Rational Subgroups of Cubed 3-Manifold Groups

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1. Introduction

In [6, §4], Gromov initiated the study of nonpositively curved polyhedral complexes. Aitchison and Rubinstein made an extensive study of cubed 3-manifolds in [1], where they described the structure of canonical immersed surfaces in such manifolds. The existence of these surfaces imply strong structural properties. For example, it follows from the work of Hass and Scott [7] that manifolds which are homotopy equivalent to such manifolds are actually homeomorphic.

The main results contained in this paper are that the fundamental group of a closed 3-manifold which admits a cubing of nonpositive curvature is bi-automatic (Theorem 2.8) and that the subgroups corresponding to canonical immersed surfaces are rational with respect to this bi-automatic structure (Theorem 3.5), where the terms used are explained below. The first builds on work of Skinner [12], who has shown that such groups are automatic. The additional information of bi-automaticity implies, by the results of Gersten and Short [5], that infinite cyclic subgroups as well as various other subgroups are rational. A recent result of Mihalik [9] implies that covers which correspond to these subgroups can be compactified, thus verifying Simon's conjecture (see [8, p. 112]) for the canonical surface groups.

We consider only closed 3-manifolds. Bounded 3-manifolds, as well as the case of cusped manifolds, will be dealt with in a sequel. The cubings considered here are cubings of nonpositive curvature. Some examples of such cubings are given in the final section. It is clear that many of the results carry through to higher-dimensional cubed manifolds. We shall assume a slight familiarity with [4] and recall some results due to Skinner [12].

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2. Fundamental Group of a Cubed Manifold is Bi-automatic

We begin by introducing some definitions and recalling some known results.

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Let M be a closed 3-manifold. The definition of a cubing of nonpositive curvature is given in [1, p. 5] and also in [12, §10.1, p. 193]. Basically, M admits such a cubing if it can be obtained from a finite number of identical regular Euclidean cubes by identifying faces via isometries. This is done in such a way that the geometry on M is locally Euclidean except along a co-dimension-2 complex where the curvature is nonpositive; that is, for each point v the polyhedral link contains no embedded geodesic loop of length less than 2π. As we have a cubing, this condition on the links can be verified in terms of a no-triangle condition [6, p. 122] (i.e., the polyhedral link of a vertex contains no bigons and no triangles that do not comprise a face of the link). This can be achieved by insisting that each edge belongs to at least four cubes and by disallowing certain types of edge identifications (see [1, p. 5]). From such a cubing one can construct canonical immersed incompressible surfaces in M [1, §3, p. 14]. Such a surface may be built in the following way: Begin with three squares embedded in a cube (Figure 1) and then extend to adjacent cubes in the decomposition of M via the face identification maps.

There is an induced cubing on the universal cover $\tilde{M}$ of M. A component of the canonical surface lifts to a totally geodesic embedded plane in $\tilde{M}$. Denote by Π the family of such planes, the union of which forms the full pre-image of the surface. It is shown in [1] that Π satisfies the 4-plane, 1-line conditions of Hass and Scott [7]. The cubings of M and $\tilde{M}$ give decompositions of each, from which we denote by L_i^M and L_i the collection of i-cells in M and $\tilde{M}$, respectively (following notation in [12]). We regard the edges in the decomposition as being directed, and for each edge e we label as $-e$ the same edge with the opposite orientation. Let $\mathcal{G}$ be the groupoid whose vertices are the elements of L_0^M and whose morphisms are homotopy classes of

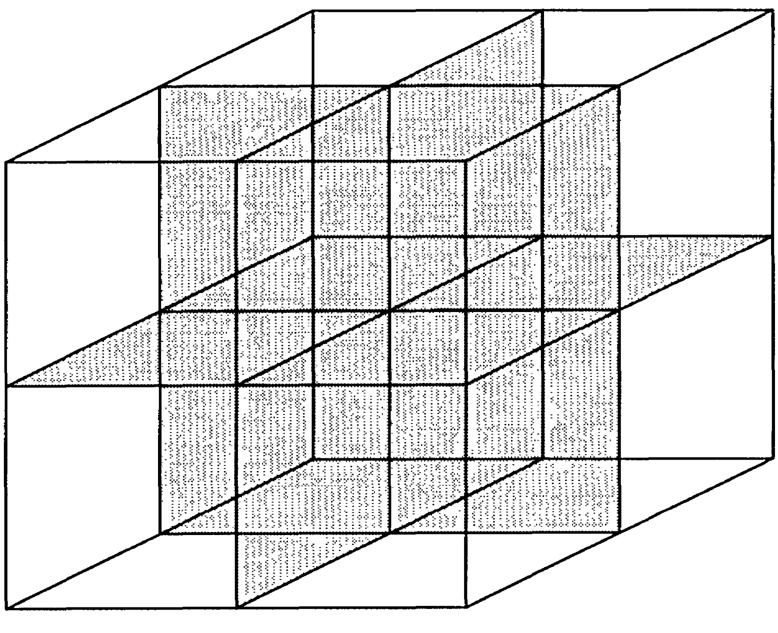

Figure 1 Section of canonical surface

edge paths in M. $\mathcal{G}$ is sometimes called the *fundamental groupoid* of L_2^M. Let $\mathcal{A} = L_1^M \cup (-L_1^M)$. Then $\mathcal{A}$ is a finite set of generators for $\mathcal{G}$, and we denote by $\mathcal{A}^*$ the free monoid on $\mathcal{A}$. We want to define a regular language $\mathcal{L}$ over $\mathcal{A}$ such that $(\mathcal{A}, \mathcal{L})$ gives a bi-automatic structure for $\mathcal{G}$. The concept of a bi-automatic groupoid is defined below in a fashion analogous to that for groups. Let Γ be a graph with a metric d obtained by declaring each edge to be of length 1. Two paths $w_1, w_2 : \mathbb{R} \to \Gamma$ are said to be *k-fellow travelers* if $d(w_1(t), w_2(t)) \leq k$ for all t. There is a weaker condition that we will make use of in Section 3: w_1 and w_2 are known as *asynchronous* k-fellow travelers if there exists a nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ such that $d(w_1(t), w_2(f(t))) \leq k$ for all t. Recall that the Cayley graph of a groupoid, with respect to a set generating set $\mathcal{A}$ and a base point v_0, has as vertices the set of morphisms based at v_0. Given two morphisms $f : v_0 \to x$ and $g : v_0 \to y$, there is an edge labeled by $a \in \mathcal{A}$ from f to g if the composition fa is defined and equal to g. We denote by $\Gamma(\mathcal{G}, \mathcal{A})$ the Cayley graph of $\mathcal{G}$ with respect to $\mathcal{A}$, and omit mention of the basepoint v_0 since the exact basepoint is immaterial in what follows. As composition is not always defined in a groupoid, we need to consider a subset $\mathcal{L}(\mathcal{G}, \mathcal{A}) \subseteq \mathcal{A}^*$ consisting of all words that label paths in $\Gamma(\mathcal{G}, \mathcal{A})$. Let $\pi : \mathcal{L}(\mathcal{G}, \mathcal{A}) \to \mathcal{G}$ be the map given by evaluation (where as usual we are identifying $\mathcal{G}$ with its set of morphisms).

DEFINITION (see [4, p. 248]). Let G be a groupoid, with A a finite set of generators for G. We say that G is *bi-automatic* if there is a regular language L over A, and a constant $k \geq 1$, such that:

(i) $L \subseteq \mathcal{L}(\mathcal{G}, \mathcal{A})$.
(ii) $\pi : L \to \mathcal{G}$ is surjective.
(iii) Let γ and γ' be paths in $\Gamma(\mathcal{G}, \mathcal{A})$ that are labeled by elements of L. If γ, γ' are paths from x to y and from x' to y' respectively, then γ and γ' are K-fellow travelers where $K = k(d(x, x') + d(y, y') + 1)$.

Take the natural metric on L_1 such that each edge has unit length. An edge in L_1 will be labeled by the element of $\mathcal{A}$ to which it projects. In this way L_1 may be regarded as the Cayley graph of the groupoid $\mathcal{G}$. Notice that if a word in $\mathcal{L}(\mathcal{G}, \mathcal{A}) \subset \mathcal{A}^*$ represents a path in L_1, it also represents all images of that path under the group of covering transformations of $\tilde{M}$.

The following notation will be useful.

DEFINITION. For $z \in L_0$, denote by poly(z) the closure of the component of $\tilde{M} - \bigcup_{P \in \Pi} P$ which contains z.

Notice that all vertices of poly(z) are of degree 3, and all faces are of degree greater than 3. We have the following lemma.

LEMMA 2.1 (Skinner). *Let $x \in L_0$ and $P_1, P_2 \in \Pi$ be such that*

$$F_1 = P_1 \cap \text{poly}(x) \neq \emptyset;$$
$$F_2 = P_2 \cap \text{poly}(x) \neq \emptyset.$$

Then

(i) *If P_1 and P_2 both separate x from $y \in L_0$, then $P_1 \cap P_2 \neq \emptyset$.*
(ii) *If $P_1 \cap P_2 \neq \emptyset$ then $F_1 \cap F_2 \neq \emptyset$.*

Proof. Assume $P_1 \cap P_2 = \emptyset$. Then x lies in the component of $\tilde{M} - (P_1 \cup P_2)$ which is between P_1 and P_2, and so at most one of P_1 and P_2 can separate x from y.

To prove the second part, we use the result that the family of planes given by Π satisfies the 1-line and 4-plane properties. Recall that this means that if two planes intersect then they do so along a line, and that given any four planes in Π, at least one pair is disjoint. Assume that $P_1 \cap P_2 \neq \emptyset$; let $e_1, \ldots, e_k$ be the edges of F_1 (labeled cyclically) and let $Q_1, \ldots, Q_k$ be the planes such that $Q_i \cap F_1 = e_i$. We want to show that $P_2 = Q_i$ for some i. Assume that this is not the case. If $P_2 \cap Q_i = \emptyset$ for all i then any path in P_1 which connects $P_1 \cap P_2$ to F_1 must intersect one (or more) of the lines given by $P_1 \cap Q_i$; say it meets $P_1 \cap Q_1$. It follows from the assumption that $P_2 \cap Q_1 = \emptyset$ that any path from P_2 to F_1 must cross the plane Q_1, which contradicts the hypothesis that $P_2 \cap \text{poly}(x) \neq \emptyset$.

So assume that $P_2 \cap Q_1 \neq \emptyset$ and consider the four planes P_1, P_2, Q_1, Q_2. Applying the 4-plane property, we see that it must be the case that $P_2 \cap Q_2 = \emptyset$. Similarly $P_2 \cap Q_k = \emptyset$. But then one of Q_2 or Q_k must separate P_2 from poly(x), which contradicts our assumption. ❑

Let γ be a geodesic in L_1 with endpoints $x, y \in L_0$. If z is another vertex in γ (which is not equal to y) then it follows, from Lemma 2.1 and the fact that poly(z) has degree-3 vertices, that there are exactly three possibilities:

(a) One face, F, of poly(z) extends to a plane $P \in \Pi$ separating z and y.
(b) Two faces, F_1 and F_2, extend to planes $P_1, P_2 \in \Pi$ which separate z and y.
(c) Three faces, F_1, F_2, and F_3, extend to planes $P_1, P_2, P_3 \in \Pi$ which separate z and y.

We shall call such faces *separating faces* where it is clear from the context which vertices are being separated. Given a path γ in L_1, we will use $\iota(\gamma)$ and $\tau(\gamma)$ to denote the initial and terminal vertices (respectively), and we shall call γ a *geodesic* if it has minimal length among all paths from $\iota(\gamma)$ to $\tau(\gamma)$. Using the above result and the notion of cone type as introduced by Cannon [3], it is possible to show that the language

$$\mathcal{L} = \{w \in \mathcal{A}^* \mid w \text{ gives a geodesic edge path in } L_1\}$$

is regular (see [12, §11.5]). This language can be shown to satisfy the first two conditions of the definition [12, §11.2.1, p. 228ff], but it does not satisfy the third. To achieve this it is necessary to restrict to a sublanguage of the language of all geodesic paths.

DEFINITION. A geodesic path $\gamma = e_1 \ldots e_n$ is a *normal geodesic* if for each vertex z in γ, which is not equal to y, one of the following holds:

(i) z is of type (a) above, and the edge e_i of γ based at z crosses P;
(ii) z is of type (b), and the subarc $e_i e_{i+1}$ of γ based at z either crosses P_1 then P_2 or it crosses P_2 then P_1;
(iii) z is of type (c), and the subarc $e_i e_{i+1} e_{i+2}$ of γ based at z crosses each of P_1, P_2, P_3 in some order.

Loosely speaking, normal geodesics travel as diagonally as possible. Observe that if $\gamma = e_1 \ldots e_n$ is a normal geodesic, then any subpath $\delta = e_i \ldots e_{i+k}$ is also a normal geodesic. Let

$$\mathcal{L}(\mathcal{G}) = \{w \in \mathcal{A}^* \mid w \text{ represents a normal geodesic edge path in } L_1\}.$$

This language was studied in [12, Chap. 11], from where we quote the following results; for clarity we present (somewhat different) proofs.

LEMMA 2.2 (Skinner). *$\mathcal{L}(\mathcal{G})$ is a regular language over $\mathcal{A}$.*

Proof. We use the result that the language $\mathcal{L}$ of all geodesic words is regular. First it will be shown that if a word represents a geodesic which fails to be normal, then there is a subword of length 4 or less which gives a nonnormal geodesic. Let $\gamma = \gamma_1 \ldots \gamma_n$ be a geodesic edge path in L_1 and assume that γ is not a normal geodesic. It follows that for at least one vertex on γ the conditions of the definition are not satisfied. Let $z = \iota(\gamma_i)$ be the closest such vertex to $\tau(\gamma)$. Notice that we must have $i \le n-2$. There must be either two or three faces of poly(z) which separate (z from $\tau(\gamma)$); denote by Π the set of planes corresponding to these faces, and let $P_0 \in \Pi$ be the plane crossed by γ_i. Then the planes in $\Pi \backslash P_0$ must give separating faces of poly($\iota(\gamma_{i+1})$), and so the segment $\gamma_{i+1}\gamma_{i+2}\gamma_{i+3}$ must cross each of the planes in $\Pi \backslash P_0$ (by the choice of z). It follows that each of the planes in Π separates z from $\tau(\gamma_{i+3})$, and therefore $\gamma_i\gamma_{i+1}\gamma_{i+2}\gamma_{i+3}$ is a geodesic which is not normal (if $i+3 > n$ we consider only the first three edges of the segment); see Figure 2.

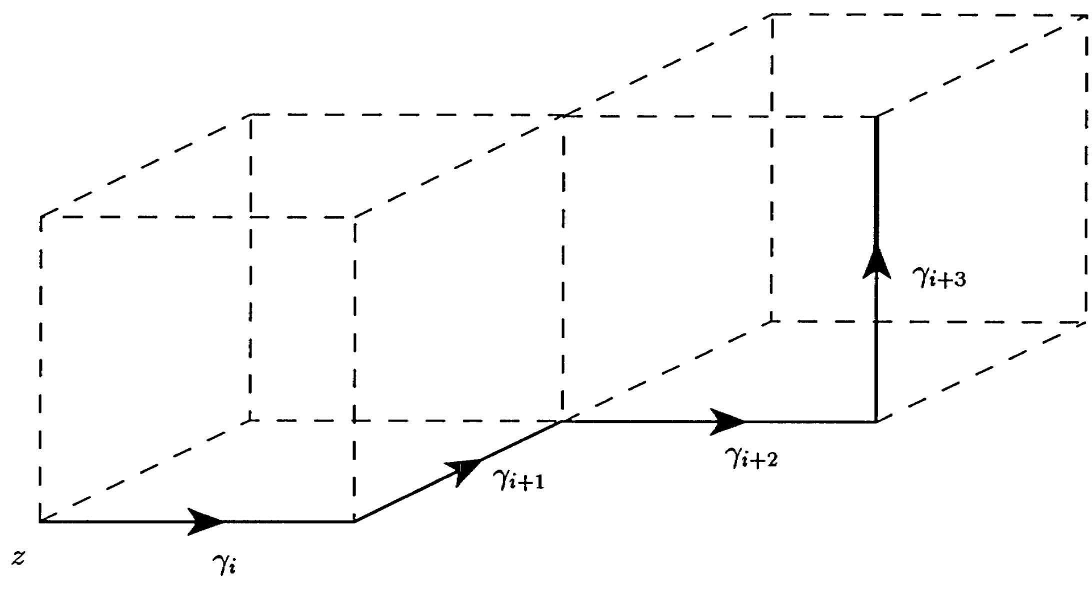

Figure 2 An example of a geodesic segment which is not normal; in this case, poly(z) has three separating faces.

To ensure that a geodesic word is normal it is enough to exclude all subwords of length 4 which do not themselves correspond to normal geodesics. Let $\mathcal{P} \subset \mathcal{A}^*$ be the set of all such words. Then $\mathcal{P}$ is finite. For each $w \in \mathcal{P}$, let $\mathcal{L}_w$ be the regular language consisting of all words in $\mathcal{A}^*$ that contain w as a subword. Then

$$\mathcal{L}(\mathcal{G}) = \mathcal{L} \Big\backslash \bigcup_{w \in \mathcal{P}} \mathcal{L}_w = \mathcal{L} \cap \Big(\bigcup_{w \in \mathcal{P}} \neg \mathcal{L}_w \Big)$$

is also regular since the union, intersection, and complement of a regular language are regular. □

LEMMA 2.3 (Skinner). *The map* $\pi: \mathcal{L}(\mathcal{G}) \to \mathcal{G}$ *is surjective.*

Proof. We will show that if $x, y \in L_0$ are any two vertices then there is a normal geodesic with initial vertex x and terminal vertex y. Let γ be a geodesic from x to y. If γ is normal then we are done. Otherwise, let $z = \iota(\gamma_i)$ be the first vertex that fails to satisfy the conditions of the definition (so the first $i-1$ vertices do satisfy the conditions of the definition). Let k $(2 \le k \le 3)$ be the number of separating faces of poly(z), and define Π and P_0 as in the previous proof. Let $1 \le j \le |\Pi|-1$ be such that γ_{i+j} is the first edge after z that does not cross one of the planes in Π. Choose $k-j$ edges $\gamma'_{i+j}, \ldots, \gamma'_{i+k}$ so that $\gamma_i \ldots \gamma_{i+j-1}\gamma'_{i+j} \ldots \gamma'_{i+k}$ is a geodesic that crosses all of the planes from Π in some order. Then $\gamma_1 \ldots \gamma_i \ldots \gamma_{i+j-1}\gamma'_{i+j} \ldots \gamma'_{i+k}$ is a normal geodesic. Notice that $d(\tau(\gamma'_{i+k}), y) = d(\tau(\gamma_{i+k}), y)$. Choose a geodesic δ from $\tau(\gamma'_{i+k})$ to y. We now have a path $\gamma' = \gamma_1 \ldots \gamma_i \ldots \gamma_{i+j-1}\gamma'_{i+j} \ldots \gamma'_{i+k}\delta$, from x to y, which is a geodesic (since it has the same length as γ) and for which the first i vertices satisfy the conditions for normal geodesics. By repeating the above procedure at most $d(x, y)-2$ times, we obtain a normal geodesic from x to y. See Figure 3. □

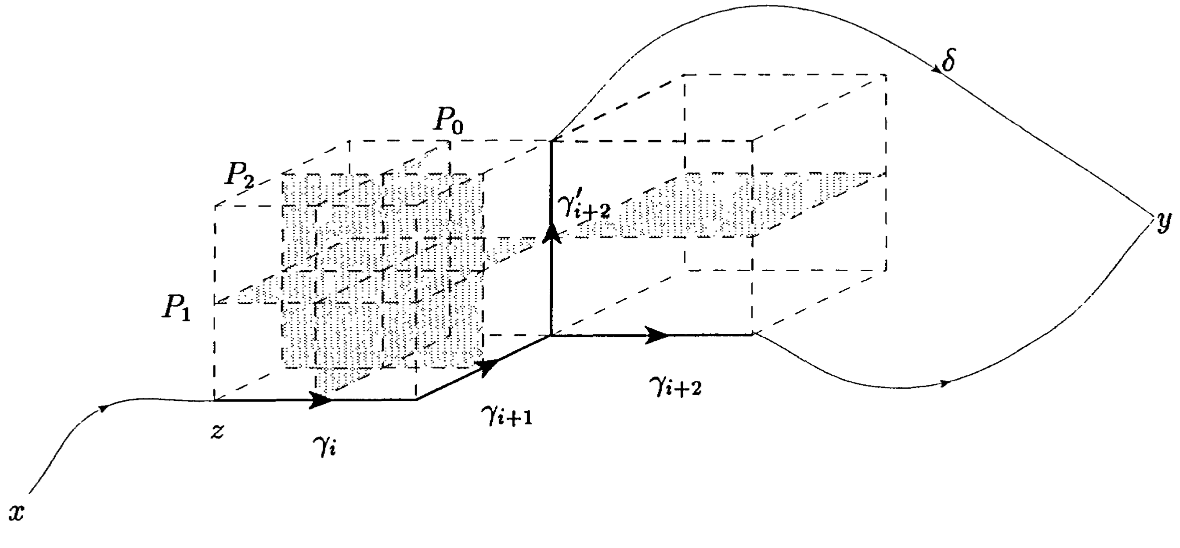

Figure 3 Constructing a normal geodesic

The remainder of this section is devoted to showing that the language $\mathcal{L}(\mathcal{G})$ actually gives a bi-automatic structure for $\mathcal{G}$. It then follows from a result in [4] that the fundamental group of M is bi-automatic. The first two conditions of the definition have already been established. It remains to verify condition (iii). If $x, x' \in L_0$ are two vertices with $d(x, x') = 1$ then there is a unique plane which separates them, which we will denote by P_x.

LEMMA 2.4 (Skinner). *Let $x, y, x', y' \in L_0$ be such that $d(x, x') = 1$ and $d(y, y') = 1$, and let γ and γ' be normal geodesics from x to y and x' to y' respectively. If $P_x = P_y$, then γ and γ' are 3-fellow travelers. Moreover, the result remains true if a path of length 2 is prepended to one of γ or γ'.*

Proof. Let P denote the plane $P_x = P_y$. A plane $Q \in \Pi$ (including P) separates x from y if and only if Q separates x' from y'. It follows that length(γ) = length(γ'). We shall argue by induction on this length. The result is trivial for length 0, so assume that the geodesics have length $n > 0$ and that the result is true for all cases where the length is less than n. Let $\gamma = \gamma_1 \dots \gamma_n$ and $\gamma' = \gamma'_1 \dots \gamma'_n$, and let $Q \neq P$ be a plane such that $Q \cap \text{poly}(x) \neq \emptyset$ and Q separates x from y. Then Q also separates x' from y' and, moreover, $Q \cap P \neq \emptyset$. So by Lemma 2.1, $F' = Q \cap \text{poly}(x') \neq \emptyset$ and is a separating face of poly(x'). There are three possibilities, determined by the number of faces of poly(x) that separate x from y:

(1) If $Q \cap \text{poly}(x)$ is the only face of poly(x) separating x from y, then $Q \cap \text{poly}(x')$ is the only separating face of poly(x'). Thus $\gamma_2 \dots \gamma_n$ and $\gamma'_2 \dots \gamma'_n$ are 3-fellow travelers by the inductive hypothesis, and hence so too are γ and γ' (see Figure 4). It is clear that prepending two edges to one of γ or γ' does not alter the result.

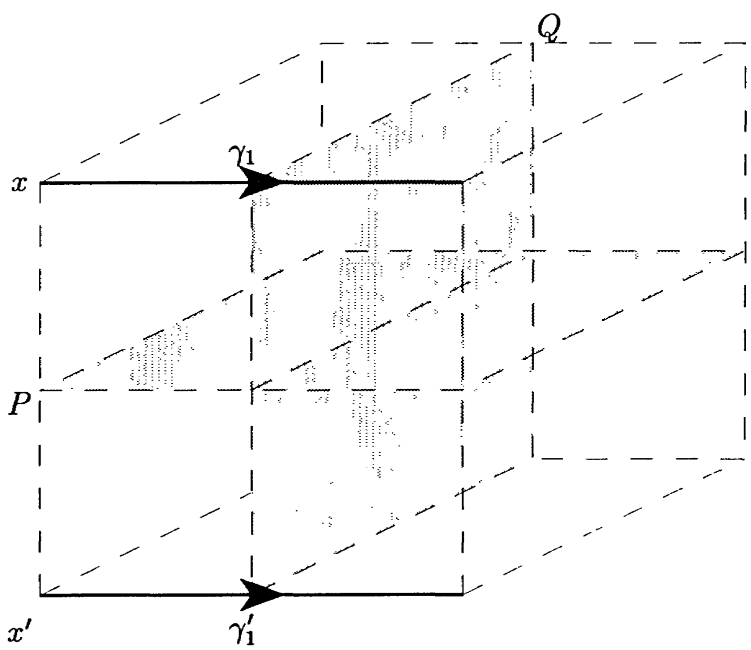

Figure 4 $Q \cap \text{poly}(x)$ is the only separating face

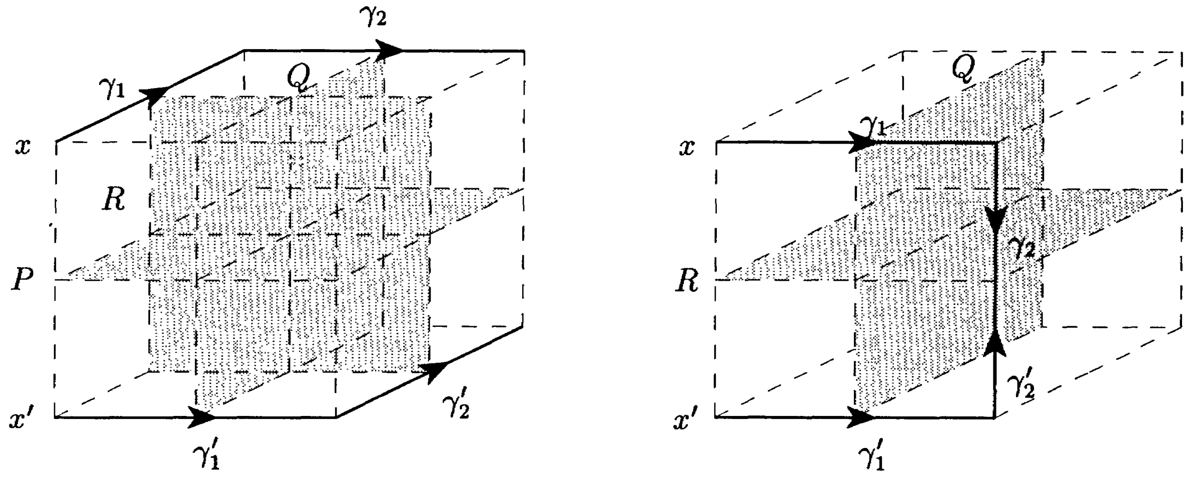

Figure 5 $Q \cap \text{poly}(x)$ and $R \cap \text{poly}(x)$ separate x from y

(2) If two faces of poly(x) separate x from y, then let them be given by $Q \cap \text{poly}(x)$ and $R \cap \text{poly}(x)$. Then, as explained above, $Q \cap \text{poly}(x')$ and $R \cap \text{poly}(x')$ are (the only) separating faces of poly(x'). Because γ and γ' are normal, they must both have Q and R as the first two planes crossed.

The normal geodesics $\gamma_3 \dots \gamma_n$ and $\gamma_3' \dots \gamma_n'$ satisfy the inductive hypothesis; also, the segments $\gamma_1\gamma_2$ and $\gamma_1'\gamma_2'$ are (at worst) 3-fellow travelers in whichever order they cross the planes (see Figure 5). Therefore γ and γ' are 3-fellow travelers. It is possible that one of the planes Q or R is equal to P. If we prepend a path σ to γ where length(σ) $= 2$, then the required result follows from the inductive hypothesis and the observation that σ and $\gamma_1'\gamma_2'$ are 3-fellow travelers.

(3) If three faces of poly(x) separate x from y, then one of them must be $P \cap \text{poly}(x)$. This is because each must intersect P, and it is not possible to have four planes which each meet poly(x) and pairwise intersect. So, assume that $P \cap \text{poly}(x)$ is a separating face, as are two other faces $Q \cap \text{poly}(x)$ and $R \cap \text{poly}(x)$. Then the normal geodesics $\gamma_4 \dots \gamma_n$ and $\gamma_4' \dots \gamma_n'$ are 3-fellow travelers by induction, and the initial segments $\gamma_1\gamma_2\gamma_3$ and $\gamma_1'\gamma_2'\gamma_3'$ are 3-fellow travelers in whichever order they cross the planes P, Q, and R. It follows that γ and γ' are 3-fellow travelers (see Figure 6). Again we note that if σ has length equal to 2 and $\tau(\sigma) = x$, then σ and $\gamma_1'\gamma_2'$ are 3-fellow travelers.
This completes the proof. □

We will also need the following variation of the preceding result.

LEMMA 2.5. *Let $x, y, x', y' \in L_0$ be such that x and x' are diagonally opposite vertices of a single square and $d(y, y') = 1$. Let $\gamma = \gamma_1 \dots \gamma_m$ and $\gamma' = \gamma_1' \dots \gamma_n'$ be normal geodesics from x to y and x' to y', respectively. Suppose that there is a plane $P \in \Pi$ such that P separates x from x' and y from y'. Assume further that $P \cap \gamma = P \cap \gamma' = \emptyset$. Then γ and γ' are 3-fellow travelers. Moreover, the result remains true if a path of length 1 is prepended to one of γ or γ'.*

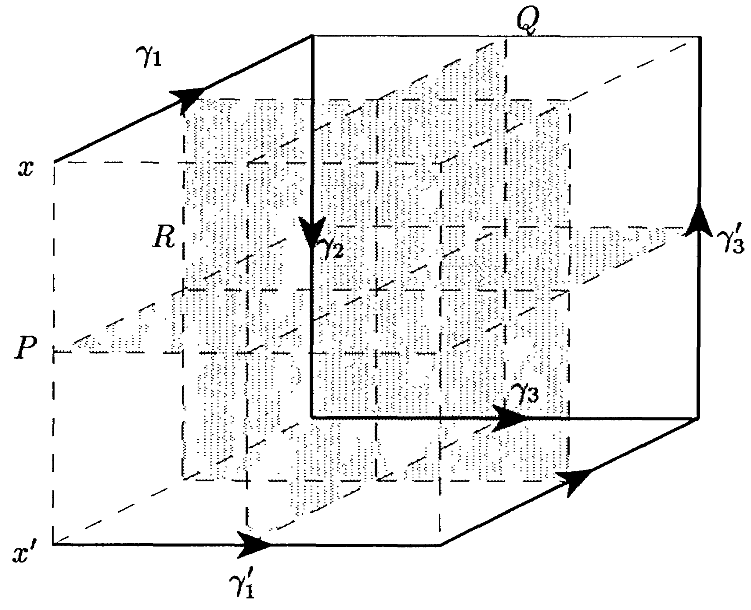

Figure 6 $P \cap \text{poly}(x)$, $Q \cap \text{poly}(x)$, and $R \cap \text{poly}(x)$ separate x from y

Proof. Denote by Q the plane other than P which separates x from x'. It must be the case that Q separates exactly one of the pairs x and y or x' and y'. We shall assume the latter, so $n = m+1$. We proceed by induction on the maximum length. The result is clearly true in the case where $m = 0$ and $n = 1$, so assume that $n > 1$ and that the result is true for all cases where the maximum length is less than n (note that we include the statement which concerns prepending a path of length 2 as part of the inductive hypothesis). If Q is the only face of poly(x') which separates, then γ_1' is the edge dual to this face and we apply Lemma 2.4 to γ and $\gamma_2' \dots \gamma_n'$. If $R \cap \text{poly}(x')$ is another face which separates, then we apply the inductive hypothesis to γ and $\gamma_3' \dots \gamma_n'$. □

We are now ready to establish the main result.

PROPOSITION 2.6. *Let $\gamma = \gamma_1 \dots \gamma_m$ and $\gamma' = \gamma_1' \dots \gamma_n'$ be normal geodesics with $d(\iota(\gamma), \iota(\gamma')) = 1$ and $d(\tau(\gamma), \tau(\gamma')) = 1$. Then γ and γ' are 3-fellow travelers.*

Proof. Let $P_x, P_y \in \Pi$ be the planes separating $x = \iota(\gamma)$ from $x' = \iota(\gamma')$ and $y = \tau(\gamma)$ from $y' = \tau(\gamma')$ respectively (see Figure 7). Let x_i be the terminal vertex of γ_i and let x_i' be the terminal vertex of γ_i'. Observe that a plane $Q \in \Pi$ which is equal to neither P_x nor P_y separates x from y if and only if it separates x' from y'. Therefore the difference in length between γ and γ' is either 0 or 2. This is because each of P_x and P_y separate either x from x' or y from y' (but not both), so either they separate the same pair, in which case the length difference is 2, or they separate different pairs, in which case the length difference is 0. We shall proceed by induction on the sum of the two lengths.

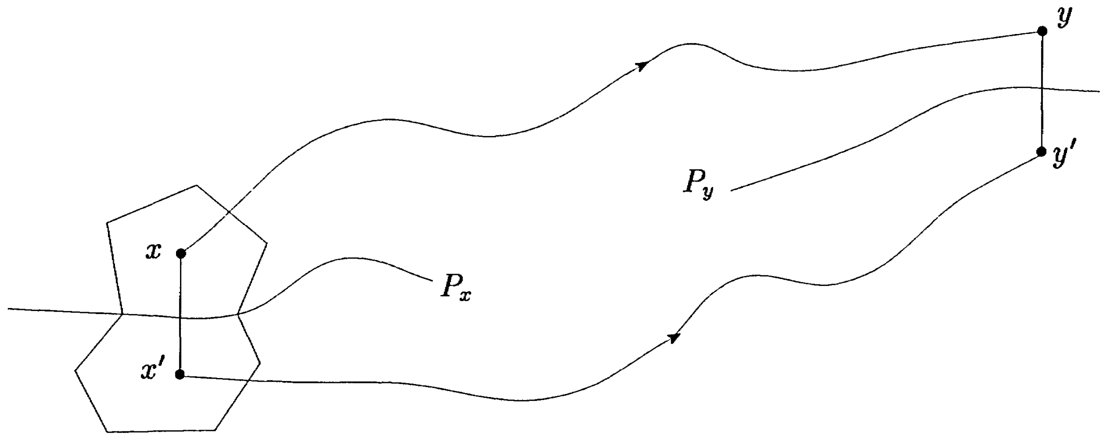

Figure 7 Two paths in $\tilde{M}$

The result is clear for the case where both of the paths have length 0, or one path has length 0 and the other has length 2. Assume that $n+m>0$. Assume that if ρ and ρ' satisfy the conditions of the statement and $\text{length}(\rho)+\text{length}(\rho')<n+m$ then ρ and ρ' are 3-fellow travelers; assume further that if σ is a path of length 2 such that $\sigma\rho$ and ρ' also satisfy the conditions of the statement then $\sigma\rho$ and ρ' are 3-fellow travelers. If $P_x=P_y$ then γ and γ' are 3-fellow travelers, by Lemma 2.4. If P_x separates neither x and y nor x' and y', then $P_x\cap\gamma=P_x\cap\gamma'=\emptyset$ and so we must have that $P_x=P_y$. Without loss of generality it will be assumed that P_x separates x' and y'. As a result, it is not possible to find a σ such that $\sigma\gamma'$ is a geodesic with $d(\iota(\sigma),\iota(\gamma))=1$. Hence we need only check the case in which a path of length 2 is prepended to γ; in what follows, σ denotes such a path. Consider as follows the number of faces of poly(x') that separate x' from y'.

Case 1: If $P_x\cap X'$ is the only separating face, then since γ' is a geodesic, γ_1' must be the edge dual to this face. This is precisely the edge joining x' and x, and so $\gamma_2'\dots\gamma_n'$ is a normal geodesic from x to y'. Hence $\gamma_2'\dots\gamma_n'$ and $\gamma_2\dots\gamma_n$ are normal geodesics which begin and end a distance 1 apart (see Figure 8). The initial segments γ_1 and γ_1' are clearly 1-fellow travelers. It follows that γ and γ' are 3-fellow travelers. Also, if $\sigma=\sigma_1\sigma_2$ is a path such that $\sigma\gamma$ and γ satisfy the conditions of the proposition then clearly σ_1 and γ_1' are 3-fellow travelers, and from the inductive hypothesis we have that $\sigma_2\gamma_1(\gamma_2\dots\gamma_m)$ and $\gamma_2'\dots\gamma_n'$ are 3-fellow travelers. It follows that $\sigma\gamma$ and γ' are 3-fellow travelers.

Case 2: Assume that $P_x\cap$poly(x') and one other face of poly(x') separate x' and y'. Then $d(x,x_2')=1$ (see Figure 9). Applying the inductive hypothesis to γ and $\gamma_3'\dots\gamma_n'$, we see that γ and γ' are 3-fellow travelers. If we prepend σ to γ, then clearly σ and $\gamma_1'\gamma_2'$ are 3-fellow travelers. From the inductive hypothesis we see that γ and $\gamma_3'\dots\gamma_n'$ are 3-fellow travelers. Hence $\sigma\gamma$ and γ' are 3-fellow travelers.

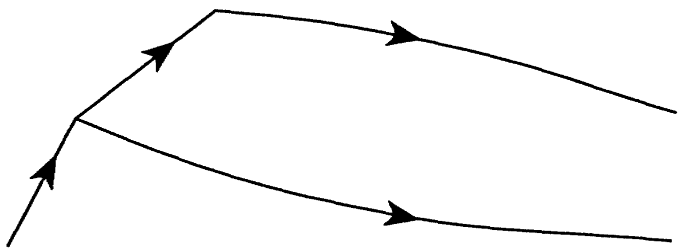

Figure 8 poly(x) has one separating face

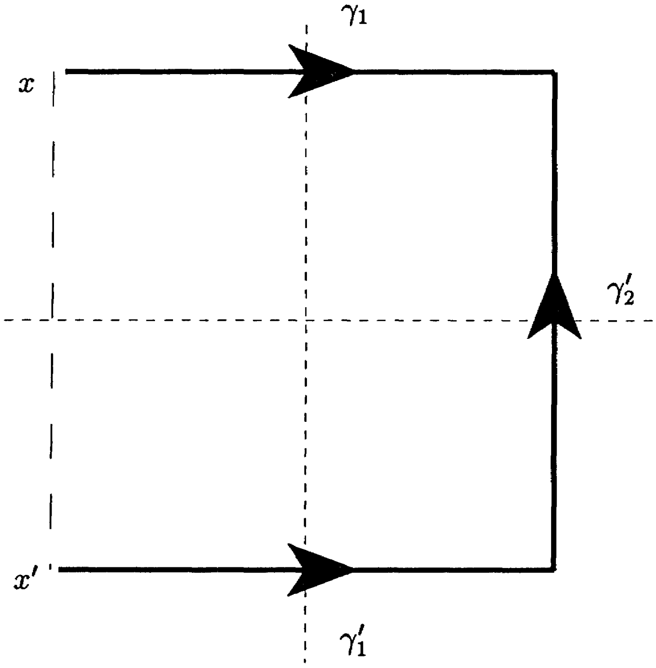

Figure 9 Two planes separate x' and y'

Case 3: Assume that $P_x \cap \text{poly}(x')$ and two other faces of poly(x') separate x' and y'. Call these faces $Q \cap \text{poly}(x')$ and $R \cap \text{poly}(x')$. If one, say R, is equal to P_y then $Q \cap \text{poly}(x)$ is a separating face of poly(x). Observe that poly(x) has at most two separating faces. If $Q \cap \text{poly}(x)$ is the only separating face then, by applying Lemma 2.4 to $\gamma_2 \dots \gamma_m$ and $\gamma_4' \dots \gamma_n'$, we see that $\gamma_2 \dots \gamma_m$ and $\gamma_2'\gamma_3'(\gamma_4' \dots \gamma_n')$ are 3-fellow travelers. Clearly γ_1 and γ_1' are 3-fellow travelers, and therefore so too are γ and γ'. If σ is prepended to γ, observe that $\sigma\gamma_1$ and $\gamma_1'\gamma_2'\gamma_3$ must be 3-fellow travelers, as are $\gamma_2 \dots \gamma_m$ and $\gamma_4' \dots \gamma_n'$ (Lemma 2.4). Therefore $\sigma\gamma$ and γ' and 3-fellow travelers.

It may be that there is one other face of poly(x) that separates. Applying Lemma 2.5 to $\gamma_3 \dots \gamma_m$ and $\gamma_4' \dots \gamma_n'$, we have that $\gamma_3 \dots \gamma_m$ and $\gamma_3'(\gamma_4' \dots \gamma_n')$ are 3-fellow travelers. Clearly $\gamma_1\gamma_2$ and $\gamma_1'\gamma_2'$ are 3-fellow travelers and so γ and γ' are 3-fellow travelers. If σ is prepended to γ, observe that $\sigma\gamma_1$ and $\gamma_1'\gamma_2'\gamma_3'$ are 3-fellow travelers as are $\gamma_2(\gamma_3 \dots \gamma_m)$ and $\gamma_4' \dots \gamma_n'$ (Lemma 2.5). Therefore $\sigma\gamma$ and γ' are 3-fellow travelers.

If neither Q nor R is equal to P_y then both $Q \cap \text{poly}(x)$ and $R \cap \text{poly}(x)$ are separating faces of poly(x). There may be a third face of poly(x) which separates, but in any case $\gamma_4 \ldots \gamma_m$ and $\gamma_4' \ldots \gamma_n'$ are normal geodesics which begin and end a distance 1 apart and the initial segments of length 3 are 3-fellow travelers. Therefore γ and γ' are 3-fellow travelers. If σ is prepended to γ, notice that $\sigma\gamma_1$ and $\gamma_1'\gamma_2'\gamma_3'$ are 3-fellow travelers. Applying the inductive hypothesis to $\gamma_4 \ldots \gamma_m$ and $\gamma_4' \ldots \gamma_n'$, we have that $\gamma_2\gamma_3(\gamma_4 \ldots \gamma_m)$ and $\gamma_4' \ldots \gamma_n'$ are 3-fellow travelers. Hence $\sigma\gamma$ and γ' are 3-fellow travelers.

This exhausts the possibilities and so completes the proof. □

We remark that it is possible to show that if two geodesics are within a Hausdorff distance of k, then they are $2k$-fellow travelers. This could then be used to shorten the proof slightly at the expense of increasing the fellow-traveler constant to 6.

It is now straightforward to verify that the structure given by $\mathcal{L}(\mathcal{G})$ is bi-automatic.

THEOREM 2.7. *$\mathcal{G}$ is a bi-automatic groupoid.*

Proof. As already stated, conditions (i) and (ii) of the definition are satisfied. We will verify condition (iii) of the definition by using Proposition 2.6. Let γ, γ', x, y, x', and y' be as in the definition. Let $\delta^x = \delta_1^x \ldots \delta_m^x$ and $\delta^y = \delta_1^y \ldots \delta_n^y$ be geodesics joining x to x' and joining y to y', respectively. Let $x_i = \iota(\delta_i^x)$ for $1 \le i \le m$ and $x_i = x'$ for $i > m$. Similarly, $y_i = \iota(\delta_i^y)$ for $1 \le i \le n$ and $y_i = y'$ for $i > n$. For each i such that $1 \le i \le \max(m, n)$, choose a normal geodesic γ_i joining x_i to y_i and let $\gamma_i = \gamma'$ for $i = \max(m, n) + 1$; see Figure 10. Then, for each i, γ_i and γ_{i+1} are 3-fellow travelers. It follows that, for a fixed t, $\gamma(t)$ and $\gamma'(t)$ can be joined by a path ϵ where $l(\epsilon) \le$

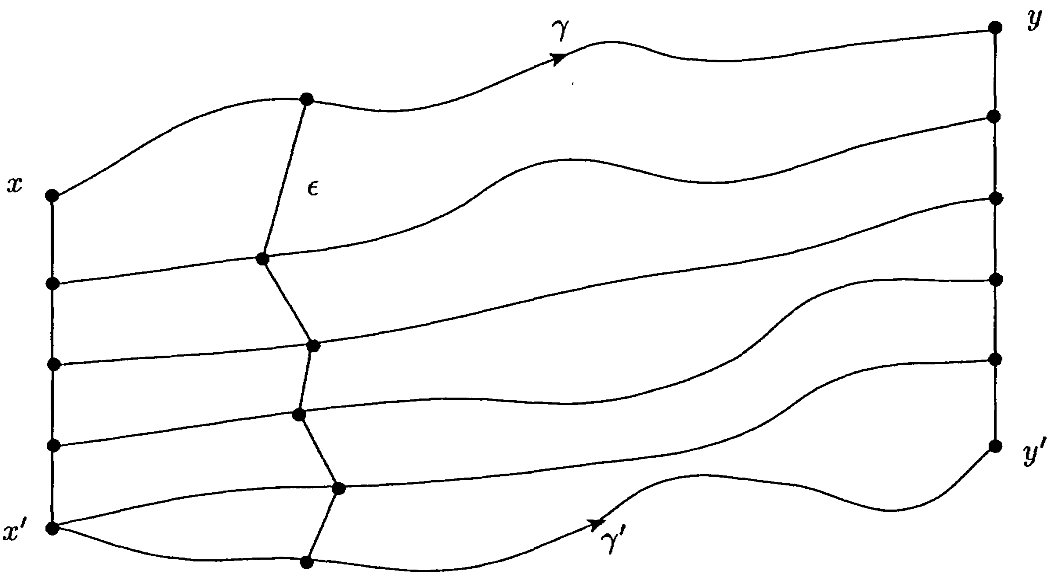

Figure 10 γ and γ' are fellow travelers

$3(\max(m,n)+1)$. Thus, $d(\gamma(t),\gamma'(t)) \leq 3(d(x,x')+d(y,y')+1)$ and it follows that $\mathcal{G}$ is bi-automatic. □

From Proposition 11.1.5 in [4], we have the following theorem as an immediate consequence.

THEOREM 2.8. *The fundamental group of a closed 3-manifold with a cubing of nonpositive curvature is bi-automatic.*

3. Canonical Surface Groups Are Rational

We recall that if $(\mathcal{A},\mathcal{L})$ is a rational structure for a group G, then a subset H of G is *$\mathcal{L}$-rational* if the set of words in $\mathcal{L}$ that project to elements of H is a regular sublanguage of $\mathcal{A}^*$. For subgroups this is equivalent to the following concept: H is *$\mathcal{L}$-quasiconvex* if there exists $k \geq 0$ such that, for each word w in $\mathcal{L}$ that projects to an element of H, the path in $\Gamma(G,\mathcal{A})$ corresponding to w lies within a k-neighborhood of the set of vertices that represent elements of H.

Consider the canonical surface defined above, and let S be a component of this surface. It will be shown that the corresponding subgroup of $\pi_1(M)$ is rational with respect to the bi-automatic structure $\mathcal{L}(\mathcal{G})$.

From the description of the canonical surface, one obtains a second, finer cubing of M in the obvious way—that is, each cube in the original cubing is divided into eight smaller cubes. Let $\mathcal{G}'$ be the groupoid obtained as for $\mathcal{G}$ (so $\mathcal{G}$ is a subgroupoid of $\mathcal{G}'$), and define $\mathcal{A}'$ and $\mathcal{L}(\mathcal{G}')$ in an analogous manner.

Fix a vertex p in $\mathcal{G}$ and a vertex s in $\mathcal{G}'$ such that s lies in the surface S, and denote by $\mathcal{L}_p(\mathcal{G})$ the sublanguage of $\mathcal{L}(\mathcal{G})$ consisting of words which project to paths in M that begin and end at p. Similarly define $\mathcal{L}_s(\mathcal{G}') \subseteq \mathcal{L}(\mathcal{G}')$.

As in [4, p. 247], an element of $\mathcal{A}$ (corresponding to a directed edge in L_1^M) may be regarded as a generator for the group $\mathcal{G}_p$. Choose a maximal tree T in the 1-skeleton of the original cubing, and a maximal tree T' in the finer cubing, such that $T \subseteq T'$. An edge e represents the element of $\mathcal{G}_p$ defined by following T from p to the initial vertex of e, following along e, and then returning within T to p. Then $\mathcal{L}_p(\mathcal{G})$ gives a bi-automatic structure for $\mathcal{G}_p \cong \pi_1(M)$. Similarly, $\mathcal{L}_s(\mathcal{G}')$ gives a bi-automatic structure for $\mathcal{G}'_s \cong \pi_1(M)$.

The main result of this section, Theorem 3.5, relies on the concept of equivalent automatic structures for a group, as introduced by Neumann and Shapiro in [11]. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two regular languages over alphabets $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then $\mathcal{L}_1 \cup \mathcal{L}_2$ is a regular language over the alphabet $\mathcal{A}_1 \cup \mathcal{A}_2$.

DEFINITION. If both $\mathcal{L}_1$ and $\mathcal{L}_2$ give automatic structures for a group G, we say that the two structures are *equivalent* if $\mathcal{L}_1 \cup \mathcal{L}_2$ gives an asynchronous automatic structure for G.

PROPOSITION 3.1. *The languages $\mathcal{L}_p(\mathcal{G})$ and $\mathcal{L}_s(\mathcal{G}')$ yield equivalent bi-automatic structures for $\pi_1(M)$.*

Proof. From the definition we have that p is also a vertex of $\mathcal{G}'$. We first show the following two propositions.

PROPOSITION 3.2. *The languages $\mathcal{L}_p(\mathcal{G}')$ and $\mathcal{L}_s(\mathcal{G}')$ yield equivalent bi-automatic structures for $\pi_1(M)$.*

Proof. Consider $\Gamma(\pi_1(M),\mathcal{A}')$, the Cayley graph of $\pi_1(M)$ with respect to $\mathcal{A}'$. Let $w_s \in \mathcal{L}_s(\mathcal{G}')$ and $w_p \in \mathcal{L}_p(\mathcal{G}')$ be words over $\mathcal{A}'$ which map to paths in $\Gamma(\pi_1(M),\mathcal{A})$ such that $d(\bar{w}_s, \bar{w}_p) \leq 1$. By regarding $\mathcal{A}'$ as generators for $\mathcal{G}'$, we may also consider w_s and w_p as paths in $\Gamma(\mathcal{G}',\mathcal{A}')$. As such, they have endpoints separated by a distance of less than twice the diameter of T' and are therefore fellow travelers (with constant $4(4\,\mathrm{diam}(T')+1)$). It follows that the paths in $\Gamma(\pi_1(M),\mathcal{A}')$ are fellow travelers.

Recall that ϕ is a (k,ϵ) *quasi-isometric embedding* if, for all x_1, x_2: $(1/k)d(x_1,x_2)-\epsilon \leq d(\phi(x_1),\phi(x_2)) \leq kd(x_1,x_2)+\epsilon$ (note that ϕ need not be injective). We thus have a quasi-isometric embedding $\phi:\Gamma(\mathcal{G}',\mathcal{A}')\to \Gamma(\pi_1(M),\mathcal{A}')$ given as follows: We regard $\pi_1(M)$ as being given by a vertex group of $\mathcal{G}'$, and we have base point p for $\Gamma(\mathcal{G}',\mathcal{A}')$. A vertex of $\Gamma(\mathcal{G}',\mathcal{A}')$ is a morphism based at p and so has as representative some word $w \in \mathcal{L}(\mathcal{G}')$ over $\mathcal{A}'$. As described above, $\mathcal{A}'$ may also be regarded as a set of generators for $\mathcal{G}'_p$ or $\mathcal{G}'_s$. In either case, w will be sent to the same element of $\pi_1(M)$. This defines the map ϕ. Given $x_1, x_2 \in \Gamma(\mathcal{G}',\mathcal{A}')$, we have:

$$d(\phi(x_1),\phi(x_2)) \leq d(x_1,x_2) \leq \mathrm{diam}(T')\,d(\phi(x_1),\phi(x_2)).$$

The equivalence of $\mathcal{L}_p(\mathcal{G}')$ and $\mathcal{L}_s(\mathcal{G}')$ then follows from the fact that they are both sublanguages of $\mathcal{L}(\mathcal{G}')$; this yields a bi-automatic structure for $\mathcal{G}'$, which therefore has the fellow traveler property. □

We also have the following proposition.

PROPOSITION 3.3. *The languages $\mathcal{L}_p(\mathcal{G})$ and $\mathcal{L}_p(\mathcal{G}')$ yield equivalent bi-automatic structures for $\pi_1(M)$.*

Proof. The proof is very similar to that of Proposition 3.2. Let $w \in \mathcal{L}_p(\mathcal{G})$ and $w' \in \mathcal{L}_p(\mathcal{G}')$ be two words which map to paths in $\Gamma(\pi_1(M),\mathcal{A}\cup\mathcal{A}')$ such that $d(\bar{w},\bar{w}') \leq 1$. There is a word $v \in \mathcal{L}_p(\mathcal{G}')$ with length twice that of w and such that v is within a distance of 1 from w inside $\Gamma(\pi_1(M),\mathcal{A}\cup\mathcal{A}')$. This word is obtained by rewriting each element of $\mathcal{A}$ as the product of two elements from $\mathcal{A}'$ in the obvious way. Then v and w' are fellow travelers within $\Gamma(\pi_1(M),\mathcal{A}')$. Hence w and w' are asynchronous fellow travelers within $\Gamma(\pi_1(M),\mathcal{A}\cup\mathcal{A}')$. Notice that we need to use the fact that the definition of equivalence requires only an asynchronous structure. □

Combining Propositions 3.2 and 3.3 gives the required result that $\mathcal{L}_p(\mathcal{G})$ and $\mathcal{L}_s(\mathcal{G}')$ are equivalent structures for $\pi_1(M)$. □

To show that $\pi_1(S)$ is a $\mathcal{L}_p(\mathcal{G})$-rational subgroup of $\pi_1(M)$, we combine Proposition 3.1 with the following.

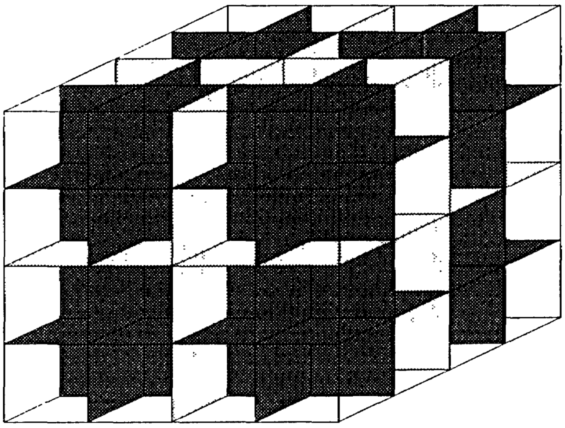

Figure 11 Further subdivision of the cubing

THEOREM 3.4. *$\pi_1(S)$ is a $\mathcal{L}_s(\mathcal{G}')$-rational subgroup of $\pi_1(M)$.*

Proof. Recall that S is a component of the canonical surface obtained from the first cubing. Choose a lift $\tilde{s}$ of s and denote by P_S the lift of S which contains $\tilde{s}$. The 1-skeleton of the cubing of $\tilde{M}$ may be thought of as the Cayley graph $\Gamma(\mathcal{G}',\mathcal{A}')$ with base point $\tilde{s}$. Vertices are labeled by morphisms in $\mathcal{G}'$ with initial vertex $\tilde{s}$.

Consider the second, finer cubing of M. One may further subdivide each cube to produce again a canonical immersed incompressible surface in M (see Figure 11). As in the definition of Π, let Π' denote the family of lifts of components of this surface to $\tilde{M}$. There are two vertices of the second cubing which are a distance 1 from $\tilde{s}$ and do not lie in P_S. Let $P_1', P_2' \in \Pi'$ be the planes which separate $\tilde{s}$ from each of these vertices respectively. From the construction of the canonical surfaces it is clear that any other vertex (of the second cubing) in P_S also lies between P_1' and P_2'. It follows then that a geodesic with initial vertex in P_S which leaves P_S must cross one of P_1' or P_2'. Such a geodesic cannot therefore return to P_S, as geodesics cross planes in Π' at most once [12, Lemma 11.5.2, p. 241]. Thus a geodesic edge path that begins and ends in P_S must lie entirely within P_S. In particular, every normal geodesic based at $\tilde{s}$ lies within P_S. Hence, in $\Gamma(\mathcal{G}',\mathcal{A}')$, each word in $\mathcal{L}_s(\mathcal{G}')$ that projects to an element of $\pi_1(S)$ must lie within a bounded distance of $\pi_1(S)$. This means that $\pi_1(S)$ is $\mathcal{L}_s(\mathcal{G}')$-quasiconvex and is therefore $\mathcal{L}_s(\mathcal{G}')$-rational (see e.g. [5, Thm. 2.2]). □

REMARK. As seen in the preceding proof, normal geodesics which begin and end on a given plane lie entirely in that plane; that is, they are in some sense totally geodesic.

We can now apply Proposition 2.7 in [11] to combine Theorem 3.4 and Proposition 3.1 to yield the following theorem.

THEOREM 3.5. *$\pi_1(S)$ is a $\mathcal{L}_p(\mathcal{G})$-rational subgroup of $\pi_1(M)$.*

Combining the with two results contained in [5], we collect here a list of some rational subgroups of the fundamental group of a nonpositively cubed 3-manifold.

THEOREM 3.6. *The following subgroups of* $\pi_1(M)$ *are rational:*

(1) $\pi_1(S)$, *the subgroup corresponding to a canonical surface in* M;
(2) *the centralizer of a finite subset of* $\pi_1(M)$;
(3) *polycyclic subgroups of* $\pi_1(M)$.

We remark that recent work of Neumann [10] shows that, for the second and third results of Theorem 3.6, it suffices to have an asynchronous bicombing.

We also observe that since $\pi_1(S)$ is rational it follows from work of Mihalik [9] that the covering of M corresponding to $\pi_1(S)$ is a missing boundary manifold; that is, it can be embedded in a compact manifold in such a way that the complement is a subset of the boundary of the compact manifold.

4. Examples

A. Some examples of 3-manifolds which admit cubings of nonpositive curvature are given in [1]. It is shown there that if M is defined by taking a branched cover of S^3 over the 5_2 knot or the Whitehead link, with all components of the branch set having degree at least 4, then M has a cubing of nonpositive curvature. M has such a cubing also if M is a branched cover of S^3 over the Borromean rings with all components of the branch set having degree at least 2.

B. In [2], Aitchison and Rubinstein give a procedure for obtaining cubings on certain surgered manifolds. We give here a specific example. Consider the link 6_1^3 (Figure 12) and the corresponding 4-valent planar graph. Two-color the regions of this graph as shown, and then split each vertex so as to isolate the shaded regions. The resulting graph may be considered as the 1-skeleton of a polyhedron (an hexagonal prism). Let P and P' be two copies of this polyhedron, and identify the faces as follows:

(i) Corresponding faces that come from a shaded region are identified by a rotation of $\pi/2$ in a positive (anticlockwise) sense when viewed from the interior of P.

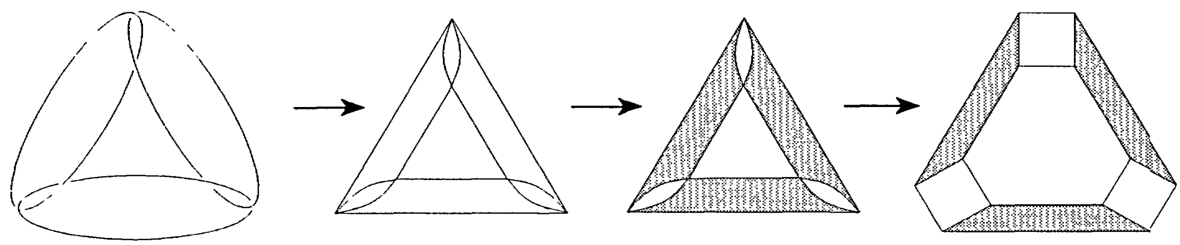

Figure 12 Steps toward obtaining a cubical decomposition of the canonically surged manifold M

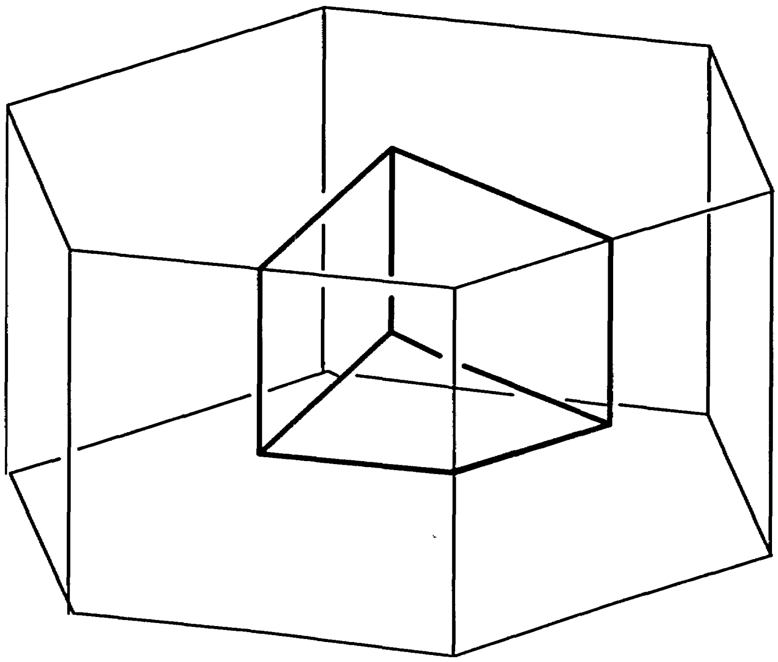

Figure 13 A hexagonal prism may be subdivided into twelve cubes, one of which is shown; all the cubes share a common vertex at the centre of the prism

(ii) Pairs of faces that correspond to unshaded regions are identified via a rotation $-\pi$ or $-2\pi/3$ if the face has degree 4 or 6, respectively.

In this way, one obtains a manifold M which is defined by $(+2, +2, +2)$-surgery on 6_1^3 [2]. To obtain a cubing of M, observe that a hexagonal prism may be subdivided into twelve cubes as shown in Figure 13. Hence we have a decomposition of M into twenty-four cubes. If we declare each cube to be Euclidean then we obtain a cubing of nonpositive curvature on M. Note that M is Haken.

C. Given a manifold with a cubing, one can obtain other cubed manifolds by taking branched covers. Let M be a manifold with a cubing of nonpositive curvature. If σ is a closed, embedded edge path in M which is locally geodesic and null-homologous in M, then any branched cover of M over σ has a cubing of nonpositive curvature.

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