the main result of the next section. The main fact behind the conversion of our problem to the one about $M_{\varphi'}$ is the existence of the natural unitary operator $R: D \to B$ that takes f to f'.

2. Multipliers with Compact Self-Commutators

We begin by showing that if φ is a multiplier of D, then $\bar{\varphi}$ multiplies D into the harmonic Dirichlet space. As usual, $\|\varphi\|_\infty \overset{\text{def}}{=} \sup_{z \in U} |\varphi(z)|$.

LEMMA 1. *If $\varphi \in M(D)$ and $f \in D$, then $\bar{\varphi} f \in D_h$.*

Proof. By assumption, $\varphi \in M(D)$ so $\|\varphi\|_\infty < \infty$. We have

$$\begin{aligned}
&|\overline{\varphi(e^{i\theta})} f(e^{i\theta}) - \overline{\varphi(e^{i\xi})} f(e^{i\xi})|^2 \\
&\quad \le 2|\overline{\varphi(e^{i\theta})} f(e^{i\theta}) - \overline{\varphi(e^{i\theta})} f(e^{i\xi})|^2 + 2|\overline{\varphi(e^{i\theta})} f(e^{i\xi}) - \overline{\varphi(e^{i\xi})} f(e^{i\xi})|^2 \\
&\quad \le 2\|\varphi\|_\infty^2 |f(e^{i\theta}) - f(e^{i\xi})|^2 + 2|(\varphi(e^{i\theta}) - \varphi(e^{i\xi})) f(e^{i\xi})|^2 \\
&\quad \le 2\|\varphi\|_\infty^2 |f(e^{i\theta}) - f(e^{i\xi})|^2 \\
&\qquad + 2(2\|\varphi\|_\infty^2 |f(e^{i\theta}) - f(e^{i\xi})|^2 + 2|\varphi(e^{i\theta}) f(e^{i\theta}) - \varphi(e^{i\xi}) f(e^{i\xi})|^2) \\
&\quad \le 6\|\varphi\|_\infty^2 |f(e^{i\theta}) - f(e^{i\xi})|^2 + 4|\varphi(e^{i\theta}) f(e^{i\theta}) - \varphi(e^{i\xi}) f(e^{i\xi})|^2,
\end{aligned}$$

and hence

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\overline{\varphi(e^{i\theta})}f(e^{i\theta})-\overline{\varphi(e^{i\xi})}f(e^{i\xi})}{e^{i\theta}-e^{i\xi}}\right|^2\frac{d\theta}{2\pi}\frac{d\xi}{2\pi}$$
$$\leq 6\|\varphi\|_\infty^2\int_0^{2\pi}\int_0^{2\pi}\left|\frac{f(e^{i\theta})-f(e^{i\xi})}{e^{i\theta}-e^{i\xi}}\right|^2\frac{d\theta}{2\pi}\frac{d\xi}{2\pi}$$
$$+4\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\varphi(e^{i\theta})f(e^{i\theta})-\varphi(e^{i\xi})f(e^{i\xi})}{e^{i\theta}-e^{i\xi}}\right|^2\frac{d\theta}{2\pi}\frac{d\xi}{2\pi}.$$

Since φf and f are in D, both integrals on the right-hand side are finite, and hence $\|\bar\varphi f\|_{D_h}<\infty$. □

For $\varphi\in M(D)$, Lemma 1 allows us to define an operator $T_{\bar\varphi}: D\to D$ by $T_{\bar\varphi}f=P(\bar\varphi f)$ where P is the projection map from D_h to D.

LEMMA 2. *Let $\varphi\in M(D)$.*

(a) *The operator $T_{\bar\varphi}$ is unitarily equivalent to the adjoint of multiplication by φ on B.*

(b) *$M_\varphi^*-T_{\bar\varphi}=M_{\varphi'}^*R$, where $M_{\varphi'}: D\to B$ is multiplication by φ' and R: $D\to B$ is a unitary operator.*

Proof. (a) Let

$$e_n(z)=\frac{z^2}{\sqrt{n}}\quad\text{and}\quad e_n^B(z)=\sqrt{n}z^{n-1}\quad\text{for } z\in U,\ n=1,2,\ldots.$$

It is easy to check that $(e_n)_{n=1}^\infty$ form an orthonormal basis in D and that $(e_n^B)_{n=1}^\infty$ form an orthonormal basis in B. Let R be the unitary operator from D to B which takes f to f', and let N_φ denote multiplication by φ on B. Finally let $\langle\ ,\ \rangle_D$ and $\langle\ ,\ \rangle_B$ be the inner products in D and B (respectively) and let $\varphi=\sum_{n=0}^\infty a_nz^n\in M(D)$.

Direct computation shows that

$$\langle T_{\bar\varphi}e_n,e_m\rangle_D=\langle P_{\bar\varphi}e_n,e_m\rangle_D=\begin{cases}\dfrac{m\bar a_{n-m}}{\sqrt{n}\sqrt{m}} & \text{if } m\leq n,\\ 0 & \text{if } m>n,\end{cases}\tag{1}$$

and

$$\langle R^*N_\varphi^*Re_n,e_m\rangle_D=\langle Re_n,\varphi Re_m\rangle_B=\langle e_n^B,\varphi e_m^B\rangle_B=\begin{cases}\dfrac{m\bar a_{n-m}}{\sqrt{n}\sqrt{m}} & \text{if } m\leq n,\\ 0 & \text{if } m>n.\end{cases}$$

Hence

$$T_{\bar\varphi}=R^*N_\varphi^*R.\tag{2}$$

(b) Part (a) gives

$$\langle(M_\varphi^*-T_{\bar\varphi})f,g\rangle_D=\langle f,\varphi g\rangle_D-\langle f',\varphi g'\rangle_B=\langle f',\varphi' g\rangle_B=\langle M_{\varphi'}^*Rf,g\rangle_D$$

as desired. □

We need the following lemma, whose proof can be found in [AS, Thm. 9].

LEMMA 3. *Let φ be a holomorphic function on U and let $M_{\varphi'}$ be the operator of multiplication by φ'.*

(a) *If $M_{\varphi'}: D \to B$ is bounded then*

$$\sup_{|z|<1} |\varphi'(z)| \left(\log \frac{1}{1-|z|^2}\right)^{1/2} (1-|z|^2) \le \|M_{\varphi'}\|.$$

(b) *If $M_{\varphi'}: D \to B$ is compact then*

$$|\varphi'(z)| \left(\log \frac{1}{1-|z|^2}\right)^{1/2} (1-|z|^2) \to 0 \quad \textit{as } |z| \to 1.$$

The proof of Theorem 1 uses the following form of Fuglede's theorem: Let a, b, c be elements of some C^*-algebra. If a and b are normal and $ac = cb$ then $a^*c = cb^*$. (For a reference see [Ru, Thms. 12.16 & 12.41].)

Now we are ready to prove the main result of this section.

THEOREM 1. *Let $\varphi \in M(D)$. Then M_φ is essentially normal if and only if $M_{\varphi'}: D \to B$ is compact.*

Proof. Since $\varphi \in M(D)$, $M_{\varphi'}: D \to B$ is bounded; by Lemma 3(a),

$$|\varphi'(z)|(1-|z|^2) \to 0 \quad \text{as } |z| \to 1.$$

This, as was shown by Axler [Ax, Prop. 3 & Thm. 7], implies that the operator N_φ of multiplication by φ on B is essentially normal.

Sufficiency: By Lemma 2(b), $M_\varphi - T_{\bar\varphi}^* = R^* M_{\varphi'}$; hence our assumption, Lemma 2(a), and the remark made above imply that M_φ is a compact perturbation of an essentially normal operator.

Necessity: Denote $K = R^* M_{\varphi'}$, where R is the unitary operator taking f to f'. By Lemma 2(b),

$$M_\varphi^* - T_{\bar\varphi} = K^* \tag{3}$$

and

$$M_\varphi T_{\bar\varphi}^* = K. \tag{4}$$

We clearly have

$$N_\varphi M_{\varphi'} = M_{\varphi'} M_\varphi,$$

and thus

$$R^{-1} N_\varphi R R^{-1} M_{\varphi'} = R^{-1} M_{\varphi'} M_\varphi. \tag{5}$$

Since $R^{-1} = R^*$, (2) and (5) imply that

$$T_{\bar\varphi}^* K = K M_\varphi.$$

By assumption, Lemma 2(a), and the remark made at the beginning of the proof, both $T_{\bar\varphi}$ and M_φ are normal in the Calkin algebra. Thus, using Fuglede's theorem,

$$T_{\bar{\varphi}}K = KM_{\varphi}^{*} \quad \text{and} \quad K^{*}T_{\bar{\varphi}}^{*} = M_{\varphi}K^{*} \quad \text{in the Calkin algebra.} \tag{6}$$

Equations (3), (4), and (6) imply that in the Calkin algebra

$$\begin{aligned}
0 &= M_{\varphi}^{*}M_{\varphi} - M_{\varphi}M_{\varphi}^{*} \\
&= (T_{\bar{\varphi}} + K^{*})(T_{\bar{\varphi}}^{*} + K) - (T_{\bar{\varphi}}^{*} + K)(T_{\bar{\varphi}} + K^{*}) \\
&= \underbrace{(T_{\bar{\varphi}}T_{\bar{\varphi}}^{*} - T_{\bar{\varphi}}^{*}T_{\bar{\varphi}})}_{0} + (T_{\bar{\varphi}}K - KT_{\bar{\varphi}}) + (K^{*}T_{\bar{\varphi}}^{*} - T_{\bar{\varphi}}^{*}K^{*}) + (K^{*}K - KK^{*}) \\
&= K\underbrace{(M_{\varphi}^{*} - T_{\bar{\varphi}})}_{K^{*}} + \underbrace{(M_{\varphi} - T_{\bar{\varphi}}^{*})}_{K}K^{*} + (K^{*}K - KK^{*}) \\
&= KK^{*} + K^{*}K.
\end{aligned}$$

Since both KK^{*} and $K^{*}K$ are positive, they must be 0 in the Calkin algebra; hence K is compact, which forces $M_{\varphi'}$ to be compact. □

Remark. Brown, Douglas, and Fillmore [BDF] studied essentially normal operators and proved the following: If S is essentially normal and if $\operatorname{ind}(S - \lambda I) \le 0$ for all λ outside the essential spectrum of S, then S is unitarily equivalent to a compact perturbation of a subnormal operator. Here "ind" denotes the Fredholm index.

Notice that $M_{\varphi} - \lambda I = M_{\varphi - \lambda}$ has trivial kernel (if φ is nonconstant), so $\operatorname{ind}(M_{\varphi - \lambda I}) \le 0$ for all λ not in the essential spectrum of M_{φ}. Theorem 1 states that if M_{φ} is essentially normal then $M_{\varphi'}$ is compact, and since $M_{\varphi} = R^{*}N_{\varphi}R + R^{*}M_{\varphi'}$, M_{φ} is unitarily equivalent to a compact perturbation of the multiplication on B—one of the main examples of subnormal operators. Thus Theorem 1 gives an explicit example of the phenomena discovered by Brown, Douglas, and Fillmore.

In [St, Thms. 1.1 & 2.3], Stegenga found a description of all analytic φ such that $\varphi' D \subset B$ in terms of boundary behavior of φ. His result says that $M_{\varphi'}$: $D \to B$ is bounded if and only if

$$\int_{\bigcup S(I_j)} |\varphi'|^2 \, dA = O(\operatorname{Cap}(\textstyle\bigcup I_j)),$$

where (I_j) is any finite collection of disjoint subarcs on the circle, $S(I)$ denotes the "square" in the disc with side I, and Cap denotes the logarithmic capacity.

In [RW, Cor. 3.1], Rochberg and Wu proved that compactness of $M_{\varphi'}$ is equivalent to a "little-o" version of the Stegenga condition. This, together with Theorem 1, yields the following corollary.

Corollary 1. *Let $\varphi \in M(D)$. The operator M_{φ} is essentially normal if and only if*

$$\int_{\bigcup S(I_j)} |\varphi'|^2 \, dA = o(\operatorname{Cap}(\textstyle\bigcup I_j)),$$

where (I_j) is a finite collection of disjoint subarcs on the circle and $S(I)$ is the "square" in the disc with side I.

3. Multipliers with Noncompact Self-Commutators

In this section we will show that there are multipliers of D for which the corresponding multiplication operator is not essentially normal. By Theorem 1, it is enough to construct $\varphi \in M(D)$ such that the operator $M_{\varphi'}: D \to B$ of multiplication by φ' is not compact. We will do this in two steps. Theorem 2 shows the existence of a function φ holomorphic on U, with $M_{\varphi'}: D \to B$ bounded and

$$|\varphi'(z)|\left(\log \frac{1}{1-|z|^2}\right)^{1/2}(1-|z|^2) \not\to 0 \quad \text{as } |z| \to 1.$$

For such a φ, the operator $M_{\varphi'}$ is not compact (see Lemma 3). A method of making φ bounded without losing any of its properties is given by Corollary 2. As a result, we will get a multiplier φ with noncompact $M_{\varphi'}$.

The extended Dirichlet space $\mathfrak{D}$ is the Hilbert space of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on U such that

$$\int_U |f'(z)|^2 \frac{dA}{\pi} < \infty.$$

The norm on $\mathfrak{D}$ is defined by

$$\|f\|_{\mathfrak{D}}^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} + \int_U |f'(z)|^2 \frac{dA}{\pi} = \sum_{n=0}^{\infty} (n+1)|a_n|^2,$$

where $d\theta$ denotes the usual Lebesgue measure and $f(e^{i\theta})$ is the nontangential limit of f ($d\theta$ almost everywhere). It is clear that $\mathfrak{D}$ and D differ only by one dimension and that the norm $\|\ \|_{\mathfrak{D}}$ restricted to D is equivalent to $\|\ \|_D$. Thus, the operator $M_{\varphi'}$ from D to B is compact (bounded) if and only if it is compact (bounded) as an operator from $\mathfrak{D}$ to B. For technical reasons, the next theorem uses $\mathfrak{D}$ instead of D.

THEOREM 2. *Let $0 < c < 1$. Then there exists φ analytic in U, as well as a sequence $(z_n) \subset U$ converging to 1, such that:*

(1) $|\varphi'(z_n)|(\log(1/(1-|z_n|^2)))^{1/2}(1-|z_n|^2) \to c$ *as $n \to \infty$; and*
(2) $\|M_{\varphi'}\|_{\mathfrak{D}\to B} \le 1$, *where $\|\ \|_{\mathfrak{D}\to B}$ denotes the norm of $M_{\varphi'}$ as a multiplication by φ' from $\mathfrak{D}$ to B.*

We will need a few more lemmas before proving Theorem 2. We will adopt the following notation:

$$k_w(z) = \frac{1}{\bar{w}z} \log \frac{1}{1-\bar{w}z} \quad \text{and} \quad K_w(z) = \frac{1}{(1-\bar{w}z)^2}.$$

It is easy to check that

$$f(w) = \langle f, k_w \rangle_{\mathfrak{D}} \quad \text{for all } f \in \mathfrak{D}$$

and

$$f(w) = \langle f, K_w \rangle_B \quad \text{for all } f \in B.$$

The functions k_w and K_w are called the *reproducing kernels* for $\mathfrak{D}$ and B, respectively. It is not hard to see that, for any finite set of distinct points $w_1, w_2, \ldots, w_n$ in U, the corresponding families (k_{w_i}) and (K_{w_i}) are linearly independent and that the norms of k_w and K_w are

$$\|k_w\|_{\mathfrak{D}} = (k_w(w))^{1/2} = \frac{1}{|w|}\left(\log \frac{1}{1-|w|^2}\right)^{1/2}$$

and

$$\|K_w\|_B = (K_w(w))^{1/2} = \frac{1}{1-|w|^2}.$$

Notice that if

$$|\varphi'(z_n)| = c\frac{\|K_{z_n}\|_B}{\|k_{z_n}\|_{\mathfrak{D}}}$$

for some sequence $(z_n) \subset U$ converging to 1, then condition 1 of the theorem is clearly satisfied. Moreover, if $M_{\varphi'}\colon \mathfrak{D} \to B$ is bounded then

$$\begin{aligned}\langle M_{\varphi'}^* K_w, f\rangle_{\mathfrak{D}} &= \langle K_w, \varphi' f\rangle_B = \overline{\langle \varphi' f, K_w\rangle_B} = \overline{\varphi'(w) f(w)} = \overline{\varphi'(w)\langle f, k_w\rangle_{\mathfrak{D}}} \\ &= \langle \overline{\varphi'(w)} k_w, f\rangle_{\mathfrak{D}}\end{aligned}$$

for all $f \in \mathfrak{D}$, and hence

$$M_{\varphi'}^* K_w = \overline{\varphi'(w)} k_w.$$

This suggests that we may specify the values of φ' using the operator $M_{\varphi'}^*$. More precisely, we will construct a sequence $(z_n) \subset U$ converging to 1 and an operator $\Lambda^c\colon B \to \mathfrak{D}$ with $\|\Lambda^c\| \le 1$ and

$$\Lambda^c K_{z_n} = c\frac{\|K_{z_n}\|_B}{\|k_{z_n}\|_{\mathfrak{D}}} k_{z_n} \quad \text{for } n = 1, 2, 3, \ldots$$

in such a way that $\Lambda^c = M_{\varphi'}^*$ for some φ. This will give us a function φ and a sequence (z_n) with all the required properties. The idea just described, as well as many techniques used in the proof of Theorem 2, come from the preprint of Marshall and Sundberg [MS].

First we prove the following lemma.

LEMMA 4. *Let* $0 < c < 1$. *Then there exists a sequence* $(z_n) \subset U$ *such that* $z_n \to 1$ *and the operators* $\Lambda_n^c\colon \operatorname{span}(K_{z_1}, \ldots, K_{z_n}) \to \operatorname{span}(k_{z_1}, \ldots, k_{z_n})$ *defined by*

$$\Lambda_n^c K_{z_i} = c\frac{\|K_{z_i}\|_B}{\|k_{z_i}\|_{\mathfrak{D}}} k_{z_i} \quad \textit{for } i = 1, 2, \ldots, n$$

satisfy $\|\Lambda_n^c\| \le 1$ *for all* n.

Proof. Notice that if the families $(f_i)_{i=1}^n \subset B$ and $(g_i)_{i=1}^n \subset \mathfrak{D}$ are linearly independent, and if

$$L\colon \operatorname{span}(f_1, f_2, \ldots, f_n) \to \operatorname{span}(g_1, g_2, \ldots, g_n)$$

is defined by $Lf_i = a_i g_i$, then

$$\begin{aligned}
&\|L\| \le 1\\
&\Leftrightarrow \left\| L\left(\sum_{i=1}^{n} b_i f_i\right)\right\|_{\mathfrak{D}}^2 \le \left\|\sum_{i=1}^{n} b_i f_i\right\|_B^2 \quad \text{for all } (b_i)_{i=1}^n \subset \mathbf{C}\\
&\Leftrightarrow \left\langle \sum_{i=1}^{n} b_i f_i, \sum_{i=1}^{n} b_i f_i\right\rangle_B - \left\langle \sum_{i=1}^{n} b_i a_i g_i, \sum_{i=1}^{n} b_i a_i g_i\right\rangle_{\mathfrak{D}} \ge 0 \quad \text{for all } (b_i)_{i=1}^n \subset \mathbf{C}\\
&\Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} (\langle f_i, f_j\rangle_B - a_i \bar{a}_j \langle g_i, g_j\rangle_{\mathfrak{D}}) b_i \bar{b}_j \ge 0 \quad \text{for all } (b_i)_{i=1}^n \subset \mathbf{C}.
\end{aligned}$$

Hence

$$\|L\| \le 1 \Leftrightarrow \{\langle f_i, f_j\rangle_B - a_i \bar{a}_j \langle g_i, g_j\rangle_{\mathfrak{D}}\}_{i,j=1,2,\ldots,n} \text{ is positive semidefinite,} \tag{7}$$

and all that remains is to find a sequence $(z_n) \subset U$ such that $z_n \to 1$ and the matrices

$$\begin{aligned}
&\left\{\langle K_{z_i}, K_{z_j}\rangle_B - c^2 \frac{\|K_{z_i}\|_B}{\|k_{z_i}\|_{\mathfrak{D}}} \frac{\|K_{z_j}\|_B}{\|k_{z_j}\|_{\mathfrak{D}}} \langle k_{z_i}, k_{z_j}\rangle_{\mathfrak{D}}\right\}_{i,j=1,2,\ldots,n}\\
&\quad = \left\{\|K_{z_i}\|_B \|K_{z_j}\|_B \left(\frac{\langle K_{z_i}, K_{z_j}\rangle_B}{\|K_{z_i}\|_B \|K_{z_j}\|_B} - c^2 \frac{\langle k_{z_i}, k_{z_j}\rangle_{\mathfrak{D}}}{\|k_{z_i}\|_{\mathfrak{D}} \|k_{z_j}\|_{\mathfrak{D}}}\right)\right\}_{i,j=1,2,\ldots,n}
\end{aligned}$$

are positive semidefinite for all $n = 1, 2, \ldots$. Because the matrix

$$\{\|K_{z_i}\|_B \|K_{z_j}\|_B\}_{i,j=1,2,\ldots,n}$$

is a Gramian (hence positive semidefinite) and since, by Schur's lemma [HJ, Thm. 7.5.3], the entry-by-entry product of positive semidefinite matrices is positive semidefinite, it will be enough to construct a sequence $(z_n) \subset U$ such that $z_n \to 1$ and the matrices

$$\left\{\frac{\langle K_{z_i}, K_{z_j}\rangle_B}{\|K_{z_i}\|_B \|K_{z_j}\|_B} - c^2 \frac{\langle k_{z_i}, k_{z_j}\rangle_{\mathfrak{D}}}{\|k_{z_i}\|_{\mathfrak{D}} \|k_{z_j}\|_{\mathfrak{D}}}\right\}_{i,j=1,2,\ldots,n} \tag{8}$$

are positive semidefinite for all $n = 1, 2, 3, \ldots$.

We shall define inductively a sequence (z_n) for which $1 - 1/n < z_n < 1$ and

$$\det\left\{\frac{\langle K_{z_i}, K_{z_j}\rangle_B}{\|K_{z_i}\|_B \|K_{z_j}\|_B} - c^2 \frac{\langle k_{z_i}, k_{z_j}\rangle_{\mathfrak{D}}}{\|k_{z_i}\|_{\mathfrak{D}} \|k_{z_j}\|_{\mathfrak{D}}}\right\}_{i,j=1,2,\ldots,n} > 0 \tag{9}$$

for all n. This implies that the matrices of type (8) are positive semidefinite for all n by standard linear algebra [HJ, Thm. 7.2.5].

For $n = 1$, let z_1 be any real number between 0 and 1. Then a 1×1 matrix of type (8) consists of the single entry $1 - c^2$, and (9) is clearly satisfied.

Suppose we construct $z_1, \ldots, z_{N-1}$ such that $1 - 1/i < z_i < 1$ and condition (9) is satisfied for each $n = 1, 2, \ldots, N-1$. For any real z_N, we can expand by minors along the last column to obtain

$$\det\left\{\frac{\langle K_{z_i}, K_{z_j}\rangle_B}{\|K_{z_i}\|_B\|K_{z_j}\|_B} - c^2\frac{\langle k_{z_i}, k_{z_j}\rangle_{\mathfrak{D}}}{\|k_{z_i}\|_{\mathfrak{D}}\|k_{z_j}\|_{\mathfrak{D}}}\right\}_{i,j=1,2,\ldots,N}$$
$$= (1-c^2)\det\left\{\frac{\langle K_{z_i}, K_{z_j}\rangle_B}{\|K_{z_i}\|_B\|K_{z_j}\|_B} - c^2\frac{\langle k_{z_i}, k_{z_j}\rangle_{\mathfrak{D}}}{\|k_{z_i}\|_{\mathfrak{D}}\|k_{z_j}\|_{\mathfrak{D}}}\right\}_{i,j=1,2,\ldots,N-1} + A,$$

where A is the sum of terms each of which contains a factor

$$\frac{\langle K_{z_i}, K_{z_N}\rangle_B}{\|K_{z_i}\|_B\|K_{z_N}\|_B} - c^2\frac{\langle k_{z_i}, k_{z_N}\rangle_{\mathfrak{D}}}{\|k_{z_i}\|_{\mathfrak{D}}\|k_{z_N}\|_{\mathfrak{D}}}$$
$$= \frac{(1-z_i^2)(1-z_N^2)}{(1-z_iz_N)^2} - c^2\frac{\log(1/(1-z_iz_N))}{\left(\log\frac{1}{1-z_i^2}\right)^{1/2}\left(\log\frac{1}{1-z_N^2}\right)^{1/2}}$$

for some $i=1,2,\ldots,N-1$. Each of those factors can be made as small as we want by making z_N sufficiently close to 1, so there is a z_N such that $1-1/N < z_N < 1$ and

$$|A| < (1-c^2)\det\left\{\frac{\langle K_{z_i}, K_{z_j}\rangle_B}{\|K_{z_i}\|_B\|K_{z_j}\|_B} - c^2\frac{\langle k_{z_i}, k_{z_j}\rangle_{\mathfrak{D}}}{\|k_{z_i}\|_{\mathfrak{D}}\|k_{z_j}\|_{\mathfrak{D}}}\right\}_{i,j=1,2,\ldots,N-1}.$$

This implies (9) for $n=N$. □

The next lemma helps us to extend the operators Λ_n^c and will play a crucial role in the proof of Theorem 2.

LEMMA 5. *Let $z_1, z_2, \ldots, z_n$ be any sequence of complex numbers in U. Suppose the operator*

$$S\colon \operatorname{span}(K_{z_1}, K_{z_2}, \ldots, K_{z_n}) \to \operatorname{span}(k_{z_1}, k_{z_2}, \ldots, k_{z_n}),$$

defined by $SK_{z_i} = r_i k_{z_i}$ for $i=1,2,\ldots,n$ and some collection of complex numbers $r_1, r_2, \ldots, r_n$, satisfies $\|S\|\le 1$. Then for each $z\in U$ there exists a complex number r for which the operator

$$S_r\colon \operatorname{span}(K_{z_1}, K_{z_2}, \ldots, K_{z_n}, K_z) \to \operatorname{span}(k_{z_1}, k_{z_2}, \ldots, k_{z_n}, k_z),$$

defined by $S_rK_{z_i} = r_ik_{z_i}$ for $i=1,\ldots,n$ and $S_rK_z = rk_z$, satisfies $\|S_r\|\le 1$.

Proof. Fix $z\in U$. The map $t\to\|S_t\|$ is continuous on $\mathbf{C}$ and goes to ∞ as $|t|\to\infty$. Thus there exists $r\in\mathbf{C}$ such that

$$\|S_r\| = \inf_{t\in\mathbf{C}}\|S_t\|.$$

Denote by H_n^B the subspace of $\operatorname{span}(K_{z_1}, K_{z_2}, \ldots, K_{z_n}, K_z)$ orthogonal to K_z, and by $H_n^{\mathfrak{D}}$ the subspace of $\operatorname{span}(k_{z_1}, k_{z_2}, \ldots, k_{z_n}, k_z)$ orthogonal to k_z. Let

$$P^B\colon \operatorname{span}(K_{z_1}, K_{z_2}, \ldots, K_{z_n}, K_z) \to H_n^B,$$
$$P^{\mathfrak{D}}\colon \operatorname{span}(k_{z_1}, k_{z_2}, \ldots, k_{z_n}, k_z) \to H_n^{\mathfrak{D}}$$

be the orthogonal projections, $\hat{K}_{z_i} = P^BK_{z_i}$ for $i=1,2,\ldots,n$ and $\hat{k}_{z_i} = P^{\mathfrak{D}}k_{z_i}$ for $i=1,2,\ldots,n$. It is easy to see that

$$\hat{K}_{z_i} = K_{z_i} - \frac{\langle K_{z_i}, K_z\rangle_B}{\langle K_z, K_z\rangle_B} K_z \quad \text{and} \quad \hat{k}_{z_i} = k_{z_i} - \frac{\langle k_{z_i}, k_z\rangle_{\mathfrak{D}}}{\langle k_z, k_z\rangle_{\mathfrak{D}}} k_z.$$

Let $\hat{S}\colon H_n^B \to H_n^{\mathfrak{D}}$ be defined by

$$\hat{S}\hat{K}_{z_i} = r_i \hat{k}_{z_i} \quad \text{for } i = 1, 2, \ldots, n.$$

Using the same argument as in [MS, Lemma 9], one can show that if $\|S\| \le 1$ and $\|\hat{S}\| \le 1$ then there exists an r such that $\|S_r\| \le 1$. The ideas behind the proof of this claim are due to Agler [Ag]. Thus we need only prove that $\|\hat{S}\| \le 1$. By (7), $\|\hat{S}\| \le 1$ if and only if the matrix

$$\{\langle \hat{K}_{z_i}, \hat{K}_{z_j}\rangle_B - r_i \bar{r}_j \langle \hat{k}_{z_i}, \hat{k}_{z_j}\rangle_{\mathfrak{D}}\}_{i,j=1,2,\ldots,n}$$

is positive semidefinite.

Set $z_0 = z$. For simplicity we will use the following notation:

$$K_{ij} = \langle K_{z_i}, K_{z_j}\rangle_B \quad \text{and} \quad k_{ij} = \langle k_{z_i}, k_{z_j}\rangle_{\mathfrak{D}} \quad \text{for } i, j = 0, 1, 2, \ldots, n.$$

An easy computation shows that

$$\begin{aligned}
&\langle \hat{K}_{z_i}, \hat{K}_{z_j}\rangle_B - r_i \bar{r}_j \langle \hat{k}_{z_i}, \hat{k}_{z_j}\rangle_{\mathfrak{D}} \\
&\quad = K_{ij} - \frac{K_{i0}\bar{K}_{j0}}{K_{00}} - r_i \bar{r}_j \left(k_{ij} - \frac{k_{i0}\bar{k}_{j0}}{k_{00}}\right) \\
&\quad = (K_{ij} - r_i \bar{r}_j k_{ij}) \left(1 - \frac{k_{i0}\bar{k}_{j0}}{k_{00}k_{ij}}\right) + \frac{k_{i0}\bar{k}_{j0}}{K_{00}} \left(\frac{K_{00}K_{ij}}{k_{00}k_{ij}} - \frac{K_{i0}\bar{K}_{j0}}{k_{i0}\bar{k}_{j0}}\right).
\end{aligned}$$

Because $\|S\| \le 1$, (7) implies that

$$\{K_{ij} - r_i \bar{r}_j k_{ij}\}_{i,j=1,2,\ldots,n}$$

is positive semidefinite. Marshall and Sundberg ([MS, Lemmas 10 & 11], see also [Qu, Cor. 5.3]) have shown that

$$\left\{1 - \frac{k_{i0}\bar{k}_{j0}}{k_{00}k_{ij}}\right\}_{i,j=1,2,\ldots,n}$$

is positive semidefinite. The matrix

$$\left\{\frac{k_{i0}\bar{k}_{j0}}{K_{00}}\right\}_{i,j=1,2,\ldots,n}$$

is a Gramian and hence positive semidefinite, so by Schur's lemma we need only prove that

$$\left\{\frac{K_{00}K_{ij}}{k_{00}k_{ij}} - \frac{K_{i0}\bar{K}_{j0}}{k_{i0}\bar{k}_{j0}}\right\}_{i,j=1,2,\ldots,n}$$

is positive semidefinite.

Let

$$w(z) = \left((1 - z^2)\frac{1}{z}\log\frac{1}{1-z}\right)^{-1}$$

and $w_{ij} = w(\bar{z}_i z_j)$ for all $i, j = 0, 1, \ldots, n$. Clearly

$$w_{00}w_{ij} - w_{i0}\bar{w}_{j0} = \frac{K_{00}K_{ij}}{k_{00}k_{ij}} - \frac{K_{i0}\bar{K}_{j0}}{k_{i0}\bar{k}_{j0}},$$

so we need to show that

$$\{w_{00}w_{ij} - w_{i0}\bar{w}_{j0}\}_{i,j=1,2,\ldots,n}$$

is positive semidefinite.

Write $w(z) = \sum_{n=0}^{\infty} a_n z^n$ (notice that w has a removable singularity at 0, which is the main reason for using $\mathfrak{D}$ instead of D in the statement of the theorem). One can easily prove that $a_n > 0$ for all $n = 0, 1, 2, \ldots$ (see [MS, pp. 22–23]). Using the argument from the proof of [MS, Lemma 10], we get

$$w_{00}w_{ij} - w_{i0}\bar{w}_{j0} = w_{00}\sum_{k=0}^{\infty} a_k(\bar{z}_i^k - \bar{z}_0^k)(z_j^k - z_0^k) - (w_{00} - w_{i0})(w_{00} - \bar{w}_{j0}).$$

Thus, for any complex numbers $b_1, b_2, \ldots, b_n$,

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} b_i\bar{b}_j(w_{00}w_{ij} - w_{i0}\bar{w}_{j0}) \\
&\quad = w_{00}\sum_{k=0}^{\infty} a_k\left|\sum_{i=1}^{n} b_i(\bar{z}_i^k - \bar{z}_0^k)\right|^2 - \left|\sum_{i=1}^{n} b_i(w_{00} - w_{i0})\right|^2 \\
&\quad = \underbrace{w_{00}\sum_{k=0}^{\infty} a_k\left|\sum_{i=1}^{n} b_i(\bar{z}_i^k - \bar{z}_0^k)\right|^2}_{S_1} - \underbrace{\left|\sum_{i=1}^{n} b_i\sum_{k=0}^{\infty} a_k z_0^k(\bar{z}_i^k - \bar{z}_0^k)\right|^2}_{S_2}.
\end{aligned}$$

Notice that, since $a_n > 0$ for all $n = 0, 1, 2, \ldots$, both S_1 and S_2 are nonnegative and by Hölder's inequality

$$\begin{aligned}
S_2 &= \left|\sum_{k=0}^{\infty}(a_k^{1/2}z_0^k)\left(a_k^{1/2}\sum_{i=1}^{n} b_i(\bar{z}_i^k - \bar{z}_0^k)\right)\right|^2 \\
&\le \sum_{k=0}^{\infty} a_k|z_0|^{2k}\sum_{k=0}^{\infty} a_k\left|\sum_{i=1}^{n} b_i(\bar{z}_i^k - \bar{z}_0^k)\right|^2 \\
&= w_{00}\sum_{k=0}^{\infty} a_k\left|\sum_{i=1}^{n} b_i(\bar{z}_i^k - \bar{z}_0^k)\right|^2 = S_1.
\end{aligned}$$

Hence $S_1 - S_2 \ge 0$, as needed. □

We now prove Theorem 2.

Proof of Theorem 2. Fix $c \in (0,1)$. Let (z_n) be a sequence of complex numbers promised by Lemma 4. Fix n and consider the operator Λ_n^c defined as in Lemma 4. Let $\{z'_{n+1}, z'_{n+2}, z'_{n+3}, \ldots\}$ be any countable dense set in the disk. Lemma 5 allows us to extend Λ_n^c to an operator

$$L_n^c\colon \operatorname{span}(K_{z_1}, \ldots, K_{z_n}, K_{z'_{n+1}}, K_{z'_{n+2}}, \ldots) \to \operatorname{span}(k_{z_1}, \ldots, k_{z_n}, k_{z'_{n+1}}, k_{z'_{n+2}}, \ldots)$$

in such a way that

$$L_n^c K_{z_i} = c\frac{\|K_{z_i}\|_B}{\|k_{z_i}\|_{\mathfrak{D}}} k_{z_i} \quad \text{for } i = 1, 2, 3, \dots, n,$$

$$L_n^c K_{z'_i} = \bar{r}_{z'_i} k_{z'_i} \quad \text{for } i = n+1, n+2, n+3, \dots,$$

and

$$\|L_n^c\| \leq 1.$$

Because $\operatorname{span}(K_{z_1}, K_{z_2}, \dots, K_{z_n}, K_{z'_{n+1}}, K_{z'_{n+2}}, \dots)$ is dense in B, L_n^c extends by continuity to a bounded operator from B to $\mathfrak{D}$. Moreover, for each $z \in U$, $L_n^c K_z = \bar{r}_z k_z$ with $r_{z_i} = c(\|K_{z_i}\|_B / \|k_{z_i}\|_{\mathfrak{D}}) k_{z_i}$ for $i = 1, 2, 3, \dots$. Define $\psi_n(z) = r_z$. Then

$$(L_n^{c*} f)(z) = \langle L_n^{c*} f, K_z \rangle_B = \langle f, \overline{\psi_n(z)} k_z \rangle_{\mathfrak{D}} = \psi_n(z) f(z)$$

for each $f \in \mathfrak{D}$ and $z \in U$. Thus L_n^{c*} is a multiplication by ψ_n. In particular ψ_n is analytic, and if φ_n denotes any antiderivative of ψ_n then $L_n^c = M_{\varphi'_n}^*$. The norms of L_n^{c*} are uniformly bounded by 1, so there is a subsequence of (L_n^{c*}) that converges weak* to some operator L^{c*}. Clearly there exists φ analytic in U with $L^{c*} = M_{\varphi'}$. Thus $\|M_{\varphi'}\|_{\mathfrak{D} \to B} \leq 1$ and

$$|\varphi'(z_n)| = c\frac{\|K_{z_i}\|_B}{\|k_{z_i}\|_{\mathfrak{D}}} \quad \text{for } i = 1, 2, 3, \dots. \qquad \square$$

Now we can answer the question discussed in the introduction.

COROLLARY 2. *There exists a function $\psi \in M(D)$ such that M_ψ is not essentially normal.*

Proof. Let $c \in (0, 1)$ and let φ be the function constructed in Theorem 2. Then $M_{\varphi'}\colon D \to B$ is bounded,

$$|\varphi'(z_n)| \left(\log \frac{1}{1 - |z_n|^2}\right)^{1/2} (1 - |z_n|^2) \to c \quad \text{as } n \to \infty$$

for some sequence $(z_n) \subset U$ converging to 1, and $\varphi'(z_n) \to \infty$. Let K be a compact set of positive area measure contained in the complement of $\varphi(U)$ (there is one since $\varphi \in \mathfrak{D}$). By the result of Uy [Uy, Thm. 4.1], there exists a function g analytic and bounded on the complement of K with respect to the extended plane and such that g' is bounded and $g'(\infty) > 0$. Let $\psi = g \circ \varphi$. Then ψ is bounded, $M_{\psi'}$ is bounded, and

$$|\psi'(z_n)| \left(\log \frac{1}{1 - |z_n|^2}\right)^{1/2} (1 - |z_n|^2) \not\to 0 \quad \text{as } |z_n| \to 1.$$

Thus $\psi \in M(D)$ and $M_{\psi'}$ is not compact. $\square$

Axler and Shields [AS] showed that $M(D)$ is nonseparable in the operator norm. Let W be the space of all holomorphic functions φ in U such that $M_{\varphi'}\colon D \to B$ is bounded with the operator norm. It is no surprise that W also turns out to be nonseparable.

COROLLARY 3. *The space W is nonseparable.*

Proof. Fix $c\in(0,1)$. A minor modification of Lemma 4 and the proof of Theorem 2 lead to a sequence (z_n) in the unit disc with $|z_n|>1/2$ and $z_n\to 1$ such that, for any sequence (a_n) consisting of 1s and −1s, there exists a function $\varphi\in W$ satisfying

$$\varphi'(z_n)=ca_n\frac{\|K_{z_n}\|_B}{\|k_{z_n}\|_{\mathfrak{D}}}.$$

Let (a_n) and (b_n) be any two different sequences of 1s and −1s, and let φ,ψ be the corresponding functions in W with

$$\varphi'(z_n)=ca_n\frac{\|K_{z_n}\|_B}{\|k_{z_n}\|_{\mathfrak{D}}}\quad\text{and}\quad\psi'(z_n)=cb_n\frac{\|K_{z_n}\|_B}{\|k_{z_n}\|_{\mathfrak{D}}}.$$

Then, by Lemma 3,

$$\begin{aligned}\|M_{\varphi'}-M_{\psi'}\|&\ge\sup_n|\varphi'(z_n)-\psi'(z_n)|\left(\log\frac{1}{1-|z_n|^2}\right)^{1/2}(1-|z_n|^2)\\&\ge\sup_n c|z_n||a_n-b_n|\ge c.\end{aligned}$$

Because the set of all sequences of 1s and −1s is uncountable, W must be nonseparable. □

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References

[Ag] J. Agler, *Interpolation,* preprint, 1986.

[Ax] S. Axler, *The Bergman space, the Bloch space and commutators of multiplication operators,* Duke Math. J. 53 (1986), 315–332.

[AS] S. Axler and A. L. Shields, *Univalent multipliers of the Dirichlet space,* Michigan Math. J. 32 (1985), 65–80.

[BDF] L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras,* Proceedings of a conference on operator theory (Halifax, Nova Scotia, 1983) (P. A. Fillmore, ed.), Lecture Notes in Math., 345, pp. 58–128, Springer, Berlin, 1973.

[Do] J. Douglas, *Solution of the problem of Plateau,* Trans. Amer. Math. Soc. 33 (1931), 263–321.

[HJ] R. A. Horn and C. R. Johnson, *Matrix analysis,* Cambridge Univ. Press, New York, 1985.

[MS] D. E. Marshall and C. Sundberg, *Interpolating sequences for the multipliers of the Dirichlet space,* preprint, 1993.

[Qu] P. Quiggin, *For which reproducing kernel Hilbert spaces is Pick's theorem true?* Integral Equations Operator Theory 16 (1993), 244–266.

[RW] R. Rochberg and Z. Wu, *Toeplitz operators on Dirichlet spaces,* Integral Equations Operator Theory 15 (1992), 325–342.

[Ru] W. Rudin, *Functional analysis,* 2nd ed., McGraw-Hill, New York, 1991.
[Sa] D. Sarason, *Function theory on the unit circle,* Virginia Polytechnic and State Univ., Blacksburg, VA, 1978.
[St] D. Stegenga, *Multipliers of the Dirichlet space,* Illinois J. Math. 24 (1980), 113–139.
[Ta] G. D. Taylor, *Multipliers on* D_α, Trans. Amer. Math. Soc. 123 (1966), 229–240.
[Uy] N. X. Uy, *Removable sets of analytic functions satisfying a Lipschitz condition,* Ark. Mat. 17 (1979), 19–27.

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