Nonlinear Potential Theory on the Ball, with Applications to Exceptional and Boundary Interpolation Sets

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0. Introduction

Let S denote the boundary of B_n, the unit ball in $\mathbf{C}^n$, and let $d\sigma$ be the usual rotation-invariant measure defined on S. If $f \in L^1(d\sigma)$, $\eta \in S$, and $0 < \beta < n$, then we define the non-isotropic potential

$$I_\beta f(\eta) = \int_S \frac{f(\zeta)}{|1 - \langle \eta, \zeta \rangle|^{n-\beta}}\, d\sigma. \tag{0.1}$$

We also set

$$I_\beta \mu(\eta) = \int_S \frac{d\mu}{|1 - \langle \eta, \zeta \rangle|^{n-\beta}} \tag{0.2}$$

for any finite measure μ ($\mu \in \mathfrak{M}(S)$).

If $1 < p < \infty$, let L^p_β be the space of potentials $F = I_\beta f$ where $f \in L^p(d\sigma)$ with norm

$$\|F\|_{L^p_\beta} = \|f\|_{L^p}.$$

The space L^p_β is an analog of the usual potential space defined in Euclidean space. In the case where β is an integer, L^p_β coincides with a (non-isotropic) Sobolev space.

We will also need the Hardy–Sobolev spaces H^p_β ($0 < \beta, p < \infty$) of functions F holomorphic in the unit ball. Let

$$F(z) = \sum_k f_k(z) \tag{0.3}$$

be the homogeneous polynomial expansion of F (see [Ru]) and

$$R^\beta F(z) = \sum_k (1+k)^\beta f_k(z) \tag{0.4}$$

its (radial) fractional derivative of order β. Then H^p_β is the space of all holomorphic functions F on B_n with the property that

$$\|F\|_{H^p_\beta} = \sup_{0<r<1} \|R^\beta F(r\zeta)\|_{L^p(d\sigma)} < \infty. \tag{0.5}$$

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It can be shown (this is implicit in [AC1, Lemma 2.2]) that H^p_β, $1<p<\infty$, can be identified as a closed subspace of L^p_β consisting of the boundary values

$$f(\zeta)=\lim_{r\to 1}F(r\zeta).$$

Let $z\in B_n$ and $F\in L^1(d\sigma)$. Then the invariant Poisson integral of f is defined by

$$P[f](z)=\int_S f(\zeta)\frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}}\,d\sigma.$$

For $\zeta\in S$ and $\alpha>1$, let

$$\Gamma_\alpha(\zeta)=\left\{z\colon |1-\langle z,\zeta\rangle|<\frac{\alpha}{2}(1-|z|^2)\right\} \tag{0.6}$$

be the admissible approach regions and

$$M_\alpha F(\zeta)=\{\sup|F(z)|\colon z\in\Gamma_\alpha(\zeta)\} \tag{0.7}$$

the corresponding admissible maximal functions.

The *exceptional set* $E(F)$ of a function F defined on the ball is that subset of a sphere where F fails to have a finite admissible limit (i.e., the boundary limit within the admissible region) for all $\alpha>1$.

It is easily seen that exceptional sets of the invariant Poisson integrals of Sobolev functions f ($f\in L^p_\beta$, $1<p<\infty$, $0<\beta\le n/p$) are exactly the sets of (non-isotropic) capacity zero (see [AC1]). (Note that the only interesting case in the problem of exceptional sets is $0<\beta\le n/p$, since for $\beta>n/p$ all functions from the Hardy–Sobolev space H^p_β are continuous in the closed ball $\bar{B}_n=\{z\colon |z|\le 1\}$ and hence have admissible limits everywhere on S.)

Here the corresponding capacity, $\mathrm{Cap}(E,L^p_\beta)$, of a set $E\subset S$ is defined by

$$\mathrm{Cap}(E,L^p_\beta)=\inf\{\|f\|^p_{L^p}\colon I_\beta f\ge 1 \text{ on } E;\ f\ge 0,\ f\in L^p(d\sigma)\}. \tag{0.8}$$

The situation in the case of Hardy–Sobolev spaces H^p_β is much more complicated. As was shown in [Ah], for $0<p\le 1$, a function in H^p_β has admissible limits almost everywhere with respect to the Hausdorff capacity H_m with $m=n-\beta p$. The capacity H_m for $E\subset S$ is defined by

$$H_m(E)=\inf\sum\delta_j^m, \tag{0.9}$$

where the infimum is taken over all coverings of E by unions of Koranyi balls $B(\zeta_j,\delta_j)=\{\eta\in S\colon |1-\langle\eta,\zeta_j\rangle|<\delta_j\}$. On the other hand [C1], every compact E with $H_m(E)=0$ is the exceptional set for a function $F\in H^p_\beta$.

The case $1<p<\infty$ was investigated in [Ah] and [AC1]. It was shown that every function in H^p_β has admissible limits with an exceptional set of zero capacity $\mathrm{Cap}(\cdot,L^p_\beta)$. On the other hand, as was discovered by Ahern [Ah], at least for $1<p\le 2$ and $n-\beta p\ge 1$ there exist compact subsets $E\subset S$ such that $\mathrm{Cap}(E,L^p_\beta)=0$ but E is not exceptional for any Hardy–Sobolev function.

Note that if $n-\beta p=1$ and $1<p\le 2$, one can set $E=T^1$ to get an example of a non-exceptional set of zero L^p_β-capacity. (If $p>2$, it follows from a result of Ullrich [Ul] that T^1 is exceptional for H^p_β, $n-\beta p=1$. Nevertheless, non-exceptional sets of L^p_β-capacity zero exist for all $1<p<\infty$ and $n-\beta p\geq 1$. Some examples of this type recently constructed by the authors are given in [CV].)

One of the main goals of this paper is to prove the following conjecture (see [AC1]).

CONJECTURE 1. *Let $p>1$ and $0\leq n-\beta p<1$. If K is a compact subset of S such that* $\operatorname{Cap}(K, L^p_\beta)=0$, *then K is exceptional for the space H^p_β.*

In [AC1], Conjecture 1 was shown to be true in some special cases, including the case where the set E is contained in a complex tangential manifold or nonsingular curve. It was also shown to be true for the Hilbert case $p=2$. A more general conjecture on exceptional sets stated in [AC1] is still open.

CONJECTURE 2. *Let $p>1$ and $0<\beta p\leq n$. If a compact set $K\subset S$ is a peak set for the ball algebra A such that* $\operatorname{Cap}(K, L^p_\beta)=0$, *then K is exceptional for the space H^p_β.*

Conjecture 1 follows from Conjecture 2 since, for $n-\beta p<1$, any compact set K such that $\operatorname{Cap}(K, L^p_\beta)=0$ is a peak set for A by the Davie-Øksendal theorem (see [Ru, p. 221]).

One of the possible approaches to the problem of exceptional sets is to develop an analog of the *nonlinear potential theory* (see [AH; HW; Ma]) for the spaces of holomorphic functions on the ball. (Nonlinear potentials appear when $p\neq 2$.)

In this paper we make use of some ideas of nonlinear potential theory to construct, for any compact subset $E\subset S$ of positive capacity, $\operatorname{Cap}(E, L^p_\beta)>0$, a holomorphic function

$$\phi=\phi_E\in H^p_\beta \quad (0\leq n-\beta p<1)$$

such that $\|\phi\|^p_{H^p_\beta}\leq C\operatorname{Cap}(E, L^p_\beta)$ and $\Re\phi(\eta)\geq 1$ on E. Here C depends only on β, p, and n; $\phi(\eta)$ is the admissible boundary value of ϕ at $\eta\in E$, and $\Re\phi$ denotes the real part of ϕ.

In $\mathbf{R}^n$, there are two alternative definitions of nonlinear potentials [AH]. A direct analog of the first one for positive measures μ on the sphere ($\mu\in\mathfrak{M}^+(S)$) is given by

$$\mathcal{G}^{\beta p}\mu=I_\beta(I_\beta\mu)^{p'-1}, \tag{0.10}$$

where $p'=p/(p-1)$ for $1<p<\infty$ and $0<\beta\leq n/p$. We recall that the energy of μ is defined by

$$\mathcal{E}^\beta_p(\mu)=\|I_\beta\mu\|^{p'}_{L^{p'}(d\sigma)}. \tag{0.11}$$

It can also be estimated by means of $\mathcal{G}^{\beta p}\mu$ (see [AH]):

$$c_1\mathcal{E}^\beta_p(\mu)\leq\int_S \mathcal{G}^{\beta p}\mu\, d\mu\leq c_2\mathcal{E}^\beta_p(\mu), \tag{0.11'}$$

where c_1 and c_2 are independent of μ.

An alternative definition of a nonlinear potential (in the Euclidean case) is due to Hedberg and Wolff [HW]:

$$\mathcal{W}^{\beta p}\mu(\eta) = \int_0^1 \left[\frac{\mu(B(\eta, 1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1} \frac{dr}{1-r}, \tag{0.12}$$

where $\eta \in S$. It is easily seen that, for all $\eta \in S$,

$$\mathcal{W}^{\beta p}\mu(\eta) \le C \mathcal{I}^{\beta p}\mu(\eta). \tag{0.13}$$

The fundamental Wolff inequality states that, in the average, the converse is also true:

$$\mathcal{E}_p^\beta(\mu) = \int_S \mathcal{I}^{\beta p}\mu \, d\mu \le C \int \mathcal{W}^{\beta p}\mu \, d\mu. \tag{0.14}$$

This means that $\mathcal{I}\mu$ and $\mathcal{W}\mu$ lead to equivalent notions of the energy and the corresponding capacities, although they may have different pointwise behavior. (Actually, as was shown by D. Adams, Wolff's inequality follows from some earlier estimates of Muckenhoupt and Wheeden [MW]. But, as mentioned below, Wolff's original proof has its own advantages and can be used to obtain some stronger estimates of that type.)

Both nonlinear potentials have deep applications to many difficult problems of the theory of Sobolev spaces and partial differential equations (cf. [AH], [HW], and [MV].) In fact, different applications require certain modifications of potentials $\mathcal{I}\mu$ and $\mathcal{W}\mu$. (For instance, one can replace measures of the balls in the definition of $\mathcal{W}^{\beta p}\mu$ by convolutions with smooth functions with compact support; see [HW].)

For holomorphic functions in the ball, the Hedberg-Wolff potential $\mathcal{W}\mu$ seems to be more promising. However, it is based on regularization by means of functions with compact support. In the case of the spaces H_β^p, one deals with non-isotropic holomorphic kernels giving rise to potentials

$$J_\beta f(\eta) = \int \frac{f(\zeta)\, d\sigma(\zeta)}{(1-\langle \eta, \zeta \rangle)^{n-\beta}}, \tag{0.15}$$

which cannot be represented by means of convolutions with compactly supported functions.

We consider two holomorphic modifications of the Hedberg-Wolff potential (one of them is more suitable for $p > 2$ and the other one for $p \le 2$), and the corresponding estimates of energies of measures distributed on the unit sphere S. These potentials, whose real parts are bounded from below by $\mathcal{W}^{\beta p}$, seem to provide an adequate tool for studying Hardy-Sobolev spaces H_β^p in the case $n - \beta p < 1$ which is addressed in Conjecture 1.

Let $\mu \in \mathfrak{M}^+$, $1 < p < \infty$, and $0 < \beta \le n/p$. Denote by $B(\zeta, r)$ the anisotropic ball centered at $\zeta \in S$, of radius r. For any $\lambda > 0$ and $z \in B_n$, we set

$$\mathcal{U}_\lambda^{\beta p}\mu(z) = \int_0^1 \int_S \left[\frac{\mu(B(\zeta, 1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1} \frac{(1-r)^{\lambda-n}}{(1-r\langle z, \zeta\rangle)^\lambda} \, d\sigma(\zeta) \, \frac{dr}{1-r}. \tag{0.16}$$

If $0 < \lambda < 1$, we set

$$\mathcal{V}_\lambda^{\beta p}\mu(z) = \int_0^1 \left[\int_S \frac{(1-r)^{\lambda+\beta p-n}}{(1-r\langle z, \zeta\rangle)^\lambda} \, d\mu(\zeta)\right]^{p'-1} \frac{dr}{1-r}. \tag{0.17}$$

Clearly, both potentials are holomorphic functions in the ball. Note that the Hedberg–Wolff potential may be considered as an analog of the radial Littlewood–Paley g-function (see [FJW]). Then the potentials $\mathcal{U}_\lambda^{\beta p}\mu$ and $\mathcal{V}_\lambda^{\beta p}\mu$ are analogs of the tangential g_λ-function.

The estimates based on the Fefferman–Stein vector-valued maximal theorem show that, for $\lambda > n$, the potential $\mathcal{U}_\lambda^{\beta p}\mu$ belongs to the Hardy–Sobolev space H_β^p, and its Sobolev norm is equivalent to the corresponding energy of the measure μ. In fact, an estimate like this holds for any distribution μ on S. The main problem is to obtain similar inequalities for the smallest possible value of λ in order to guarantee that the real part of the potential is positive. (What we need, at the very least, is $\lambda < 1$, which can be true only for a positive measure μ.)

It can be shown that Conjecture 1 follows from the following crucial estimates (Section 1), combined with the analogs of some known facts on nonlinear potentials and capacities for the ball (see [AH] and [AC1]).

THEOREM 1. *Let $\mu \in \mathfrak{M}^+$. Suppose $p > 1$, $0 < \beta \le n/p$, and $1 > \lambda > n - \beta p$.*

(a) *If $p \le 2$, then*

$$\|\mathcal{U}_\lambda^{\beta p}\mu\|_{H_\beta^p}^p \le C\mathcal{E}_p^\beta(\mu). \tag{0.18}$$

(b) *If $p \ge 2$, then*

$$\|\mathcal{V}_\lambda^{\beta p}\mu\|_{H_\beta^p}^p \le C\mathcal{E}_p^\beta(\mu). \tag{0.19}$$

Note that one can try to use $\mathcal{U}_\lambda^{\beta p}\mu$ instead of $\mathcal{V}_\lambda^{\beta p}\mu$ for $p > 2$. However, it is easy to show that in this case (0.18) is true only if $\lambda > n - \beta p'$, whereas (0.19) holds for the wider range $\lambda > n - \beta p$. On the other hand, one cannot use $\mathcal{V}_\lambda^{\beta p}\mu$ for $p < 2$ in the problem of exceptional sets because its real part is no longer positive when $p' - 1 > 1$.

We observe that Theorem 1 looks more natural in the more general framework of Triebel spaces F_β^{pq} (see [Tr]). A complete characterization of the positive cone of $F_{-\beta}^{pq}$ for $\beta > 0$, $0 < q \le \infty$, was obtained recently by Jawerth, Perez, and Welland [JPW] using wavelet-type decompositions of Triebel spaces due to Frazier and Jawerth (see [FJW]).

We propose another approach to the result of [JPW] which makes use of some ideas of Wolff's original proof in a simplified form. It can be easily extended to the case of the unit ball in $\mathbf{C}^n$ to get the following analog of Wolff's inequality for $0 < q \le \infty$ used in the proof of Theorem 1 (see Section 2).

THEOREM 2. *Let $\mu \in \mathfrak{M}^+(S)$, $1 < p < \infty$, $0 < q \le \infty$, and $\beta > 0$. Then*

$$\int_S \left[\int_0^1 \left(\frac{\mu(B(\zeta, 1-r))}{(1-r)^{n-\beta}} \right)^q \frac{dr}{1-r} \right]^{p'/q} d\sigma(\zeta) \le C \int_S \mathcal{W}^{\beta p}\mu \, d\mu, \tag{0.20}$$

where C is a positive constant independent of μ.

Note that, when $q = 1$, the preceding inequality is equivalent to (0.14). From (0.20) and a non-isotropic analog of a result of Muckenhoupt and Wheeden [MW], it follows that, for $0 < q \le \infty$,

$$\mathcal{E}_p^\beta(\mu) \le c_1 \|\mu\|_{F_{-\beta}^{p'q}(S)}^{p'} \le c_2 \int_S \left[\int_0^1 \left(\frac{\mu(B(\zeta, 1-r))}{(1-r)^{n-\beta}}\right)^q \frac{dr}{1-r}\right]^{p'/q} d\sigma(\zeta) \le c_3 \mathcal{E}_p^\beta(\mu)$$

for some positive constants c_1, c_2, c_3 independent of μ (cf. [AH] for $1<q<\infty$ and [JPW] for $0<q\le 1$, where a proof of these inequalities is given in the Euclidean case). Note that $(F_{-\beta}^{p'q})^* = F_\beta^{pq_1}$, where $1/p+1/p'=1$ $(1<p<\infty)$, and that $q_1 = q'$ if $1<q<\infty$ while $q_1=\infty$ if $0<q\le 1$ (see [Tr]).

Theorem 1 makes it possible (see Section 3) to characterize interpolation sets of Hardy–Sobolev functions lying in the ball algebra (see [C2]) in the spirit of the Carleson–Rudin theorem and its further development by Khrushchev and Peller [KP] and Koosis [Ko].

We recall that a compact subset K of S is said to be a *boundary interpolation set* (B.I. set) for a Banach space $\mathcal{B} \subset C(S)$ if the restriction operator $R_K\colon \mathcal{B} \to C(S)$ maps $\mathcal{B}$ onto $C(K)$. Similarly, K is a *strong boundary interpolation set* (S.B.I. set) for $\mathcal{B}$ if R_K maps the unit ball of $\mathcal{B}$ onto the unit ball of $C(K)$.

THEOREM 3. *Let $p>1$ and $0\le n-\beta p<1$. Suppose $H_\beta^p \cap C(S)$ is given the norm $\|f\| = \max(\|f\|_{H_\beta^p}, \|f\|_A)$, where A is the ball algebra. If K is a compact subset of S, then the following properties are equivalent.*

(i) $\mathrm{Cap}(K, L_\beta^p) = 0$.

(ii) *K is a B.I. set for $H_\beta^p \cap C(S)$.*

(iii) *K is a S.B.I. set for $H_\beta^p \cap C(S)$.*

Analogous statements are also true for Besov spaces B_β^p. (In the case $n=1$, for Besov spaces B_β^p, Theorem 3 was proved in [C2].) We observe that if $n>1$ then Theorem 3 is no longer true for $n-\beta p \ge 1$: It is easily seen that in this case the circle T^1 is not an interpolation set but $\mathrm{Cap}(T^1, L_\beta^p)=0$.

1. Proof of Theorem 1

In this section we prove that, for any measure $\mu \in \mathfrak{M}^+(S)$ of finite energy (i.e., $\mathcal{E}_p^\beta(\mu)<\infty$), the potential $\mathcal{U}_\lambda^{\beta p}\mu$ defined by (0.16) belongs to the Hardy–Sobolev space H_β^p if $\lambda > n-\beta p$ and $1<p\le 2$. This result remains true for $p>2$ if we replace $\mathcal{U}_\lambda^{\beta p}\mu$ by $\mathcal{V}_\lambda^{\beta p}\mu$ defined by (0.17). (See Section 0.)

We start with some auxiliary statements. The following estimate is easy to prove (see [Ru]).

PROPOSITION 1.1. *Let $s>0$, $\lambda>0$, $z \in B_n$, and $\zeta \in S$. Then*

$$\left| R^s \frac{1}{(1-\langle z,\zeta\rangle)^\lambda}\right| \le C \frac{1}{|1-\langle z,\zeta\rangle|^{\lambda+s}}. \tag{1.1}$$

Let $0\le\rho<1$, $\eta\in S$, $0<p<\infty$, $0<\beta<\infty$, and $f \in H_\beta^p$. An equivalent (quasi-) norm in H_β^p (via the Littlewood–Paley g-function) is given (see [AB]) by

$$\|f\|_{H_\beta^p}^p = |f(0)|^p + \int_S \left[\int_0^1 (1-\rho)^{2(s-\beta)} |R^s f(\rho\eta)|^2 \frac{d\rho}{1-\rho}\right]^{p/2} d\sigma(\eta), \tag{1.2}$$

where $s>\beta$.

We will also use the Triebel spaces F_β^{pq} $(0<p<\infty,\ 0<q<\infty,\ -\infty<\beta<\infty)$ of holomorphic functions f in the ball B_n with (quasi-) norm

$$\|f\|_{F_\beta^{pq}}^p = |f(0)|^p + \int_S \left[\int_0^1 (1-\rho)^{(s-\beta)q} |R^s f(\rho\eta)|^q \frac{d\rho}{1-\rho}\right]^{p/q} d\sigma(\eta), \quad (1.3)$$

where $s>\beta$.

It can be shown that, as in the real-variable case [Tr], (1.2) and (1.3) each define equivalent norms for different values of $s>\beta$. It is also easy to see that $F_\beta^{p2}=H_\beta^p$ and $F_\beta^{pp}=B_\beta^p$, where B_β^p is the corresponding Besov space (see [AB; AC2]).

We will need some known imbeddings for the Hardy–Sobolev, Besov, and Triebel spaces (see [AB; FJW; Pe; Tr]) which are collected in the following proposition.

PROPOSITION 1.2. *Let* $1<p<\infty$ *and* $-\infty<\beta<\infty$.

(a) *If* $p\le 2$, *then*

$$c_1\|f\|_{H_\beta^p}^p \le \|f\|_{B_\beta^p} \le c_2\|f\|_{F_\beta^{p1}}. \quad (1.4)$$

(b) *If* $p\ge 2$, *then*

$$c_1\|f\|_{B_\beta^p}^p \le \|f\|_{H_\beta^p} \le c_2\|f\|_{F_\beta^{p1}} \quad (1.5)$$

for some positive constants c_1, c_2 *independent of the function* f *holomorphic in the ball* B_n.

Let $1<p<\infty$, $1/p+1/p'=1$, and $0<\beta\le n/p$. For $\mu\in\mathfrak{M}^+(S)$, $0<\lambda<\infty$, and $z\in B_n$, we set

$$\mathcal{U}_\lambda^{\beta p}\mu(z) = \int_0^1\int_S \left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1} \frac{(1-r)^{\lambda-n}}{(1-r\langle z,\zeta\rangle)^\lambda}\, d\sigma(\zeta)\, \frac{dr}{1-r}. \quad (1.6)$$

We will need an estimate of $\mathcal{U}_\lambda^{\beta p}\mu(z)$ in terms of the linear potential

$$I_{2\beta}\mu(z) = \int_S \frac{d\mu(\zeta)}{|1-\langle z,\zeta\rangle|^{n-2\beta}} \quad (1.7)$$

for $p=2$.

PROPOSITION 1.3. *Let* $\mu\in\mathfrak{M}^+$, $\lambda>n-2\alpha$, *and* $0<\alpha<n/2$. *Then*

$$|\mathcal{U}_\lambda^{\alpha 2}\mu(z)| \le C I_{2\alpha}\mu(z), \quad (1.8)$$

where C *is independent of* $z\in B_n$ *and* μ.

Proof. Applying Fubini's theorem, we get

$$\begin{aligned} |\mathcal{U}_\lambda^{\alpha 2}\mu(z)| &\le \int_0^1\int_S \frac{(1-r)^{2\alpha+\lambda-2n-1}\mu(B(\zeta,1-r))}{|1-r\langle z,\zeta\rangle|^\lambda}(1-r)^{\lambda-n}\, dr\, d\sigma(\zeta) \\ &\le C\int_0^1\int_S (1-r)^{2\alpha+\lambda-2n-1}\, dr\, d\mu(\eta) \int_{B(\eta,1-r)} \frac{d\sigma(\zeta)}{|1-r\langle z,\zeta\rangle|^\lambda}. \end{aligned}$$

It is easily seen that, for $\zeta\in B(\eta,1-r)$, we have $|1-r\langle z,\zeta\rangle| \ge c|1-r\langle z,\eta\rangle|$. Hence

$$\int_{B(\eta,1-r)} \frac{d\sigma(\zeta)}{|1-r\langle z,\zeta\rangle|^{\lambda}} \le C\frac{(1-r)^n}{|1-r\langle z,\eta\rangle|^{\lambda}}.$$

From this it follows that

$$|\mathcal{U}_{\lambda}^{\alpha 2}\mu(z)| \le C\int_0^1\int_S \frac{(1-r)^{2\alpha+\lambda-n-1}}{|1-r\langle z,\eta\rangle|^{\lambda}}\,dr\,d\mu(\eta) \le C\int_S \frac{d\mu(\eta)}{|1-\langle z,\eta\rangle|^{n-2\alpha}},$$

and this completes the proof. □

REMARK 1.1. It is easy to see that, for $p=2$, $0<2\alpha<n$, and $1>\lambda>n-2\alpha$, both $|\mathcal{U}_{\lambda}^{\alpha p}\mu|$ and $|\mathcal{V}_{\lambda}^{\alpha p}\mu|$ are actually pointwise equivalent to $I_{2\alpha}\mu$.

REMARK 1.2. Proposition 1.3 remains true for $\alpha=n/2$ if we replace (1.7) by the corresponding logarithmic potential.

Proof of Theorem 1. (a) Let $1<p\le 2$. Let $z=\rho\eta\in B_n$ and $\phi(z)=\mathcal{U}_{\lambda}^{\beta p}\mu(z)$. By Proposition 1.2(a), it suffices to estimate the Triebel norm,

$$\|\phi\|_{F_{\beta}^{p1}}^{p} = |\phi(0)|^p + \int_S\left[\int_0^1 (1-\rho)^{(s-\beta)}|R^s\phi(\rho\eta)|\frac{d\rho}{1-\rho}\right]^p d\sigma(\eta),$$

by the energy, $\mathcal{E}_p^{\beta}(\mu)$, for some integer $s>\beta$. By Proposition 1.1,

$$\int_0^1 (1-\rho)^{(s-\beta)}|R^s\phi(\rho\eta)|\frac{d\rho}{1-\rho}$$

$$\le C\int_0^1 (1-\rho)^{s-\beta}\int_0^1\int_S\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1}\frac{(1-r)^{\lambda-n}}{|1-\rho r\langle\eta,\zeta\rangle|^{\lambda+s}}\,d\sigma(\zeta)\frac{dr}{1-r}\frac{d\rho}{1-\rho}.$$

Note that

$$\int_0^1 \frac{(1-\rho)^{s-\beta}}{|1-\rho r\langle\eta,\zeta\rangle|^{\lambda+s}}\frac{d\rho}{1-\rho} \le C\frac{1}{|1-r\langle\eta,\zeta\rangle|^{\lambda+\beta}}.$$

Interchanging the order of integration and applying the preceding estimate, we get

$$\int_0^1 (1-\rho)^{(s-\beta)}|R^s\phi(\rho\eta)|\frac{d\rho}{1-\rho} \le C\Phi(\eta), \tag{1.9}$$

where

$$\Phi(\eta) = \int_0^1\int_S\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1}\frac{(1-r)^{\lambda-n}}{|1-r\langle\eta,\zeta\rangle|^{\lambda+\beta}}\,d\sigma(\zeta)\frac{dr}{1-r}. \tag{1.10}$$

Clearly,

$$|\phi(0)| = \int_0^1\int_S\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1}(1-r)^{\lambda-n}\,d\sigma(\zeta)\frac{dr}{1-r}$$

is bounded by $\|\Phi\|_{L^p(d\sigma)}$. Hence, it remains to prove that

$$\|\Phi\|_{L^p(d\sigma)}^p \le C\mathcal{E}_p^{\beta}(\mu). \tag{1.11}$$

Since $\lambda>n-\beta p$, we can choose a number t such that

$$n-\beta<t<\frac{\lambda+\beta-n(2-p)}{p-1}.$$

Then we have

$$\Phi(\eta)=\int_0^1\int_S\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta}}\frac{(1-r)^{t-n}}{|1-r\langle\eta,\zeta\rangle|^t}\right]^{p-1}$$
$$\times\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta}}\right]^{p'-p}\frac{(1-r)^{\lambda+\beta-n(2-p)-t(p-1)}}{|1-r\langle\eta,\zeta\rangle|^{\lambda+\beta-t(p-1)}}\,d\sigma(\zeta)\,\frac{dr}{1-r}.$$

Applying Hölder's inequality with exponents $1/(p-1)$ and $1/(2-p)$, we obtain

$$\Phi(\eta)\le C\Phi_1(\eta)^{p-1}\Phi_2(\eta)^{2-p}, \tag{1.12}$$

where

$$\Phi_1(\eta)=\int_0^1\int_S\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta}}\frac{(1-r)^{t-n}}{|1-r\langle\eta,\zeta\rangle|^t}\,d\sigma(\zeta)\,\frac{dr}{1-r}$$

and

$$\Phi_2(\eta)=\int_0^1\int_S\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta}}\right]^{p'}\frac{(1-r)^{[\lambda+\beta-n(2-p)-t(p-1)]/(2-p)}}{|1-r\langle\eta,\zeta\rangle|^{[\lambda+\beta-t(p-1)]/(2-p)}}\,d\sigma(\zeta)\,\frac{dr}{1-r}.$$

We estimate $\Phi_1(\eta)$, using Proposition 1.3 with $\alpha=\beta/2$ and $\lambda=t>n-\beta$, and obtain

$$\Phi_1(\eta)\le CI_\beta\mu(\eta).$$

Now we again apply Hölder's inequality with exponents $\gamma=1/(p-1)^2>1$ and $\gamma'=1/(p(2-p))$ to (1.12) with $I_\beta\mu$ in place of Φ_1; this yields

$$\|\Phi\|^p_{L^p(d\sigma)}\le C\|I_\beta\mu\|^{p'/\gamma}_{L^{p'}(d\sigma)}\left[\int_S|\Phi_2(\eta)|^{(2-p)p\gamma'}d\sigma(\eta)\right]^{1/\gamma'}.$$

Note that $(2-p)p\gamma'=1$. Hence, by Fubini's theorem,

$$\int_S|\Phi_2(\eta)|\,d\sigma(\eta)$$
$$\le C\int_0^1\int_S\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta}}\right]^{p'}\int_S\frac{(1-r)^{[\lambda+\beta-n(2-p)-t(p-1)]/(2-p)}}{|1-r\langle\eta,\zeta\rangle|^{[\lambda+\beta-t(p-1)]/(2-p)}}\,d\sigma(\eta)\,\frac{dr}{1-r}.$$

By our choice of t, we have $[\lambda+\beta-t(p-1)]/(2-p)>n$. Then

$$\int_S\frac{(1-r)^{[\lambda+\beta-n(2-p)-t(p-1)]/(2-p)}}{|1-r\langle\eta,\zeta\rangle|^{[\lambda+\beta-t(p-1)]/(2-p)}}\,d\sigma(\eta)\le C,$$

where C is independent of μ, r, and ζ. Combining these estimates, we see that

$$\|\Phi\|^p_{L^p(d\sigma)}\le C\|I_\beta\mu\|^{p'/\gamma}_{L^{p'}(d\sigma)}\left[\int_0^1\int_S\left(\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta}}\right)^{p'}d\sigma(\zeta)\,\frac{dr}{1-r}\right]^{1/\gamma'}.$$

By Theorem 2 with $q=p'$, we have that

$$\int_0^1\int_S\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta}}\right]^{p'}d\sigma(\zeta)\,\frac{dr}{1-r}\le C\mathcal{E}_p^\beta(\mu).$$

Also, by (0.11),

$$\|I_\beta\mu\|^{p'}_{L^{p'}(d\sigma)}=\mathcal{E}_p^\beta(\mu).$$

Thus we obtain the desired inequality

$$\|\Phi\|_{L^p(d\sigma)}^p \le C\mathcal{E}_p^\beta(\mu).$$

The proof of Theorem 1 in the case $1 < p \le 2$ is complete.

(b) Let $p > 2$. Let $\mu \in \mathfrak{M}^+$, $0 < \beta \le n/p$, and $n-\beta p < \lambda < 1$. We recall that $\mathcal{V}_\lambda^{\beta p}$ is defined by

$$\mathcal{V}_\lambda^{\beta p}\mu(z) = \int_0^1 \left[\int_S \frac{(1-r)^{\lambda+\beta p-n}}{(1-r\langle z,\zeta\rangle)^\lambda}\,d\mu(\zeta)\right]^{p'-1} \frac{dr}{1-r}.$$

Let us prove that

$$\|\mathcal{V}_\lambda^{\beta p}\mu\|_{H_\beta^p}^p \le C\mathcal{E}_p^\beta(\mu). \tag{1.13}$$

For $z=\rho\eta \in B_n$ we set $\phi(z) = \mathcal{V}_\lambda^{\beta p}\mu(z)$. By Proposition 1.2, it suffices to prove that the Triebel norm,

$$\|\phi\|_{F_\beta^{p1}}^p = |\phi(0)|^p + \int_S\left[\int_0^1 (1-\rho)^{s-\beta}|R^s\phi(\rho\zeta)|\frac{d\rho}{1-\rho}\right]^p d\sigma(\zeta), \tag{1.14}$$

is bounded by $\mathcal{E}_p^\beta(\mu)$. Since $\lambda > n-\beta p$, we have that $|\phi(0)|^p = C\mu(S)^{p'} \le C\mathcal{E}_p^\beta(\mu)$.

Let $\gamma = p'-1 < 1$. By differentiating

$$\left[\int_S \frac{d\mu(\zeta)}{(1-\langle z,\zeta\rangle)^\lambda}\right]^\gamma,$$

we derive the following estimate.

PROPOSITION 1.4. *Let s be an integer, and let $0<\gamma<1$, $0<\lambda<1$, and $z \in B_n$. Then*

$$\left|R^s\left[\int_S \frac{d\mu(\zeta)}{(1-\langle z,\zeta\rangle)^\lambda}\right]^\gamma\right| \le C\left[\int_S \frac{d\mu(\zeta)}{|1-\langle z,\zeta\rangle|^\lambda}\right]^{\gamma-1} \int_S \frac{d\mu(\xi)}{|1-\langle z,\xi\rangle|^{\lambda+s}}, \tag{1.15}$$

where C is independent of μ and z.

Proof. Let $z = r\eta$, where $0\le r<1$ and $\eta\in S$. Let

$$G(z,\lambda) = \int_S \frac{d\mu(\zeta)}{(1-\langle z,\zeta\rangle)^\lambda}$$

and

$$F(z,\lambda) = \int_S \frac{d\mu(\zeta)}{|1-\langle z,\zeta\rangle|^\lambda}.$$

Recall that $RG(z) = G(z) + r\,dG/dr(z)$. Hence, it suffices to show that

$$\left|\frac{d^s}{dr^s}G(z,\lambda)^\gamma\right| \le CF(z,\lambda)^{\gamma-1}F(z,\lambda+s). \tag{1.16}$$

Since $0<\lambda<1$, we have $|\Re G(z,\lambda)| \ge C_\lambda F(z,\lambda)$, where $C_\lambda = \cos(\pi\lambda/2)$. Using this estimate, it is easy to check by induction that the left-hand side of (1.16) is bounded by

$$CF(z,\lambda)^{\gamma}\sum\prod\left[\frac{F(z,\lambda+l_i)}{F(z,\lambda)}\right]^{k_i},$$

where C depends only on λ, γ, and s, and the sum of products is taken over all $k_i \geq 0$ and $l_i \geq 0$ such that $\sum k_i l_i = s$. Note that

$$\left[\frac{F(z,\lambda+l_i)}{F(z,\lambda)}\right]^{k_i} \leq C\left[\frac{F(z,\lambda+s)}{F(z,\lambda)}\right]^{k_i l_i/s}.$$

Estimating each term by Hölder's inequality and applying the preceding estimates, we get (1.16). The proof of Proposition 1.4 is complete. □

It follows from Proposition 1.4 with $\gamma = p'-1$ that

$$|R^s\phi(z)| \leq C\int_0^1\left[\int_S \frac{d\mu(\zeta)}{|1-r\langle z,\zeta\rangle|^{\lambda}}\right]^{p'-2} \times \int_S \frac{d\mu(\tau)}{|1-r\langle z,\tau\rangle|^{\lambda+s}}(1-r)^{(\lambda+\beta p-n)(p'-1)}\frac{dr}{1-r}.$$

Let $z = \rho\eta$. By Fubini's theorem,

$$\int_S \frac{d\mu(\tau)}{|1-r\langle z,\tau\rangle|^{\lambda+s}} \leq C\int_S d\mu(\tau)\int_{|1-r\langle z,\tau\rangle|<\delta}\frac{d\delta}{\delta^{\lambda+s+1}} \leq C\int_0^1 \frac{\mu(B(\eta,\delta))}{(\delta+1-r+1-\rho)^{\lambda+s+1}}\,d\delta.$$

Clearly, for any $\delta > 0$, we have

$$\int_S \frac{d\mu(\zeta)}{|1-r\langle z,\zeta\rangle|^{\lambda}} \geq C\int_{B(\eta,\delta)}\frac{d\mu(\zeta)}{|1-r\langle z,\zeta\rangle|^{\lambda}} \geq C\frac{\mu(B(\eta,\delta))}{(\delta+1-r+1-\rho)^{\lambda}}.$$

Since $p'-2 < 0$, we get

$$\int_S\left[\int_S \frac{d\mu(\zeta)}{|1-r\langle z,\zeta\rangle|^{\lambda}}\right]^{p'-2}\frac{d\mu(\tau)}{|1-r\langle z,\tau\rangle|^{\lambda+s}} \leq C\int_0^1 \frac{\mu(B(\eta,\delta))^{p'-1}}{(\delta+1-r+1-\rho)^{(\lambda+s+1)+(p'-2)\lambda}}\,d\delta.$$

Thus

$$|R^s\phi(z)| \leq C\int_0^1\int_0^1 \frac{(1-r)^{(\lambda+\beta p-n)(p'-1)}\mu(B(\eta,\delta))^{p'-1}}{(\delta+1-r+1-\rho)^{(\lambda+s+1)+(p'-2)\lambda}}\,d\delta\,\frac{dr}{1-r}.$$

It is easily seen that, since $\lambda > n-\beta p$,

$$\int_0^1 \frac{(1-r)^{(\lambda+\beta p-n)(p'-1)}}{(\delta+1-r+1-\rho)^{(\lambda+s+1)+(p'-2)\lambda}}\frac{dr}{1-r} \leq C\frac{1}{(\delta+1-\rho)^{(n-\beta p)(p'-1)+s+1}}.$$

Hence,

$$|R^s\phi(z)| \leq C\int_0^1 \frac{\mu(B(\eta,\delta))^{p'-1}}{(\delta+1-\rho)^{(s+1)+(n-\beta p)(p'-1)}}\,d\delta.$$

Note that $z=\rho\eta$. Integrating the preceding inequality against $d\rho/(1-\rho)$, we get

$$\int_0^1 (1-\rho)^{s-\beta}|R^s\phi(\rho\eta)|\frac{d\rho}{1-\rho} \le C\int_0^1\int_0^1 \frac{(1-\rho)^{s-\beta}\mu(B(\eta,\delta))^{p'-1}}{(\delta+1-\rho)^{(s+1)+(n-\beta p)(p'-1)}}\,d\delta\,\frac{d\rho}{1-\rho}.$$

Since $s>\beta$, we have

$$\int_0^1 \frac{(1-\rho)^{s-\beta}}{(\delta+1-\rho)^{(s+1)+(n-\beta p)(p'-1)}}\,\frac{d\rho}{1-\rho} \le \frac{C}{\delta^{(n-\beta)(p'-1)+1}}.$$

Thus,

$$\int_0^1 (1-\rho)^{s-\beta}|R^s\phi(\rho\eta)|\frac{d\rho}{1-\rho} \le C\int_0^1\left[\frac{\mu(B(\eta,\delta))}{\delta^{n-\beta}}\right]^{p'-1}\frac{d\delta}{\delta}.$$

Combining the preceding estimate, (1.14), and Theorem 2 (with $q=p'-1$), we obtain

$$\|\phi\|^p_{F^{p1}_\beta} \le C\mathcal{E}^\beta_p(\mu),$$

which gives (1.13). The proof of Theorem 1 is complete. ❐

REMARK 1.3. It follows from the proof of Theorem 1 that the theorem remains true if we replace the Sobolev norms of the corresponding potentials by Besov norms $\|\cdot\|_{B^p_\beta}$, or Triebel norms $\|\cdot\|_{F^{pq}_\beta}$, for $1<p<\infty$ and $1\le q\le\infty$.

Theorem 1 is apparently true for F^{pq}_β-spaces with $1<p<\infty$ and $0<q<1$, but we do not consider this case here.

2. Proof of Theorem 2

In this section we prove the inequality

$$\int_S\left\{\int_0^1\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta}}\right]^q\frac{dr}{1-r}\right\}^{p'/q} d\sigma(\zeta) \le C\int_S \mathcal{W}^{\beta p}\mu\,d\mu \tag{2.1}$$

for $0<q\le\infty$. We observe that, if $0<q<1$, estimate (2.1) is "stronger" than Wolff's inequality (0.14), which is equivalent to (2.1) with $q=1$. (It is easy to see that it suffices to prove (2.1) for sufficiently small q.) Note that by (0.11′) and (0.13),

$$\int_S \mathcal{W}^{\beta p}\mu\,d\mu \le C\mathcal{E}^\beta_p(\mu), \tag{2.2}$$

where $\mathcal{E}^\beta_p(\mu)$ is the energy of μ.

To prove (2.1), we will need the following obvious statement.

PROPOSITION 2.1. *Let $0<s<\infty$ and let $f\ge 0$ be an integrable function on $(0,1)$. Then*

$$\left[\int_0^1 f(t)\,dt\right]^s = s\int_0^1\left[\int_0^t f(y)\,dy\right]^{s-1} f(t)\,dt. \tag{2.3}$$

We may assume that $0 < q \leq \min(1, p'-1)$. For $\zeta \in S$, set

$$\phi(\zeta) = \left[\int_0^1 \left(\frac{\mu(B(\zeta,t))}{t^{n-\beta}}\right)^q \frac{dt}{t}\right]^{p'/q},$$

and apply (2.3) with $s = p'/q$ and $f(t) = t^{-1}[\mu(B(\zeta,t))/t^{n-\beta}]^q$. Then

$$\phi(\zeta) = \frac{p'}{q}\int_0^1 \left(\frac{\mu(B(\zeta,t))}{t^{n-\beta}}\right)^q \left[\int_0^t \left(\frac{\mu(B(\zeta,y))}{y^{n-\beta}}\right)^q \frac{dy}{y}\right]^{p'/q-1} \frac{dt}{t}.$$

Choose ϵ so that $0 < \epsilon < \beta(p-1)q$. By Hölder's inequality with exponent $p'/q - 1 > 1$, we get

$$\left[\int_0^t \left(\frac{\mu(B(\zeta,y))}{y^{n-\beta}}\right)^q \frac{dy}{y}\right]^{p'/q-1} \leq Ct^\epsilon \int_0^t \left(\frac{\mu(B(\zeta,y))}{y^{n-\beta}}\right)^{p'-q} \frac{dy}{y^{1+\epsilon}}.$$

Hence

$$\begin{aligned}\phi(z) &\leq C\int_0^1 \left(\frac{\mu(B(\zeta,t))}{t^{n-\beta}}\right)^q \int_0^t \left(\frac{\mu(B(\zeta,y))}{y^{n-\beta}}\right)^{p'-q} \frac{dy}{y^{1+\epsilon}}\frac{dt}{t^{1+\epsilon}} \\ &= C\int_0^1\int_0^t \mu(B(\zeta,y))^{p'-q}\mu(B(\zeta,t))^q \frac{dy}{y^{(n-\beta)(p'-q)+\epsilon+1}}\frac{dt}{t^{(n-\beta)q-\epsilon+1}}.\end{aligned}$$

Let $\eta \in B(\zeta,t)$; then $B(\zeta,t) \subset B(\eta,4t)$. Since $0 < y < t$, we have

$$\begin{aligned}\mu(B(\zeta,y))^{p'-q}\mu(B(\zeta,t))^q &= \mu(B(\zeta,y))^{p'-q-1}\int_{B(\zeta,y)} \mu(B(\zeta,t))^q d\mu(\eta) \\ &\leq \mu(B(\zeta,y))^{p'-q-1}\int_{B(\zeta,y)} \mu(B(\eta,4t))^q d\mu(\eta).\end{aligned}$$

Integrating the preceding estimate against $d\sigma$ and interchanging the order of integration, we have

$$\begin{aligned}&\int_S \mu(B(\zeta,y))^{p'-q}\mu(B(\zeta,t))^q d\sigma(\zeta) \\ &\quad\leq C\int_S\int_{B(\zeta,y)} \mu(B(\zeta,y))^{p'-q-1}\mu(B(\eta,4t))^q d\mu(\eta)\, d\sigma(\zeta) \\ &\quad= C\int_S \mu(B(\eta,4t))^q d\mu(\eta)\int_{B(\eta,y)} \mu(B(\zeta,y))^{p'-q-1} d\sigma(\zeta) \\ &\quad\leq Cy^n\int_S \mu(B(\eta,4y))^{p'-q-1}\mu(B(\eta,4t))^q d\mu(\eta).\end{aligned}$$

From this it follows that

$$\begin{aligned}&\int_S \phi(\zeta)\, d\sigma \\ &\leq C\int_0^1 \frac{dt}{t^{(n-\beta)q-\epsilon+1}}\int_0^t \frac{y^n\, dy}{y^{(n-\beta)(p'-q)+\epsilon+1}} \\ &\quad\times\int_S \mu(B(\eta,4y))^{p'-q-1}\mu(B(\eta,4t))^q d\mu(\eta) =\end{aligned}$$

$$= C\int_S d\mu(\eta)\int_0^1\left[\frac{\mu(B(\eta,4t))}{t^{n-\beta}}\right]^q\int_0^t\left[\frac{\mu(B(\eta,4y))}{y^{n-\beta p}}\right]^{p'-q-1}\frac{dy}{y^{1-\beta(p-1)q+\epsilon}}\frac{dt}{t^{1-\epsilon}}.$$

Now we estimate

$$F(t)=\int_0^t\left[\frac{\mu(B(\eta,4y))}{y^{n-\beta p}}\right]^{p'-q-1}\frac{dy}{y^{1-\beta(p-1)q+\epsilon}}.$$

Choose $\delta>0$ so that $0<\epsilon+\delta<\beta(p-1)q$, and again apply Hölder's inequality with exponent $(p'-1)/(p'-1-q)>1$. We have

$$F(t)\le Ct^{\beta(p-1)q-\epsilon-\delta}\left\{\int_0^t\left[\frac{\mu(B(\eta,4y))}{y^{n-\beta p}}\right]^{p'-1}y^{\delta(p'-1)/(p'-1-q)}\frac{dy}{y}\right\}^{(p'-1-q)/(p'-1)}.$$

Hence

$$\int_S\phi(\zeta)\,d\sigma\le C\int_S d\mu(\eta)\int_0^1\left[\frac{\mu(B(\eta,4t))}{t^{n-\beta}}\right]^q\Phi(4t)\frac{dt}{t^{1-\beta(p-1)q+\delta}},$$

where

$$\Phi(t)=\left\{\int_0^t\left[\frac{\mu(B(\eta,y))}{y^{n-\beta p}}\right]^{p'-1}y^{\delta(p'-1)/(p'-1-q)}\frac{dy}{y}\right\}^{(p'-1-q)/(p'-1)}.$$

Now we observe that, for all $0<t<1$,

$$\begin{aligned}\left[\frac{\mu(B(\eta,4t))}{t^{n-\beta}}\right]^q &\le Ct^{-\beta(p-1)q-\delta q/(p'-1-q)}\\ &\quad\times\left\{\int_0^{2t}\left[\frac{\mu(B(\eta,4y))}{y^{n-\beta p}}\right]^{p'-1}y^{\delta(p'-1)/(p'-q-1)}\frac{dy}{y}\right\}^{q/(p'-1)}\\ &= Ct^{-\beta(p-1)q-\delta q/(p'-1-q)}\Phi(8t)^{q/(p'-1-q)}.\end{aligned}$$

We obtain

$$\begin{aligned}\int_S\phi(\zeta)\,d\sigma &\le C\int_S d\mu(\eta)\int_0^1\Phi(8t)^{(p'-1)/(p'-1-q)}\frac{dt}{t^{1+\delta(p'-1)/(p'-1-q)}}\\ &= C\int_S d\mu(\eta)\\ &\quad\times\int_0^1\int_0^{8t}\left[\frac{\mu(B(\eta,y))}{y^{n-\beta p}}\right]^{p'-1}y^{\delta(p'-1)/(p'-1-q)}\frac{dy}{y}\frac{dt}{t^{1+\delta(p'-1)/(p'-1-q)}}.\end{aligned}$$

Interchanging the order of integration yields

$$\begin{aligned}&\int_0^1\int_0^{8t}\left[\frac{\mu(B(\eta,y))}{y^{n-\beta p}}\right]^{p'-1}y^{\delta(p'-1)/(p'-1-q)}\frac{dy}{y}\frac{dt}{t^{1+\delta(p'-1)/(p'-1-q)}}\\ &\quad\le C\int_0^1\left[\frac{\mu(B(\eta,y))}{y^{n-\beta p}}\right]^{p'-1}y^{\delta(p'-1)/(p'-1-q)}\int_{y/8}^1\frac{dt}{t^{1+\delta(p'-1)/(p'-1-q)}}\frac{dy}{y}\\ &\quad\le C\int_0^1\left[\frac{\mu(B(\eta,y))}{y^{n-\beta p}}\right]^{p'-1}\frac{dy}{y}=C\mathcal{W}^{\beta p}\mu(\eta).\end{aligned}$$

Thus

$$\int_S \phi(\zeta)\,d\sigma \le C\int_S \mathcal{W}^{\beta p}\mu\,d\mu.$$

The proof of Theorem 2 is complete. □

3. Applications to Exceptional and Interpolation Sets

In this section, we characterize exceptional and boundary interpolation sets for Hardy–Sobolev functions in the case $n-\beta p<1$.

If $0<\lambda<1$, then $\mathcal{U}_\lambda^{\beta p}\mu$ and $\mathcal{V}_\lambda^{\beta p}\mu$ are holomorphic functions on B_n with positive real parts. To prove Conjecture 1, we show that their admissible boundary limits are bounded from below on S by the Hedberg–Wolff potential

$$\mathcal{W}^{\beta p}\mu(\zeta)=\int_0^1\left[\frac{\mu(B(\zeta,1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1}\frac{dr}{1-r}.$$

PROPOSITION 3.1. *Let* $1<p<\infty$, $0<\beta\le n/p$, *and* $0<\lambda<1$.

(a) *If* $p\le 2$, *then*

$$\varliminf_{\rho\to 1}\Re\,\mathcal{U}_\lambda^{\beta p}\mu(\rho\eta)\ge C\,\mathcal{W}^{\beta p}\mu(\eta). \tag{3.1}$$

(b) *If* $2\le p<\infty$, *then*

$$\varliminf_{\rho\to 1}\Re\,\mathcal{V}_\lambda^{\beta p}\mu(\rho\eta)\ge C\,\mathcal{W}^{\beta p}\mu(\eta), \tag{3.2}$$

where C is independent of $\eta\in S$ *and* $\mu\in\mathfrak{M}^+$.

Proof. (a) Let $z=\rho\eta$, where $0\le\rho<1$ and $\eta\in S$. Since $0<\lambda<1$, we have

$$\Re\frac{1}{(1-\langle z,\zeta\rangle)^\lambda}\ge\cos\frac{\pi\lambda}{2}\,\frac{1}{|1-\langle z,\zeta\rangle|^\lambda} \tag{3.3}$$

for any $\zeta\in S$. Hence

$$\begin{aligned}&\Re\,\mathcal{U}_\lambda^{\beta p}\mu(\rho\eta)\\ &\quad\ge C\int_0^1\int_{B(\eta,\delta)}\frac{[(1-r)^{\beta p-n}\mu(B(\zeta,1-r))]^{p'-1}}{|1-r\rho\langle\eta,\zeta\rangle|^\lambda}(1-r)^{\lambda-n}\frac{dr}{1-r}\,d\sigma(\zeta),\end{aligned}$$

where $\delta=(1-r)/4$. If $\zeta\in B(\eta,\delta)$, then $B(\eta,\delta)\subset B(\zeta,1-r)$. We get

$$\int_{B(\eta,\delta)}\mu(B(\zeta,1-r))^{p'-1}d\sigma(\zeta)\ge C(1-r)^n(\mu(B(\eta,\delta)))^{p'-1}.$$

Moreover, for $\zeta\in B(\eta,\delta)$, we have $|1-r\rho\langle\eta,\zeta\rangle|^\lambda\le c(1-r\rho)^\lambda$. Thus

$$\begin{aligned}\Re\,\mathcal{U}_\lambda^{\beta p}\mu(\rho\eta)&\ge C\int_{1/4}^1\left[\frac{\mu(B(\eta,1-r))}{(1-r\rho)^{n-\beta p}}\right]^{p'-1}\frac{dr}{1-r}\\ &\ge C_1\int_{1/4}^{\rho}\left[\frac{\mu(B(\eta,1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1}\frac{dr}{1-r}.\end{aligned}$$

If $0<r<1/4$, then clearly

$$(\mu(S))^{p'-1} \le C \int_S [\mu(B(\eta, 1-r))]^{p'-1} d\sigma(\eta),$$

where C is independent of μ and r. Thus, for any $0<\rho<1$ and $\eta \in S$,

$$\Re \mathcal{U}_\lambda^{\beta p}\mu(\rho\eta) \ge C \int_0^{1/4} \left[\frac{\mu(B(\eta, 1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1} \frac{dr}{1-r}.$$

We complete the proof of (3.1) by combining the previous estimates and passing to the limit as $\rho \to 1$.

(b) Let $2 \le p < \infty$. Then, since $p'-1<1$ and $0<\lambda<1$, we have

$$\Re \mathcal{V}_\lambda^{\beta p}\mu(\rho\eta) \ge C \int_0^1 \left[\int_{B(\eta,1-r)} \frac{(1-r)^{\lambda+\beta p-n}}{|1-r\rho\langle\eta,\zeta\rangle|^\lambda} d\mu(\zeta)\right]^{p'-1} \frac{dr}{1-r}.$$

If $\zeta \in B(\eta, 1-r)$, then $|1-r\rho\langle\eta,\zeta\rangle|^\lambda \le C(1-\rho r)^\lambda$. Hence

$$\begin{aligned}\Re \mathcal{V}_\lambda^{\beta p}\mu(\rho\eta) &\ge C \int_0^1 \left[\frac{\mu(B(\eta,1-r))}{(1-r)^{n-\beta p-\lambda}(1-r\rho)^\lambda}\right]^{p'-1} \frac{dr}{1-r} \\ &\ge C \int_0^\rho \left[\frac{\mu(B(\eta,1-r))}{(1-r)^{n-\beta p}}\right]^{p'-1} \frac{dr}{1-r},\end{aligned}$$

which yields (3.2). The proof of Proposition 3.1 is complete. □

We are now in a position to prove Conjecture 1 (cf. [AC1]). Let $p>1$ and $0 \le n-\beta p < 1$. Let K be a compact subset of S and let $\mathrm{Cap}(K, L_\beta^p)=0$. Then there exists a collection of open nested sets G_k such that $K \subset G_k$ and

$$\mathrm{Cap}(G_k, L_\beta^p) \le C2^{-k}, \quad k=1,2,\ldots.$$

Let ν_k be the capacitary (equilibrium) measure of G_k (see [AH]). Then $\mathcal{E}^{\beta p}(\nu_k) \le C2^{-k}$. We choose λ so that $n-\beta p<\lambda<1$ and set

$$F(z) = \sum_{k=1}^\infty F_k(z), \tag{3.4}$$

where $F_k = \mathcal{U}_\lambda^{\beta p}\nu_k$ if $1<p\le 2$ and $F_k = \mathcal{V}_\lambda^{\beta p}\nu_k$ if $p>2$.

By Theorem 1, $\|F_k\|_{H_\beta^p} \le C2^{-k/p}$. Hence $F \in H_\beta^p$. It follows from Proposition 3.1 that, for all $\eta \in S$,

$$\varliminf_{\rho\to 1} \Re F_k(\rho\eta) \ge C\mathcal{W}^{\beta p}\nu_k(\eta).$$

Since G_k is open, $\mathcal{W}^{\beta p}\nu_k(\eta) \ge 1$ everywhere on K (see [HW, p. 180]). Thus, for all $\eta \in K$, we have

$$\lim_{\rho\to 1} |F(\rho\eta)| = \infty.$$

It is easy to see that if $\eta \notin K$, then $\eta \notin E(F)$. The proof of Conjecture 1 is complete. □

To prove Theorem 3, we need the following assertion.

LEMMA 3.2. *Let $1 < p < \infty$ and $0 \le n - \beta p < 1$. Suppose that K is a compact subset of S for which* $\mathrm{Cap}(K, L^p_\beta) = 0$. *Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of holomorphic functions on B_n satisfying the following conditions.*

(i) *$\Re f_k$ is positive and continuous in the extended sense on $\bar{B}_n$, the closed ball.*
(ii) $\lim_{\rho \to 1} \Re f_k(\rho\zeta) = +\infty$ *for all* $\zeta \in K$.
(iii) *f_k is continuous on $\bar{B}_n \backslash K$.*
(iv) $\lim_{k\to\infty} f_k(z) = 0$ *for* $z \in \bar{B}_n \backslash K$.
(v) $\exp(-f_k) \in H^p_\beta \cap C(S)$.
(vi) $\lim_{k\to\infty} \|1 - \exp(-f_k)\|_{H^p_\beta} = 0$.

An analog of Lemma 3.2, for Besov spaces B^p_β in the case $n = 1$, was proved in [C2, Lemma 1].

Proof. Since $\mathrm{Cap}(K, L^p_\beta) = 0$, we may find open sets G_j such that $K \subset \bigcap G_j$ and $\mathrm{Cap}(G, L^p_\beta)_j = 2^{-j}$. Let ν_j be the capacitary measure for G_j. We choose λ so that $n - \beta p < \lambda < 1$. If $1 < p \le 2$ we set

$$f_k(z) = \sum_{j \ge k} \mathcal{U}^{\beta p}_{\lambda} \mu_j(z),$$

and if $2 < p < \infty$ we set

$$f_k(z) = \sum_{j \ge k} \mathcal{V}^{\beta p}_{\lambda} \mu_j(z).$$

It is not difficult to verify conditions (i)–(iv) (cf. [C2]). To prove (v) and (vi), we need to show that, for $1 < p < \infty$,

$$\|1 - \exp(-f_k)\|_{H^p_\beta} \le C \sum_{j \ge k} \mathcal{E}^\beta_p(\nu_j). \tag{3.5}$$

We will give a sketch of the proof. In the case $1 < p \le 2$, (3.5) follows (at least when $\beta = N$ is an integer) from the inequality

$$\left| \frac{d^N e^{-f_k(tz)}}{dt^N} \right| \le C \sum_{j \ge k} \int_0^1 \int_S \frac{(1-r)^{\lambda - n}}{|1 - tr\langle z, \zeta\rangle|^{\lambda + N}} \left[\frac{\nu_j(B(\zeta, 1-r))}{(1-r)^{n - \beta p}} \right]^{p'-1} d\sigma(\zeta) \frac{dr}{1-r}.$$

This inequality is obtained by using Hölder's inequality and the argument used to obtain the estimate (6) in [C2]. If β is non-integral, the foregoing estimate, together with the estimate

$$|R^\beta F(z)| \le C \int_0^1 (1-t)^{N - \beta - 1} |R^N F(tz)| \, dt, \quad \beta < N,$$

will suffice.

For the case $2 < p < \infty$, the needed estimate (again when $\beta = N$ is an integer) is

$$\left| \frac{d^N e^{-f_k(tz)}}{dt^N} \right| \le C \sum_{j \ge k} \int_0^1 \left[\frac{\nu_j(B(\zeta, \delta))}{\delta^{n - \beta p}} \right]^{p'-1} \frac{d\delta}{(\delta + 1 - \rho)^{N+1}}.$$

To obtain this, we let

$$V_k(z) = \sum_{j\geq k} \mathcal{V}_\lambda^{\beta p} \nu_j(z).$$

Note that, by Proposition 3.1,

$$|e^{-V_k(z)}| \leq \exp\left\{-C \sum_{j\geq k} \int_0^1 \left[\frac{\nu_j(B(\zeta,\delta))}{\delta^{n-\beta p}}\right]^{p'-1} \frac{d\delta}{\delta}\right\}. \tag{3.6}$$

Next, and this is actually the argument given in [C2, Lemma 3],

$$\left.\frac{d^N e^{-V_k(tz)}}{dt^N}\right|_{t=1} = e^{-V_k(z)}\Phi(z).$$

Here $\Phi(z)$ is a sum each term of which is a constant multiplied by a product of the form

$$\prod_s \left[\frac{d^{l_s} V_k(tz)}{dt^{l_s}}\right]^{m_s},$$

where $t = 1$ and $\sum_s l_s m_s = N$.

Now use the earlier argument (see the proof of Theorem 1(b), especially Proposition 1.4) to estimate that, at $t = 1$,

$$\left|\frac{d^{l_s} V_k(tz)}{dt^{l_s}}\right| \leq C \int_0^1 \sum_{j\geq k} \left[\frac{\nu_j(B(\zeta,\delta))}{\delta^{n-\beta p}}\right]^{p'-1} \frac{d\delta}{(\delta+1-\rho)^{l_s+1}}.$$

Thus, by Hölder's inequality,

$$\left|\prod \left[\frac{d^{l_s} V_k(tz)}{dt^{l_s}}\right]^{m_s}\right| \leq C\left\{\int_0^1 \sum_{j\geq k} \left[\frac{\nu_j(B(\zeta,\delta))}{\delta^{n-\beta p}}\right]^{p'-1} \frac{d\delta}{\delta+1-\rho}\right\}^M$$
$$\times \int_0^1 \sum_{j\geq k} \left[\frac{\nu_j(B(\zeta,\delta))}{\delta^{n-\beta p}}\right]^{p'-1} \frac{d\delta}{(\delta+1-\rho)^{N+1}},$$

where $M = \sum m_s - 1$. Together with (3.6), this gives the desired estimate. The case where β is not an integer is handled exactly as before.

The rest of the proof of Theorem 3 follows the lines of a similar assertion for $B_\beta^p(T^1)$ and is based on Lemma 3.2. (See the details in [C2].)

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