A Characterization of Virtual Poincaré Duality Groups

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An important open question in geometric group theory and the topology of manifolds asks whether every Poincaré duality group is realized as the fundamental group of a closed aspherical manifold. This note establishes an analogous property for virtual Poincaré duality groups following ideas and results of [10; 11; 12] and gives an approximate solution to the realization question for Poincaré duality groups.

We recall that a group Γ is a Poincaré duality group if Γ is of type FP and, for some $n>0$, $H^i(\Gamma;\mathbf{Z}\Gamma)=0$ for $i\neq n$ and $H^n(\Gamma;\mathbf{Z}\Gamma)$ is an infinite cyclic group [1; 7; 4]. (For finitely presented groups, these conditions are equivalent to the topological condition that the Eilenberg–MacLane space $B\Gamma$ is a Poincaré complex.) Such a group satisfies Poincaré duality in the form

$$H^i(\Gamma;M)\cong H_{n-i}(\Gamma;\tilde{M}),$$

where M is a $\mathbf{Z}\Gamma$ module, $\tilde{M}=H^n(\Gamma;\mathbf{Z}\Gamma)\otimes_{\mathbf{Z}}M$, and the tensor product is given the diagonal Γ-action. (The infinite cyclic group $D=H^n(\Gamma;\mathbf{Z}\Gamma)$ is customarily called the *dualizing module* and has an associated orientation character $w:\Gamma\to\{1,-1\}$ such that $\gamma\cdot x=w(\gamma)x$ for each $\gamma\in\Gamma$ and each $x\in D$; $\tilde{M}$ is therefore described as "M twisted by the orientation character".)

Recall also that a group G is said to have a *virtual* property if and only if G possesses a finite-index subgroup H with that property. Thus, a group G is of finite virtual cohomological dimension if some subgroup H of finite index in G has finite cohomological dimension, and G is a virtual Poincaré duality group if some finite-index subgroup H is a Poincaré duality group.

The fundamental group of any closed aspherical manifold is a Poincaré duality group, and every known Poincaré duality group is of this form. Since many virtual Poincaré duality groups contain elements of finite order, we can not expect such groups to be realized as fundamental groups of aspherical manifolds.

THEOREM 1. *Let Γ be a finitely presented group of finite virtual cohomological dimension. Then Γ is a virtual Poincaré duality group if and only if there exists a closed PL manifold M with fundamental group Γ and universal cover $\tilde{M}$ such that $\tilde{M}$ is homotopy equivalent to a finite complex.*

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Proof. Γ may be written as an extension

$$1 \to \Gamma_0 \to \Gamma \to G \to 1,$$

with G a finite group and the cohomological dimension of Γ_0 finite. We recall that any Poincaré duality group has finite cohomological dimension and that the fundamental group of a closed manifold is finitely presented, so our algebraic hypotheses exclude no groups of interest for the theorem.

If a manifold M with the prescribed properties exists, then [12] shows that Γ is of type FP(∞). (See [2] and [4] for details on this finiteness property.) The finite-index subgroup Γ_0 of Γ is therefore a finitely presented group of finite cohomological dimension which is of type FP(∞), so the Eilenberg-MacLane space $B\Gamma_0$ is a finitely dominated CW complex (see [4, pp. 199, 205]. The fibration

$$\tilde{M} \to \tilde{M}/\Gamma_0 \to B\Gamma_0$$

has finitely dominated fiber, base, and total space, and the total space is a Poincaré complex, so the Gottlieb–Quinn fibering theorem [6; 8] shows that $\tilde{M}$ and $B\Gamma_0$ are also Poincaré complexes. Thus Γ_0 is a Poincaré duality group [7] and Γ is a virtual Poincaré duality group.

Conversely, if Γ is a finitely presented virtual Poincaré duality group then we may write Γ as an extension $1 \to \Gamma_0 \to \Gamma \to G \to 1$, with G a finite group and $\Gamma_0 < \Gamma$ an orientable Poincaré duality group of finite index. G acts freely and simplicially on a simply connected closed PL manifold V. (For example, take a faithful representation $\rho: G \to SU(m)$ for some $m \geq 2$, and let G act on $SU(m)$ by left translation.) The action of Γ_0 on $E\Gamma_0$ induces an action of Γ on the product $(E\Gamma_0)^{|G|}$ (see [10] for more on this induced action). A fibration results:

$$B\Gamma_0 \simeq \Gamma_0\backslash(E\Gamma_0)^{|G|} \to (E\Gamma_0)^{|G|} \times_\Gamma V \to G\backslash V.$$

Since $B\Gamma_0$ is a finitely dominated Poincaré complex and $G\backslash V$ is a closed PL manifold, the fiber, base, and total space of this fibration are finitely dominated and two of the three are Poincaré complexes. The Gottlieb–Quinn fibering theorem then shows that the total space $X := (E\Gamma_0)^{|G|} \times_\Gamma V$ is a Poincaré complex.

Let $r > 1$. The product formula for the Wall finiteness obstruction [5] shows that $X \times S^{2r-1}$ is homotopy equivalent to a finite Poincaré complex Y. Let $n = \text{dimension}(Y)$ and let N be a regular neighborhood of a simplicial imbedding of Y in $\mathbf{R}^{n+d}$ for $d \gg n+1$. Consider the Spivak normal fibration of Y, $F(\iota) \to \partial N \xrightarrow{\iota} N \simeq Y$, where $F(\iota) \simeq S^{d-1}$ [9; 3]. We have an induced fibration of universal covers,

$$\begin{array}{ccccc}
F(\iota) & \longrightarrow & \partial\tilde{N} & \longrightarrow & \tilde{N} \\
\simeq\downarrow & & \| & & \downarrow\simeq \\
S^{d-1} & \longrightarrow & \partial\tilde{N} & \longrightarrow & \tilde{X} \times S^{2r-1} \\
 & & & & \downarrow\simeq \\
 & & & & V \times S^{2r-1},
\end{array}$$

in which $\partial\tilde{N}$ is homotopy equivalent to a finite complex, so $M=\partial N$ is a closed PL manifold with the required properties. □

The theorem also gives a characterization of Poincaré duality groups:

COROLLARY 2. *Let Γ be a finitely presented group of finite cohomological dimension. Then Γ is a Poincaré duality group if and only if there exists a closed PL manifold M with fundamental group Γ and universal cover $\tilde{M}$ such that $\tilde{M}$ is homotopy equivalent to a finite complex.*

Proof. If Γ is a Poincaré duality group, then the argument for Theorem 1 shows that the required closed manifold M exists.

Conversely, given such a manifold M, consider the fibration

$$\tilde{M}\to M\to B\Gamma,$$

which has finitely dominated total space and fiber. In [11] Γ is shown to be of type FP(∞), so by [4, pp. 199, 205] $B\Gamma$ is also finitely dominated. The Gottlieb-Quinn theorem now shows that $B\Gamma$ is a Poincaré complex, so Γ is a Poincaré duality group. □

The corollary may be viewed as an approximate solution to the conjecture that Poincaré duality groups are realized by closed aspherical manifolds, and is consistent with that conjecture. The closed PL manifold V with free, simplicial G-action required above may be taken to be s-connected for arbitrarily large s (e.g., take $G\backslash W$ to be the boundary of a regular neighborhood in a high-dimensional Euclidean space of the $(s+1)$-skeleton of a BG complex) and the dimensions of the odd-dimensional spheres S^{2r-1} used above are arbitrary, so for any $s\geq 0$ we may construct M so that a classifying map $M\to B\Gamma$ is an s-equivalence; in this sense we have an arbitrarily good approximation to the realization conjecture. The manifold M may also be constructed so that $\tilde{M}$ has the homotopy type of a spherical fibration over a product of spheres, with Γ acting trivially on $H_*(\tilde{M})$, so although we do not obtain an aspherical manifold by this method, the homotopy type that is produced can be relatively uncomplicated as well as highly connected in the universal cover.

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