Point Evaluations for $P^t(\mu)$ and the Boundary of Support(μ)

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1. Introduction

Let G be a bounded, simply connected region in the complex plane $\mathbf{C}$, and for $1 \le t < \infty$ let μ be a finite, positive Borel measure with support in $\bar{G}$ such that each function in $P^t(\mu)$, the closure of the polynomials in $L^t(\mu)$, has an analytic continuation to G. When can we be assured that there exists λ in ∂G and $r > 0$ such that any function in $P^t(\mu)$ has an analytic continuation beyond G to $G \cup \{z : |z-\lambda| < r\}$? J. Brennan has answered this for many weighted area measures (see [B2, Thm. 4]). In this paper we examine this question and others like it with very few, if any, restrictions on μ, though we are, almost of necessity, more specific about G. As one might expect, J. Thomson's recent theorem on point evaluations (see [Th, Thm. 5.8] or Theorem 2.1 in this paper) is useful to us here.

If $1 \le t < \infty$ and μ is a finite, positive Borel measure with compact support in $\mathbf{C}$, then abpe$(P^t(\mu))$ denotes the collection of analytic bounded evaluations for the polynomials with respect to the $L^t(\mu)$ norm, and is the largest open subset of $\mathbf{C}$ to which every function in $P^t(\mu)$ has an analytic continuation. Let G be a bounded, simply connected region and let μ have support in $\bar{G}$ such that $G \subseteq$ abpe$(P^t(\mu))$. Among such measures are those of the form $d\mu = w\,dm_2|_G + h\,d\omega_G$, where m_2 is a 2-dimensional Lebesgue measure on $\mathbf{C}$, $\omega_G := \omega(\cdot, G, z_0)$ is harmonic measure on ∂G evaluated at some z_0 in G, $0 \le w \in L^1(m_2|_G)$, $0 \le h \in L^1(\omega_G)$, and either w is positive and continuous or $\int \log(h)\, d\omega_G > -\infty$. If Rat$(\bar{G}) \not\subseteq P^t(\mu)$, where Rat$(\bar{G})$ is the collection of rational functions with poles off the closure of G, then $\mathbf{C} \setminus \bar{G}$ has bounded components and at least one of them, along with G, is contained in abpe$(P^t(\mu))$. Under these circumstances, if $\mu|_{\partial G}$ is small enough, then [Th, Thm. 5.8] implies that $(\partial G) \cap$ abpe$(P^t(\mu)) \neq \emptyset$. However, the size of $\mu|_{\partial G}$ does not alone determine whether or not ∂G meets abpe$(P^t(\mu))$. For instance, there are measures μ_1 and μ_2 having the properties of μ described earlier such that Rat$(\bar{G}) \not\subseteq P^t(\mu_i)$ $(i = 1, 2)$, $\mu_1|_{\partial G} = \mu_2|_{\partial G}$, and $(\partial G) \cap$ abpe$(P^t(\mu_1)) = \emptyset$; in fact,

$$[\text{abpe}(P^t(\mu_1))] \setminus (\partial G) = [\text{abpe}(P^t(\mu_2))] \setminus (\partial G)$$

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and yet $(\partial G) \cap \mathrm{abpe}(P^t(\mu_2))$ is large (see Examples 3.3 and 3.4). When looking for "sharp" conditions on μ which ensure that $(\partial G) \cap \mathrm{abpe}(P^t(\mu)) \neq \emptyset$, it is helpful, if not imperative, to be more specific about G. In this paper we concern ourselves primarily with crescents, though much of what we do has application to many regions other than crescents; crescents are among the simplest regions for which our problem here is nontrivial. Actually, certain questions concerning $\mathrm{abpe}(P^t(\mu))$ when μ is 2-dimensional Lebesgue measure on some crescent are quite old and were addressed by Brennan in [B2]. Also, in [A1] this author had limited success in determining $\mathrm{abpe}(P^t(\mu))$ when μ is harmonic measure on some crescent.

Let G be a crescent, $1 \le t < \infty$, and μ be a finite, positive Borel measure with support in $\bar{G}$ such that $G \subseteq \mathrm{abpe}(P^t(\mu))$. If $\mathrm{Rat}(\bar{G}) \subseteq P^t(\mu)$ then we easily get that $\mathrm{abpe}(P^t(\mu)) = G$ (Proposition 3.1). In the case where $\mathrm{Rat}(\bar{G}) \not\subseteq P^t(\mu)$, we first find conditions on μ which ensure that $(\partial G) \cap \mathrm{abpe}(P^t(\mu)) \neq \emptyset$ (Proposition 3.2 and Theorem 3.7). Later we establish a lemma (Lemma 3.8) which is a type of Szegö's theorem (but which reaches beyond Szegö's theorem), and we use this lemma, along with some recent results of Olin and Yang [OY], to find conditions on μ which ensure that $\mathrm{abpe}(P^t(\mu)) = \mathrm{int}(\bar{G}\hat{\ })$ (the interior of the polynomially convex hull of the closure of G—the largest that $\mathrm{abpe}(P^t(\mu))$ could possibly be; see Theorems 3.11 and 3.13). We also give examples which show that, if μ does not satisfy our conditions, then it is possible that $(\partial G) \cap \mathrm{abpe}(P^t(\mu)) = \emptyset$ even though $G \subseteq \mathrm{abpe}(P^t(\mu))$ and $\mathrm{Rat}(\bar{G}) \not\subseteq P^t(\mu)$ (Examples 3.4 and 3.15).

2. Preliminaries

Let μ be a finite, positive Borel measure with compact support in $\mathbf{C}$, and let $1 \le t < \infty$. A point z in $\mathbf{C}$ is called a *bounded point evaluation* for the polynomials with respect to the $L^t(\mu)$ norm if there exists a positive constant c such that $|p(z)| \le c\|p\|_{L^t(\mu)}$ for all polynomials p; let $\mathrm{bpe}(P^t(\mu))$ denote the collection of all such points. Notice that if $z \in \mathrm{bpe}(P^t(\mu))$, then by the Hahn–Banach and Riesz Representation theorems there exists k_z in $L^s(\mu)$ $(1/s + 1/t = 1)$ such that $p(z) = \int p(\zeta) k_z(\zeta)\, d\mu(\zeta)$ for all polynomials p. For z in $\mathrm{bpe}(P^t(\mu))$ and f in $P^t(\mu)$ let $\hat{f}(z) := \int f(\zeta) k_z(\zeta)\, d\mu(\zeta)$. It is a straightforward exercise to show that $\hat{f} = f$ a.e. μ on $\mathrm{bpe}(P^t(\mu))$. A point z in $\mathbf{C}$ is called an *analytic bounded point evaluation* for the polynomials with respect to the $L^t(\mu)$ norm if there exist positive constants M and r such that $|p(w)| \le M\|p\|_{L^t(\mu)}$ for all polynomials p and all w such that $|z-w| < r$; $\mathrm{abpe}(P^t(\mu))$ denotes the collection of all points z of this sort. Notice that $\mathrm{abpe}(P^t(\mu))$ is a bounded open subset of $\mathbf{C}$, $\mathrm{abpe}(P^t(\mu)) \subseteq \mathrm{bpe}(P^t(\mu))$, and the components of $\mathrm{abpe}(P^t(\mu))$ are simply connected. Also, if $f \in P^t(\mu)$, then $\hat{f}$ is analytic on $\mathrm{abpe}(P^t(\mu))$.

J. Thomson has established the following theorem [Th, Thm. 5.8].

THEOREM 2.1. *If μ is a finite, positive Borel measure with compact support in $\mathbf{C}$, then there is a Borel partition $\{\Delta_i\}_{i=0}^{\infty}$ of the support of μ such that*

$$P^t(\mu) = L^t(\mu|_{\Delta_0}) \oplus \left(\bigoplus_{i=1}^{\infty} P^t(\mu|_{\Delta_i}) \right),$$

where for $i \geq 1$,

(a) *$P^t(\mu|_{\Delta_i})$ contains no nontrivial characteristic function,*

(b) *$W_i :=$ abpe$(P^t(\mu|_{\Delta_i}))$ is simply connected and $\Delta_i \subseteq \bar{W}_i$ (the $\bar{W}_i$s are the components of* abpe$(P^t(\mu))$), *and*

(c) *the mapping $f \mapsto \hat{f}$ is one-to-one on $P^t(\mu|_{\Delta_i})$, and under this mapping the Banach algebras $P^t(\mu|_{\Delta_i}) \cap L^\infty(\mu|_{\Delta_i})$ and $H^\infty(W_i)$ are algebraically and isometrically isomorphic and weak-star homeomorphic.*

We now return to the case where G is a bounded, simply connected region and μ is a finite, positive Borel measure with support in $\bar{G}$ such that $G \subseteq$ abpe$(P^t(\mu))$. If Rat$(\bar{G}) \not\subseteq P^t(\mu)$, then there must be a bounded component Ω of $\mathbf{C} \backslash \bar{G}$ such that $z \mapsto 1/(z-\zeta) \notin P^t(\mu)$ for each ζ in Ω. Using a standard argument, we show that $\Omega \subseteq$ abpe$(P^t(\mu))$. Fix ζ_0 in Ω, and choose g in $L^s(\mu)$ $(1/s + 1/t = 1)$ such that $\int pg\,d\mu = 0$ for every polynomial p and yet $\int g(z)/(z-\zeta_0)\,d\mu(z) \neq 0$. Now, $\hat{g}(\zeta) := \int g(z)/(z-\zeta)\,d\mu(z)$ is analytic on Ω and not identically zero there. So, for each ζ' in Ω there exist positive constants δ and r such that $\{\zeta : |\zeta - \zeta'| \leq r\} \subseteq \Omega$ and $|\hat{g}(\zeta)| \geq \delta$ for $|\zeta - \zeta'| = r$. Notice that if $\zeta \in \Omega$ and p is a polynomial, then

$$p(\zeta)\hat{g}(\zeta) = \int \frac{p(z)g(z)}{z-\zeta}\,d\mu(z).$$

Therefore, if p is a polynomial and $|\zeta - \zeta'| = r$, then

$$\begin{aligned} |p(\zeta)| &\leq \frac{1}{\delta} \cdot \int \frac{|p(z)||g(z)|}{|z-\zeta|}\,d\mu(z) \\ &\leq \text{const} \cdot \|g\|_{L^s(\mu)} \cdot \|p\|_{L^t(\mu)}. \end{aligned}$$

By the maximum principle, $\zeta' \in$ abpe$(P^t(\mu))$; we conclude that

$$\Omega \subseteq \text{abpe}(P^t(\mu)).$$

If $\Omega \not\subseteq$ abpe$(P^t(\mu|_{\partial\Omega}))$, equivalently $z \mapsto 1/(z-\zeta) \in P^t(\mu|_{\partial\Omega})$ for some ζ in Ω (this is what we meant in the introduction by $\mu|_{\partial G}$ "small enough"), then by Theorem 2.1(b), though $\Omega \subseteq$ abpe$(P^t(\mu))$, it cannot itself be a full component of abpe$(P^t(\mu))$, and so $\emptyset \neq (\partial\Omega) \cap$ apbe$(P^t(\mu)) \subseteq (\partial G) \cap$ abpe$(P^t(\mu))$.

Let $G_0 = \{z : |z| < 1\} \backslash \{z : |z - \frac{1}{2}| \leq \frac{1}{2}\}$. A *crescent* is a bounded, simply connected region G in $\mathbf{C}$ for which there exists a conformal mapping ϕ from G_0 onto G that extends to a homeomorphism from $\overline{G_0}$ onto $\bar{G}$; the *multiple boundary point* of this crescent G is $\phi(1)$.

3. Point Evaluations and ∂G

For the rest of this paper, G is a crescent, $1 \leq t < \infty$, and μ is a finite, positive Borel measure with support in $\bar{G}$ such that $G \subseteq$ abpe$(P^t(\mu))$. Also, if U is a bounded, simply connected region in $\mathbf{C}$ and $z_0 \in U$, then we let ω_U or $\omega(\cdot, U, z_0)$ denote harmonic measure on ∂U evaluated at z_0. Most of the time we opt for the ω_U notation since our results are independent of the choice of z_0 in U; by Harnack's inequality, $\omega(\cdot, U, z_1)$ and $\omega(\cdot, U, z_2)$ are boundedly equivalent for any two points z_1 and z_2 in U.

PROPOSITION 3.1. *If* $\mathrm{Rat}(\bar{G}) \subseteq P^t(\mu)$, *then* $\mathrm{abpe}(P^t(\mu)) = G$.

Proof. Choose ζ in $\mathbf{C}\setminus\bar{G}$. If $\zeta \in \mathrm{bpe}(P^t(\mu))$, then there exists k_ζ in $L^s(\mu)$ $(1/s+1/t=1)$ such that $p(\zeta)=\int p(z)k_\zeta(z)\,d\mu(z)$ for each polynomial p. Yet, by our hypothesis, $z \mapsto 1/(z-\zeta) \in P^t(\mu)$, and so we get the contradiction that $1=\int k_\zeta(z)\,d\mu(z)=0$. Therefore, $(\mathbf{C}\setminus\bar{G})\cap \mathrm{bpe}(P^t(\mu))=\emptyset$, which gives us that $G \subseteq \mathrm{abpe}(P^t(\mu)) \subseteq \bar{G}$. Since $\mathrm{abpe}(P^t(\mu))$ is open and G is a crescent, the proof is complete. □

If G is a crescent, then let d_G be the multiple boundary point of G and Ω_G be the bounded component of $\mathbf{C}\setminus\bar{G}$. With μ as before, let $\mu^0 := \mu|_{\partial\Omega_G}$, and $\mu^0 = \mu_a^0 + \mu_s^0$ be the Lebesgue decomposition of μ^0 with respect to ω_{Ω_G}; $\mu_a^0 \ll \omega_{\Omega_G}$, and $\mu_s^0 \perp \omega_{\Omega_G}$.

PROPOSITION 3.2. *If* $\mathrm{Rat}(\bar{G}) \not\subseteq P^t(\mu)$ *and* $\int \log(d\mu_a^0/d\omega_{\Omega_G})\,d\omega_{\Omega_G} = -\infty$, *then* $(\partial G)\cap \mathrm{abpe}(P^t(\mu)) \neq \emptyset$.

Proof. By [Gm; Chap. 5, Thms. 8.1 & 8.2], $\int \log(d\mu_a^0/d\omega_{\Omega_G})\,d\omega_{\Omega_G} = -\infty$ if and only if $z \mapsto 1/(z-\zeta) \in P^t(\mu^0)$ for some ζ in Ω_G. The rest of the proof follows from the discussion in Section 2. □

Unfortunately, Proposition 3.2 is not very useful. The hypothesis that $\int \log(d\mu_a^0/d\omega_{\Omega_G})\,d\omega_{\Omega_G} = -\infty$ is much more stringent than it needs to be in many cases; the next example suggests this. Some of the ideas in the next two examples can be traced back to M. V. Keldysh (see [Me]).

EXAMPLE 3.3. Let E be a crescent such that $\Omega_E = \{z: |z|<1\}$, and let Γ be the outer boundary of E (i.e., the boundary of the polynomially convex hull of the closure of E). Let $\mu = m_2|_E + m$, where m_2 is 2-dimensional Lebesgue measure and m is normalized Lebesgue measure on $\{z: |z|=1\}$. Choose λ in $\partial\Omega_E$, $\lambda \neq d_E$, and let $C=\{\zeta: |\zeta-\lambda| = \frac{1}{2}\,\mathrm{dist}(\lambda,\Gamma)\}$. Now C intersects $\partial\Omega_E$ in two points—call these points λ_1 and λ_2, and let $f(z)=(z-\lambda_1)^3\cdot(z-\lambda_2)^3$. For z in E, let $r(z)=\mathrm{dist}(z,\partial E)$ and $\Delta(z)=\{\zeta: |\zeta-z|<r(z)\}$. If $z \in E$ and p is a polynomial, then

$$p(z) = \frac{1}{\pi r^2(z)}\cdot\int_{\Delta(z)} p\,dm_2.$$

Therefore,

$$\begin{aligned} |p(z)| &\le \frac{1}{\pi r^2(z)}\cdot\int_{\Delta(z)} |p|\,dm_2 \\ &\le \frac{1}{(\pi r^2(z))^{1/t}}\cdot\|p\|_{L^t(\mu)} \\ &\le \frac{[\mathrm{diam}(E)]^2}{r^2(z)}\cdot\|p\|_{L^t(\mu)}. \end{aligned} \tag{3.3.1}$$

Similarly, if $z \in \Omega_E$ and p is a polynomial, then using the Poisson kernel for Ω_E at z we have

$$|p(z)| \le \frac{2}{(1-|z|)} \cdot \|p\|_{L^t(\mu)}. \tag{3.3.2}$$

By (3.3.1) and (3.3.2),

$$|fp| \le \text{const} \cdot \|p\|_{L^t(\mu)}$$

on C. It follows that $\lambda \in \text{abpe}(P^t(\mu))$, and hence $(\partial\Omega_E)\backslash d_E \subseteq \text{abpe}(P^t(\mu))$ (indeed, $\text{abpe}(P^t(\mu)) = \text{inside}(\Gamma)$). However, $\mu_a^0 = m$, and so μ_a^0 and ω_{Ω_G} are boundedly equivalent—clearly, $\int \log(d\mu_a^0/d\omega_{\Omega_G})\, d\omega_{\Omega_G} > -\infty$.

One of the objectives of this paper is to modify Proposition 3.2 so that it applies to examples like the one above. However, we cannot weaken the integral condition in its hypothesis and expect the conclusion to hold without adding some new restriction(s); this point is illustrated by our next example.

Example 3.4. Let E and Γ be as in the last example, and for $n = 1, 2, 3, \ldots$ let $E_n = (\text{inside}(\Gamma))\backslash\{z : |z| \le 1 + 1/n\}$. Let $\{\theta_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers in $[0, 2\pi)$. By Runge's theorem (for $n = 1, 2, 3, \ldots$) there is a polynomial p_n such that $|p_n([1+1/(n+1)]e^{i\theta_k})| \ge n$ if $1 \le k \le n$, and $|p_n(z)| \le 1$ if $z \in E_n$ or $|z| \le 1$. Let $\{c_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that $c_n|p_n(z)|^t \le 1$ for all z in E. Define μ on $\bar{E}$ by $\mu|_\Gamma \equiv 0$, $\mu|_{E_1} = m_2|_{E_1}$ (m_2 is 2-dimensional Lebesgue measure), $\mu|_{(E_{n+1}\backslash E_n)} = c_n m_2|_{(E_{n+1}\backslash E_n)}$, and $\mu|_{\{z:|z|=1\}}$ is normalized Lebesgue measure. Notice that μ is a finite, positive Borel measure with support in $\bar{E}$, $E \subseteq \text{abpe}(P^t(\mu))$, and $\text{Rat}(\bar{E}) \not\subseteq P^t(\mu)$. Furthermore, there is a positive constant M such that $\|p_n\|_{L^t(\mu)} \le M$ for all n, and yet $\max\{|p_n(z)| : |z - \lambda| \le r\} \to \infty$ as $n \to \infty$ for any fixed λ in $\partial\Omega_E$ and any $r > 0$. Hence $\text{abpe}(P^t(\mu)) = E \cup \{z : |z| < 1\}$ and so $(\partial E) \cap \text{abpe}(P^t(\mu)) = \emptyset$. Notice, though, that $\mu|_{\partial E}$ is the same here as in Example 3.3.

As usual, let $1 \le t < \infty$, G be a crescent, and μ be a finite, positive Borel measure with support in $\bar{G}$ such that $G \subseteq \text{abpe}(P^t(\mu))$. For z in G (and with μ and t understood), let

$$M(z) := \sup\{|p(z)| : p \text{ is a polynomial and } \|p\|_{L^t(\mu)} = 1\}.$$

Example 3.4 suggests that if we want to modify Proposition 3.2 by weakening the integral condition in its hypothesis, then we need some restriction on the rate of growth of $M(z)$ as z in G approaches $\partial\Omega_G$ in order to get that $(\partial G) \cap \text{abpe}(P^t(\mu)) \ne \emptyset$. It turns out that we need only restrict the rate of growth of $M(z)$ as z in G approaches certain points in $\partial\Omega_G$. Our first step in defining a growth condition on $M(z)$ is a description of these points.

Let Γ be a Jordan arc or Jordan curve in $\mathbf{C}$. Following [BCGJ], we say that Γ has a *tangent* at λ (or λ is a *tangent point* of Γ) if $\lambda \in \Gamma$ and there exists θ, $0 \le \theta < \pi$, with the property that for each $\epsilon > 0$ there exists $r > 0$ such that whenever $z \in \Gamma$ and $0 < |z - \lambda| < r$, then either $|\theta - \arg(z - \lambda)| < \epsilon$ or $|\theta + \pi - \arg(z - \lambda)| < \epsilon$. As in [BCGJ], let T be the collection of tangent points of Γ. It is a straightforward exercise to show that T is an $F_{\sigma\delta}$ subset of $\mathbf{C}$.

When it is understood that we are working with a generic crescent G, we hereafter let T (with no reference to G) denote the collection of tangent points of $\partial\Omega_G$. If G is a crescent, $\lambda \in T$ and $\lambda \neq d_G$, then define $S(\lambda, G)$ to be the collection of segments of the form $[a,\lambda] := \{(1-x)a + x\lambda : 0 \le x \le 1\}$, where $[a,\lambda]\backslash\{\lambda\} \subseteq G$ and $(1-x)a+x\lambda$ approaches λ nontangentially in G as $x \to 1$. Notice that $S(G,\lambda) \neq \emptyset$, since $\lambda \in T$ and $\lambda \neq d_G$. We are now in a position to give our growth condition on $M(z)$.

As usual, $1 \le t < \infty$, G is a crescent, and μ is a finite, positive Borel measure with support in $\bar{G}$ such that $G \subseteq \operatorname{abpe}(P^t(\mu))$. For z in G, let $r(z) := \operatorname{dist}(z, \partial G)$. Suppose that $\lambda \in T$ and $\lambda \neq d_G$. We say that μ satisfies the *segment condition* for G and t at λ, if there exists $n \ge 1$ such that

$$\int_{[a,\lambda]} \log(M(z))\, r^n(z)\, d|z| < \infty$$

whenever $[a,\lambda] \in S(G,\lambda)$.

REMARK. Our growth condition on $M(z)$ is not too far from a log-log condition, which is common in the literature (see e.g. [Be, p. 413; Bl; Le, Thm. XLIII, p. 127; KM]). Also, if we are willing to include in $S(G,\lambda)$ certain continuously differentiable Jordan arcs that have tangential approach in G to λ, then the n that appears in our growth condition can be made dependent only on the arc over which we are integrating; at present, n evidently depends on λ.

PROPOSITION 3.5. *Let $1 \le t < \infty$ and let G be a crescent. Define μ on $\bar{G}$ by $d\mu = w\, dm_2|_G + h\, d\omega_G$, where m_2 is 2-dimensional Lebesgue measure, $0 \le w \in L^1(m_2|_G)$, and $0 \le h \in L^1(\omega_G)$. If either $\int \log(h)\, d\omega_G > -\infty$ or $w = |F|$, where $0 \not\equiv F$ is analytic on G, then μ satisfies the segment condition for G and t at every λ in T ($\lambda \neq d_G$).*

Proof. Let us first consider the case where $\int \log(h)\, d\omega_G > -\infty$; we may assume here that $0 \le h \le 1$ and that $w \equiv 0$. Since $0 \le h \le 1$ and $\int \log(h)\, d\omega_G > -\infty$, there is a bounded analytic function f on G such that $f \circ \phi$ is an outer function (ϕ is a conformal map from $\mathbf{D} := \{z : |z| < 1\}$ onto G) and $|f| = h$ a.e. ω_G. Let $[a,\lambda]$ be a segment in $S(G,\lambda)$. Since $\lambda \in T$ and $\lambda \neq d_G$, by an argument involving [AKS, Lemma 2.8], there is a positive constant c such that $|z-\lambda|^2\, d|z| \le c\, d\omega(z, G\backslash[a,\lambda], w_0)$ on $[a,\lambda]$. Therefore, since f is analytic on G and $|f| \le 1$,

$$\begin{aligned} 0 &\ge \int_{[a,\lambda]} \log(|f(z)|)\, r^2(z)\, d|z| \\ &\ge c \cdot \int_{[a,\lambda]} \log(|f(z)|)\, d\omega(z, G\backslash[a,\lambda], w_0) > -\infty. \end{aligned} \tag{3.5.1}$$

For z in G, $\omega_G := \omega(\cdot, G, z_0)$ and $\omega(\cdot, G, z)$ are boundedly equivalent; in fact,

$$\frac{d\omega(\cdot, G, z)}{d\omega_G} \le \exp\left[2 \cdot \int_\gamma \frac{1}{r(\zeta)}\, d|\zeta|\right] \quad \text{where } r(\zeta) = \operatorname{dist}(\zeta, \partial G)$$

for any rectifiable arc $\gamma: [0,1] \to G$ with $\gamma(0) = z_0$ and $\gamma(1) = z$ (see the Appendix for this estimate). So, if $z \in G$ and p is a polynomial, then

$$|p(z)f(z)| \le \int |p(\zeta)f(\zeta)|\, d\omega(\zeta, G, z) = \int |p(\zeta)| \frac{d\omega(\zeta, G, z)}{d\omega_G(\zeta)} h(\zeta)\, d\omega_G(\zeta)$$
$$\le \exp\left[2 \cdot \int_\gamma \frac{1}{r(\zeta)}\, d|\zeta|\right] \cdot \|p\|_{L^t(\mu)}.$$

Therefore,

$$M(z) \le \frac{1}{|f(z)|} \cdot \exp\left[2 \cdot \int_\gamma \frac{1}{r(\zeta)}\, d|\zeta|\right]$$

for any rectifiable arc $\gamma: [0,1] \to G$ such that $\gamma(0) = z_0$ and $\gamma(1) = z$. Consequently, if $[a, \lambda] \in S(G, \lambda)$ (by Harnack's inequality we may assume that $a = z_0$), then

$$\int_{[a,\lambda]} \log(M(z))\, r^2(z)\, d|z|$$
$$\le \int_{[a,\lambda]} \log\left(\frac{1}{|f(z)|} \exp\left[2 \cdot \int_{[a,z]} \frac{1}{r(\zeta)}\, d|\zeta|\right]\right) r^2(z)\, d|z|$$
$$= \int_{[a,\lambda]} \log\left(\frac{1}{|f(z)|}\right) r^2(z)\, d|z| + 2 \cdot \int_{[a,\lambda]} \left(\int_{[a,z]} \frac{1}{r(\zeta)}\, d|\zeta|\right) r^2(z)\, d|z|. \tag{3.5.2}$$

Since $(1-x)a + x\lambda$ approaches λ nontangentially in G as $x \to 1$, there is a positive constant M such that $r(z)/r(\zeta) \le M$ whenever $\zeta \in [a, z]$ and $z \in [a, \lambda]$. From this, (3.5.1), and (3.5.2), it follows that

$$\int_{[a,\lambda]} \log(M(z))\, r^2(z)\, d|z| < \infty,$$

and therefore μ satisfies the segment condition for G and t at λ.

The other case is that $w = |F|$, where $0 \not\equiv F$ is analytic in G; we may assume here that $h \equiv 0$. As before, if $z \in G$ then let $r(z) = \operatorname{dist}(z, \partial G)$ and $\Delta(z) = \{\zeta : |\zeta - z| < r(z)\}$. Now if $z \in G$ and p is a polynomial, then

$$p(z)F(z) = \frac{1}{\pi r^2(z)} \cdot \int_{\Delta(z)} pF\, dm_2.$$

So, if $F(z) \neq 0$, then

$$|p(z)| \le \frac{1}{\pi |F(z)| r^2(z)} \cdot \int_{\Delta(z)} |p| w\, dm_2$$
$$\le \frac{c}{|F(z)| r^2(z)} \cdot \|p\|_{L^t(\mu)},$$

where c is a positive constant that depends only on t. Therefore,

$$M(z) \le \frac{c}{|F(z)| r^2(z)} \tag{3.5.3}$$

for all but at most countably many points z in G.

Now choose $[a,\lambda]$ in $S(G,\lambda)$, and let $[b,\lambda]$ be another segment in $S(G,\lambda)$ such that $[a,\lambda]\cap[b,\lambda]=\{\lambda\}$; let the angle subtended by $[a,\lambda]$ and $[b,\lambda]$ in G be $\pi/3$. Let γ be a Jordan arc in G with endpoints a and b such that $\Gamma :=$ $[a,\lambda]\cup[b,\lambda]\cup\gamma$ is a Jordan curve, and let $V=\text{inside}(\Gamma)$. By [AKS, Lemma 2.8], near λ, $d\omega_V$ is boundedly equivalent to $|z-\lambda|^2\,d|z|$ on $[a,\lambda]$ and $[b,\lambda]$. Using this and slightly modifying the proof of [AKS, Lemma 2.7], we get that F is in the Hardy space $H^1(V)$. Hence $\log|F|\in L^1(\omega_V)$. By (3.5.3) and the bounded equivalence between $d\omega_V$ and $|z-\lambda|^2\,d|z|$ on $[a,\lambda]$ near λ, we conclude that

$$\int_{[a,\lambda]}\log(M(z))\,r^2(z)\,d|z|<\infty.$$

So, once again, $n=2$ does the job, and we see that μ satisfies the segment condition for G and t at λ. □

LEMMA 3.6. *Let $1\le t<\infty$, and let G be a crescent. Suppose that λ_1 and λ_2 are distinct tangent points of $\partial\Omega_G$, where $\lambda_1\neq d_G\neq\lambda_2$, and let Γ be the unique Jordan arc in $\partial\Omega_G$ that has endpoints λ_1 and λ_2 such that $d_G\notin\Gamma$. If μ satisfies the segment condition for G and t at both λ_1 and λ_2, and if* $\text{Rat}(\bar{G})\not\subseteq P^t(\mu)$, *then* $\Gamma\backslash\{\lambda_1,\lambda_2\}\subseteq\text{abpe}(P^t(\mu))$.

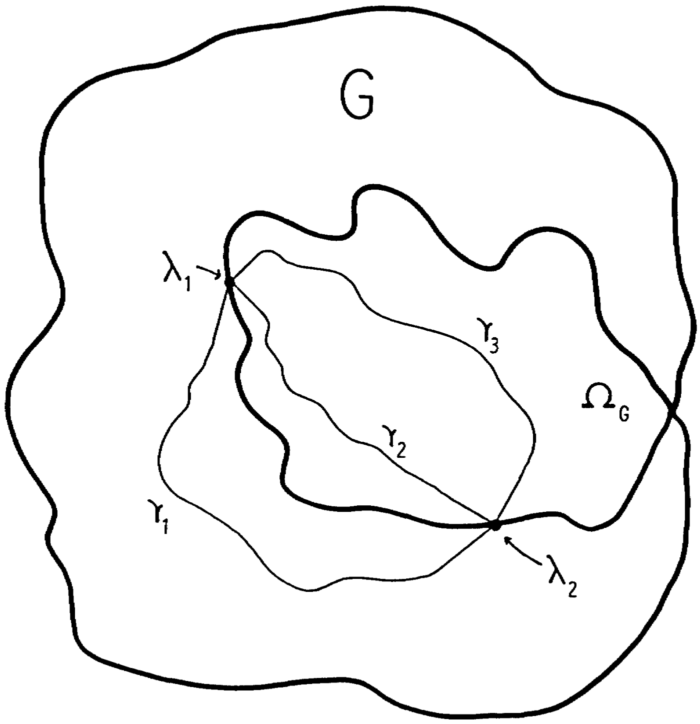
Proof. Our argument here has much in common with the proofs of [A1, Thm. 7], [Be, Cor. of Thm. D, p. 384], and [Le, Lemma 27.1, p. 135]. Among other things, we construct a certain piecewise smooth Jordan curve α such that $\Gamma\subseteq\overline{\text{inside}(\alpha)}$ and inside(α) has a small angle at both λ_1 and λ_2. Actually, this angle requirement on α is at variance with the construction in [Le, Lemma 27.1, p. 135], where a cusp is needed. Indeed, since $\partial\Omega_G$ is not necessarily smooth in a neighborhood of either λ_1 and λ_2, we lack the freedom to produce certain cusps at λ_1 and λ_2—this is what forces us to adopt a growth condition on $M(z)$ that is more restrictive than a log-log condition.

Now we return to the proof of the lemma. Since μ satisfies the segment condition for G and t at λ_j $(j=1,2)$, there exists $n_j\ge1$ such that

$$\int_{[a,\lambda_j]}\log(M(z))\,r^{n_j}(z)\,d|z|<\infty \tag{3.6.1}$$

whenever $[a,\lambda_j]\in S(G,\lambda_j)$. Since λ_1 and λ_2 are in T, we can find smooth Jordan arcs γ_1, γ_2, and γ_3 (as usual, we let γ_i denote both the Jordan arc and its trace $\gamma_i([0,1])$) such that:

(i) γ_i has endpoints λ_1 and λ_2 $(i=1,2,3)$;
(ii) $\gamma_1\backslash\{\lambda_1,\lambda_2\}\subseteq G$ and $\gamma_i\backslash\{\lambda_1,\lambda_2\}\subseteq\Omega_G$ $(i=2,3)$;
(iii) in some neighborhood of λ_j $(j=1,2)$, γ_1 coincides with some segment in $S(G,\lambda_j)$, and likewise γ_i $(i=2,3)$ coincides with some segment that has nontangential approach in Ω_G to λ_j;
(iv) $\alpha:=\gamma_1\cup\gamma_2$ and $\beta:=\gamma_2\cup\gamma_3$ are Jordan curves; and
(v) letting $V=\text{inside}(\alpha)$ and $W=\text{inside}(\beta)$, V forms an angle of $\pi/(n_j+1)$ at λ_j and W forms an angle of $\pi/2$ at λ_j, $j=1,2$ (see figure).



A simple conformal mapping argument (see e.g. the proof of [AKS, Lemma 2.8]) shows that $d\omega_V$ is boundedly equivalent to $|z-\lambda_1|^{n_1}\cdot|z-\lambda_2|^{n_2}d|z|$ on α and that $d\omega_W$ is boundedly equivalent to $|z-\lambda_1||z-\lambda_2|d|z|$ on β. Choose ζ_0 in Ω_G. Since $\text{Rat}(\bar{G}) \not\subseteq P^t(\mu)$, there exists g in $L^s(\mu)(1/s+1/t=1)$ such that $\int pg\,d\mu=0$ for every polynomial p and yet $\int g(z)/(z-\zeta_0)\,d\mu(z)\neq 0$. For $\zeta\notin\bar{G}$, let

$$\hat{g}(\zeta)=\int\frac{g(z)}{z-\zeta}\,d\mu(z).$$

Notice that $\hat{g}$ is analytic on Ω_G and not identically zero there. Moreover, there is a positive constant M such that, for any ζ in W,

$$|\hat{g}(\zeta)|\leq\frac{M}{|\zeta-\lambda_1||\zeta-\lambda_2|}.$$

Since $d\omega_W$ is boundedly equivalent to $|\zeta-\lambda_1||\zeta-\lambda_2|d|\zeta|$ on β,

$$\frac{1}{(\zeta-\lambda_1)(\zeta-\lambda_2)}$$

is in the Hardy space $H^1(W)$. Therefore, $0\not\equiv\hat{g}\in H^1(W)$, and so $\log|\hat{g}|\in L^1(\omega_W)$. Since $\omega_V|_{\gamma_2}\leq\text{const}\cdot\omega_W$, we have that $\log|\hat{g}|\in L^1(\omega_V|_{\gamma_2})$. Now, if $\zeta\in\Omega_G$ and p is a polynomial, then

$$p(\zeta)\hat{g}(\zeta)=\int\frac{p(z)g(z)}{z-\zeta}\,d\mu(z).$$

Therefore, since $\hat{g}$ is nonzero a.e. ω_V on γ_2, γ_2 had nontangential approach in Ω_G to λ_j $(j=1,2)$, $d\omega_V$ is boundedly equivalent to $|\zeta-\lambda_1|^{n_1}\cdot|\zeta-\lambda_2|^{n_2}d|\zeta|$

on γ_2, and $\log|\hat{g}| \in L^1(\omega_V|_{\gamma_2})$, there are positive constants c_1 and c_2 such that for each polynomial p,

$$\begin{aligned}\int_{\gamma_2} &\log|p(\zeta)|\,d\omega_V(\zeta)\\ &\le \int_{\gamma_2} \log\left(\frac{1}{|\hat{g}(\zeta)|}\cdot\int \frac{|p(z)||g(z)|}{|z-\zeta|}\,d\mu(z)\right)d\omega_V(\zeta)\\ &\le \int_{\gamma_2} \log\left(\frac{1}{|\hat{g}(\zeta)|}\right)d\omega_V(\zeta) + \int_{\gamma_2}\left(\int \frac{|p(z)||g(z)|}{|z-\zeta|}\,d\mu(z)\right)d\omega_V(\zeta)\\ &\le c_1 + c_2\|g\|_{L^s(\mu)}\cdot\|p\|_{L^t(\mu)}. \end{aligned} \tag{3.6.2}$$

Furthermore, since $d\omega_V$ is boundedly equivalent to $|z-\lambda_1|^{n_1}\cdot|z-\lambda_2|^{n_2}\,d|z|$ on γ_1 and since, in a neighborhood of λ_j $(j=1,2)$, γ_1 coincides with some segment in $S(G,\lambda_j)$, we get by (3.6.1) that

$$\int_{\gamma_1} \log(M(z))\,d\omega_V(z) < \infty. \tag{3.6.3}$$

By (3.6.2), (3.6.3), and Harnack's inequality, if p is a polynomial and

$$\|p\|_{L^t(\mu)} = 1$$

then there is a positive number $C(v_0)$ (recall that $\omega_V := \omega(\cdot, V, v_0)$) depending continuously on v_0 in V such that

$$\log|p(v_0)| \le \int \log|p|\,d\omega_V \le C(v_0).$$

It follows that $V \subseteq \operatorname{abpe}(P^t(\mu))$, and thus the proof is complete. □

Our next theorem is an improvement upon Proposition 3.2 for those measures that satisfy the segment condition for G and t at most tangent points of $\partial\Omega_G$. We first introduce some notation. Recall that if $1 \le t < \infty$, G is a crescent, and μ is a finite, positive Borel measure with support in $\bar{G}$ such that $G \subseteq \operatorname{abpe}(P^t(\mu))$, then $\mu^0 := \mu|_{\partial\Omega_G}$ and $\mu^0 = \mu_a^0 + \mu_s^0$ is Lebesgue decomposition of μ^0 with respect to ω_{Ω_G}; $\mu_a^0 \ll \omega_{\Omega_G}$ and $\mu_s^0 \perp \omega_{\Omega_G}$. Let $\mu_a^0 = \mu_{a_1}^0 + \mu_{a_2}^0$ be the Lebesgue decomposition of μ_a^0 with respect to ω_G; $\mu_{a_1}^0 \ll \omega_G$ and $\mu_{a_2}^0 \perp \omega_G$. Let Λ_1 be 1-dimensional Hausdorff measure on $\mathbf{C}$ and as before, let T be the collection of tangent points of $\partial\Omega_G$.

THEOREM 3.7. *Let $1 \le t < \infty$, G be a crescent, and μ satisfy the segment condition for G and t at λ for Λ_1-almost all λ in T. If* $\operatorname{Rat}(\bar{G}) \not\subseteq P^t(\mu)$ *and* $\int \log(d\mu_{a_2}^0/d\omega_{\Omega_G})\,d\omega_{\Omega_G} = -\infty$, *then* $(\partial G) \cap \operatorname{abpe}(P^t(\mu)) \ne \emptyset$.

REMARK. By the "Λ_1-almost all" hypothesis of Theorem 3.7, if $\Lambda_1(T) = 0$ then we do not impose any segment condition here.

Proof of Theorem 3.7. We divide the proof into two cases according to whether or not $\Lambda_1(T) = 0$.

If $\Lambda_1(T) > 0$, then by our hypothesis there exist two points λ_1 and λ_2 (in fact uncountably many points) in $T\setminus\{d_G\}$ at which μ satisfies the segment condition for G and t. Applying Lemma 3.6 yields $(\partial G)\cap \text{abpe}(P^t(\mu)) \neq \emptyset$.

If $\Lambda_1(T) = 0$, then by [BCGJ, Thm.], $\omega_G \perp \omega_{\Omega_G}$. Consequently, $\mu^0_{a_2} = \mu^0_a$. By Proposition 3.2 we once again have that $(\partial G)\cap \text{abpe}(P^t(\mu)) \neq \emptyset$. □

Let E and μ be as in Example 3.3. Notice that $\mu^0_{a_2} \equiv 0$ here, and so

$$\int \log\left(\frac{d\mu^0_{a_2}}{d\omega_{\Omega_G}}\right) d\omega_{\Omega_G} = -\infty.$$

Since $\text{Rat}(\bar{E}) \not\subseteq P^t(\mu)$ and μ satisfies the segment condition for E and t at every λ in $\partial\Omega_E$ not equal to d_E (see Proposition 3.5), Theorem 3.7 tells us that $(\partial E)\cap \text{abpe}(P^t(\mu)) \neq \emptyset$. However, in Example 3.3 we showed much more than this; in fact, we showed that $\text{abpe}(P^t(\mu)) = \text{int}(\bar{E}\hat{\ })$ (the interior of the polynomially convex hull of the closure of E). In general, when can we be assured that $\text{abpe}(P^t(\mu)) = \text{int}(\bar{E}\hat{\ })$? Our next two theorems give an answer to this, but before we get to them we lay some groundwork in the form of a lemma. This lemma can be viewed as an improvement upon Szegö's theorem.

LEMMA 3.8. *Let* $\mathbf{D} := \{z : |z| < 1\}$, $A = \{z \in \mathbf{D} : \text{Im}(z) > 0\}$, $B = \{z \in \mathbf{D} : \text{Im}(z) < 0\}$, *and* $I = \{z : |z| = 1$ *and* $\text{Im}(z) \geq 0\}$. *Let m be normalized Lebesgue measure on $\partial\mathbf{D}$, η be a finite, positive Borel measure with support in $I \cup \bar{B}$, and $d\eta|_{\partial\mathbf{D}} = h\,dm + d\eta_s$ be the Lebesgue decomposition of $\eta|_{\partial\mathbf{D}}$ with respect to m; $0 \leq h \in L^1(m)$ and $\eta_s \perp m$. If $\int_I \log(h)\,d\omega_A = -\infty$, then* $\text{Rat}(I \cup \bar{B}) \subseteq$ *$P^t(\eta)$ for $1 \leq t < \infty$; that is, $P^t(\eta) = L^t(\eta|_I) \oplus P^t(\eta|_{\bar{B}\setminus\{-1,1\}})$.*

REMARKS 3.9. (1) The harmonic measure ω_A given in the statement of Lemma 3.8 has support in ∂A and, by [AKS, Lemma 2.8], is boundedly equivalent to $|z+1||z-1|\,d|z|$ on ∂A.

(2) If $\eta|_{\bar{B}}$ is substantial enough (e.g., weighted area measure on $\bar{B}$ where the weight does not decay too quickly near $[-1,1]$), then

$$P^t(\eta) = L^t(\eta) \oplus P^t(\eta|_{\bar{B}\setminus\{-1,1\}})$$

if and only if $\int_I \log(h)\,d\omega_A = -\infty$; results along these lines will appear in a subsequent paper.

Proof of Lemma 3.8. By a standard argument (see the proof of [Gr; Thm. 3.1, p. 144]) we may assume that $\eta_s \equiv 0$. Suppose that $\int_I \log(h)\,d\omega_A = -\infty$. Then there exists k in $L^1(m)$ such that $k > h$ a.e. m and $\int_I \log(k)\,d\omega_A = -\infty$, and so with no loss of generality we may assume that $h > 0$ a.e. m.

We begin by showing that for any $\epsilon > 0$ there is an outer function F in the Hardy space $H^t(\mathbf{D})$ such that:

(1) $|F|^t \geq h$ a.e. m;
(2) $|F(i/2)|^t < \epsilon$; and
(3) $|F(z)| \geq 1$ whenever $z \in \bar{B}$.

We construct F in two parts (factors). First of all, define h_0 on $\partial\mathbf{D}$ by $h_0(z) = \max(1, h(z))$, and let F_0 be an outer function such that $|F_0| = h_0^{1/t}$ a.e. m. Notice that $F_0 \in H^t(\mathbf{D})$. In preparation for the second factor let $J = \{z: \bar{z} \in I\}$, and for $0 < \delta < 1$ let $I_\delta = \{z \in I: \delta \le h(z) \le 1\}$ and $J_\delta = \{z: \bar{z} \in I_\delta\}$. Because $\int_I \log(h)\, d\omega_A = -\infty$ and we may assume that $h > 0$ a.e. m, there exists δ $(0 < \delta < 1)$ such that

$$\exp\left[\int_{I_\delta} \log(h)\, d\omega_A\right] < \frac{\epsilon}{|F_0(i/2)|^t};$$

we may assume that $\omega_A := \omega(\cdot, A, i/2)$. Define h_1 on $\partial\mathbf{D}$ by $h_1(z) = 1$ if $z \notin I_\delta \cup J_\delta$, $h_1(z) = h(z)$ if $z \in I_\delta \backslash \{-1, 1\}$, and $h_1(z) = 1/h(\bar{z})$ if $z \in J_\delta$. Let F_1 be an outer function such that $|F_1| = h_1^{1/t}$ a.e. m. Notice that $F_1 \in H^\infty(\mathbf{D})$, $|F_1| \ge 1$ a.e. m off I_δ, and $|F_1|^t = h$ a.e. m on I_δ. Moreover, if $-1 < z < 1$ (i.e., if $z \in \mathbf{D}$ and $\operatorname{Im}(z) = 0$), then $P_z(\zeta) = P_z(\bar{\zeta})$ for all ζ in $\partial\mathbf{D}$, where $P_z(\cdot)$ is the Poisson kernel on $\partial\mathbf{D}$ for z; thus, by the definition of h_1,

$$|F_1(z)|^t = \exp\left[\int \log(h_1(\zeta)) P_z(\zeta)\, dm(\zeta)\right] = 1.$$

Therefore,

$$\begin{aligned} \left|F_1\left(\frac{i}{2}\right)\right|^t &= \exp\left[\int_{\partial A} \log|F_1|^t\, d\omega_A\right] \\ &= \exp\left[\int_{I_\delta} \log(h)\, d\omega_A\right] \\ &< \frac{\epsilon}{|F_0(i/2)|^t}. \end{aligned}$$

Because F_1 is outer, $|F_1| \ge 1$ a.e. m on J, and $|F_1(z)| = 1$ whenever $-1 < z < 1$, we have also that $|F_1| \ge 1$ on $\bar{B}$. Therefore, $F := F_0 F_1$ satisfies requirements (1), (2), and (3).

Now we make use of this outer function. Choose $\epsilon > 0$ and let F be the outer function corresponding to ϵ that we just constructed. By [Gr; Thm. 7.4, p. 85], there is a sequence $\{f_n\}$ of bounded functions such that:

(a) $|f_n(z) F(z)| \le |F(i/2)|$ for all z in $\bar{\mathbf{D}}$ and $n = 1, 2, 3, \ldots$; and
(b) $f_n F \to F(i/2)$ a.e. m as $n \to \infty$.

A consequence of (a) and (b) is that $f_n(i/2) F(i/2) \to F(i/2)$ and so $f_n(i/2) \to 1$ as $n \to \infty$ $(F(i/2) \ne 0)$. Therefore, letting $H_0^\infty(\mathbf{D}) = \{f: f$ is bounded and analytic on $\mathbf{D}$ and $f(i/2) = 0\}$, we have that

$$\begin{aligned} \inf_{f \in H_0^\infty(\mathbf{D})} \int |1 - f|^t\, d\eta &\le \lim_{n \to \infty} \sup \int |f_n|^t\, d\eta \\ &\le \lim_{n \to \infty} \sup\left(\int_{\partial\mathbf{D}} |f_n F|^t\, dm + \int_{\bar{B}} |f_n F|^t\, d\eta\right) \\ &< \epsilon(1 + \eta(\bar{B})). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\inf_{f\in H_0^\infty(\mathbf{D})}\int|1-f|^t\,d\eta = 0$. Now $P^t(\eta)$ contains $H^\infty(\mathbf{D})$ (the collection of bounded analytic functions on $\mathbf{D}$) and $i/2 \notin \text{support}(\eta)$, so

$$\inf_{f\in P^t(\eta)}\int\left|\frac{1}{z-i/2}-f(z)\right|^t d\eta(z) \le \inf_{f\in H^\infty(\mathbf{D})}\int\left|\frac{1}{z-i/2}-f(z)\right|^t d\eta(z) = 0.$$

The conclusion now follows. □

The next corollary, which we state without proof, is a diversion from our theme. A special case of this corollary was established by A. Vol'berg [V, Thm. 1] who used nontrivial methods. An elementary proof follows, almost immediately, from interpreting Lemma 3.8 under Möbius transformations of the form $T(z) = e^{i\theta}\cdot(z-\alpha)/(1-\bar{\alpha}z)$, where $|\alpha| < 1$.

COROLLARY 3.10. *Let $\mathbf{D} = \{z: |z| < 1\}$, let S be a nonempty finite subset of $\partial\mathbf{D}$, and let $U = \mathbf{D}\backslash\{t\zeta: \zeta\in S$ and $\frac{1}{2}\le t\le 1\}$. Let m be normalized Lebesgue measure on $\partial\mathbf{D}$, η be a finite, positive Borel measure with support in ∂U, and $\eta|_{\partial\mathbf{D}} = \eta_a+\eta_s$ be the Lebesgue decomposition of $\eta|_{\partial\mathbf{D}}$ with respect to m; $d\eta_a = h\,dm$ $(0\le h\in L^1(m))$ and $\eta_s\perp m$. If $\int_{\partial\mathbf{D}}\log(h)\,d\omega_U = -\infty$, then $P^t(\eta) = L^t(\eta)$ for $1\le t<\infty$. Moreover, if there exists g $(0\le g\in L^1(\omega_U|_{\mathbf{D}}))$ such that* $\log(g)\in L^1(\omega_U|_{\mathbf{D}})$ *and $g\,d\omega_U|_{\mathbf{D}}\le d\eta|_{\mathbf{D}}$, then $P^t(\eta) = L^t(\eta)$ if and only if $\int_{\partial\mathbf{D}}\log(h)\,d\omega_U = -\infty$.*

QUESTION 3.11. Let α be a cross-cut of $\mathbf{D} := \{z: |z| < 1\}$; that is, $\alpha: [0,1]\to\mathbf{C}$ is a Jordan arc such that $|\alpha|\le 1$ and $|\alpha(x)| = 1$ if and only if x is either 0 or 1. Then $\mathbf{D}\backslash\alpha$ is the disjoint union of two Jordan domains V and W; let $\Gamma = (\partial V)\cap(\partial\mathbf{D})$. Let m be normalized Lebesgue measure on $\partial\mathbf{D}$ and let η be a finite, positive Borel measure with support in $\Gamma\cup\bar{W}$ such that $d\eta|_{\partial\mathbf{D}} = h\,dm$ $(0\le h\in L^1(m))$. If $\int_\Gamma\log(h)\,d\omega_V = -\infty$, do we then have $P^t(\eta) = L^t(\eta|_\Gamma)\oplus P^t(\eta|_{\bar{W}})$?

In the next two theorems, we keep the same notation used in Proposition 3.2 and Theorem 3.7.

THEOREM 3.12. *Let $1\le t<\infty$, G be a crescent, and μ be a finite, positive Borel measure with support in $\bar{G}$ such that $G\subseteq$* abpe($P^t(\mu)$). *If* Rat($\bar{G}$) $\not\subseteq$ *$P^t(\mu)$ and if, for every Jordan arc $\gamma: [0,1]\to\partial\Omega_G$ (where $\gamma(0) = d_G$) there exists x $(0<x<1)$ such that*

$$\int_{\gamma([x,1])}\log\left(\frac{d\mu_a^0}{d\omega_{\Omega_G}}\right)d\omega_{\Omega_G} = -\infty,$$

then abpe($P^t(\mu)$) $=$ int($\bar{G}\,\hat{}$).

REMARK. If the answer to Question 3.11 is Yes, then we can strengthen Theorem 3.12 to the following statement: If Rat($\bar{G}$) $\not\subseteq P^t(\mu)$, and if

$$\int_{\gamma} \log\left(\frac{d\mu_a^0}{d\omega_{\Omega_G}}\right) d\omega_{\Omega_G} = -\infty$$

whenever $\gamma\colon [0,1] \to \partial\Omega_G$ is a Jordan arc and $\gamma(0) = d_G$, then $\mathrm{abpe}(P^t(\mu)) = \mathrm{int}(\bar{G}\hat{\ })$.

Proof of Theorem 3.12. An immediate consequence of our hypothesis is that

$$\int_{\partial\Omega_G} \log\left(\frac{d\mu_a^0}{d\omega_{\Omega_G}}\right) d\omega_{\Omega_G} = -\infty,$$

and so, by Proposition 3.2, $(\partial G) \cap \mathrm{abpe}(P^t(\mu)) \neq \emptyset$. Applying Theorem 2.1, we have

$$P^t(\mu) = L^t(\mu|_{\Delta_0}) \oplus P^t(\mu|_{\Delta_1}),$$

where $P^t(\mu|_{\Delta_1})$ is irreducible and $W_1 := \mathrm{abpe}(P^t(\mu|_{\Delta_1})) = \mathrm{abpe}(P^t(\mu))$ is a bounded simply connected region such that $G \cup \Omega_G \subseteq W_1 \subseteq \mathrm{int}(\bar{G}\hat{\ })$. Since $\mathrm{abpe}(P^t(\mu|_{\Delta_1})) = \mathrm{abpe}(P^t(\mu))$, and our only objective here is to show that $\mathrm{abpe}(P^t(\mu)) = \mathrm{int}(\bar{G}\hat{\ })$, we may assume that $P^t(\mu)$ has no $L^t(\mu)$-summand. Hence $P^t(\mu) = P^t(\mu|_{\Delta_1})$, and is therefore irreducible.

Suppose that $W_1 \neq \mathrm{int}(\bar{G}\hat{\ })$ (we look for a contradiction). Then there exists a Jordan arc α contained in $\partial\Omega_G$ such that $d_G \in \alpha$ (d_G may or may not be an endpoint of α), and $W_1 = [\mathrm{int}(\bar{G}\hat{\ })] \setminus \alpha$. Let ϕ be a conformal mapping from $\mathbf{D} := \{z : |z| < 1\}$ onto W_1, and let $\psi = \phi^{-1}$. By our hypotheses there exists a Jordan arc $\gamma\colon [0,1] \to \alpha$ such that $\gamma(0) = d_G$, and for some x and y $(0 < x < y < 1)$,

$$\int_{\gamma([x,y])} \log\left(\frac{d\mu_a^0}{d\omega_{\Omega_G}}\right) d\omega_{\Omega_G} = -\infty.$$

Let Γ be a Jordan curve such that $\gamma([x,y]) \subseteq \Gamma \subseteq \Omega_G \cup \gamma([x,y])$. Now ψ maps inside(Γ) conformally onto a Jordan subdomain V of $\mathbf{D}$. Let $\beta = (\partial V) \cap (\partial \mathbf{D})$. Notice that β is a closed subarc of $\partial\mathbf{D}$, ϕ maps β homeomorphically onto $\gamma([x,y])$, and $\beta \cap \overline{\psi(G)} = \emptyset$. As in Lemma 3.8, let $I = \{z : |z| = 1$ and $\mathrm{Im}(z) \geq 0\}$ and $B = \{z : |z| < 1$ and $\mathrm{Im}(z) < 0\}$. Since we have the freedom to make adjustments in x, y, and ϕ, we may assume that $\beta \subseteq I \setminus \{-1, 1\}$ and that $\phi(G) \subseteq B$.

By Theorem 2.1(c), there exists $\tilde{\psi}$ in $P^t(\mu) \cap L^\infty(\mu)$ such that $\tilde{\psi} = \psi$ on W_1; we adopt the $\tilde{\ }$ notation found in [OY]. By [OY, Lemma 2.2], we may assume that $\tilde{\psi}(\partial W_1) \subseteq \partial\mathbf{D}$. Let m be normalized Lebesgue measure on $\partial\mathbf{D}$, and let η be the finite, positive Borel measure with support in $I \cup \bar{B}$ given by $\eta := \mu \circ \tilde{\psi}^{-1}$. From [OY, Lemma 2.1] and [Co, p. 301] we get that $\eta|_{\partial\mathbf{D}} \ll m$. Also, by [OY, Lemma 2.5, Thm. 2.6 and its proof] there is a Borel subset E of $\partial\mathbf{D}$ such that $\eta((\partial\mathbf{D}) \setminus E) = 0$, and $\tilde{\psi}^{-1}$ is defined and equals ϕ on E. Thus if F is a Borel subset of β then

$$\begin{aligned}\mu(\phi(F)) \geq \mu(\phi(F \cap E)) &= \mu(\tilde{\psi}^{-1}(F \cap E)) \\ &= \eta(F \cap E) \\ &= \eta(F).\end{aligned}$$

Therefore, $\eta|_\beta \le \mu\circ\phi|_\beta$ $(=\mu^0\circ\phi|_\beta)$. Since $\eta|_\beta \ll m$ and $\mu_a^0\circ\phi|_\beta$ is the part of the Lebesgue decomposition of $\mu\circ\phi|_\beta$ with respect to m that is absolutely continuous with respect to m (this can be seen by a change of variables under $\phi|_\beta^{-1}$), we conclude that $\eta|_\beta \le \mu_a^0\circ\phi|_\beta$. Furthermore, since $\Omega_G \subseteq W_1$, there is a positive constant c such that $\omega_{\Omega_G}|_{\partial W_1} \le c\,\omega_{W_1}$. Therefore,

$$\int_\beta \log\left(\frac{d\eta}{dm}\right) dm \le \int_\beta \log\left(\frac{d(\mu_a^0\circ\phi|_\beta)}{dm}\right) dm$$

$$= \int_{\gamma([x,y])} \log\left(\frac{d\mu_a^0}{d\omega_{W_1}}\right) d\omega_{W_1} = -\infty.$$

Since β is a closed subset of $I\backslash\{-1,1\}$, m and ω_A are boundedly equivalent on β (here, A is as in Lemma 3.8). So, by Lemma 3.8, $\mathrm{Rat}(I\cup\bar{B}) \subseteq P^t(\eta)$. It follows that $\mathrm{bpe}(P^t(\eta)) \subseteq B$. This contradicts [OY, Lemma 2.1]. Therefore, W_1 must be equal to $\mathrm{int}(\bar{G}\hat{\ })$. □

If $\mu^0 \equiv 0$, then clearly μ^0 satisfies the integral condition in the hypothesis of Theorem 3.12. Hence the following corollary is an immediate consequence of that theorem.

COROLLARY 3.13. *Let $1 \le t < \infty$, G be a crescent, and μ be a finite measure such that $d\mu = w\,dm_2|_G$, where m_2 is 2-dimensional Lebesgue measure and w is a positive, continuous weight on G. If* $\mathrm{Rat}(\bar{G}) \not\subseteq P^t(\mu)$, *then* $\mathrm{abpe}(P^t(\mu)) = \mathrm{int}(\bar{G}\hat{\ })$.

THEOREM 3.14. *Let $1 \le t < \infty$ and let G be a crescent. Suppose that μ satisfies the segment condition for G and t at λ for Λ_1-almost all λ in T, and suppose for any Jordan arc $\gamma\colon [0,1] \to \partial\Omega_G$ (where $\gamma(0) = d_G$) that there exists x $(0<x<1)$ such that $\int_{\gamma([x,1])} \log(d\mu_{a_2}^0/d\omega_{\Omega_G})\,d\omega_{\Omega_G} = -\infty$. If* $\mathrm{Rat}(\bar{G}) \not\subseteq P^t(\mu)$, *then* $\mathrm{abpe}(P^t(\mu)) = \mathrm{int}(\bar{G}\hat{\ })$.

Proof. If $\mathrm{abpe}(P^t(\mu)) \ne \mathrm{int}(\bar{G}\hat{\ })$ then by Theorem 3.7 there is a Jordan arc α contained in $\partial\Omega_G$ such that $d_G \in \alpha$ and $\mathrm{abpe}(P^t(\mu)) = [\mathrm{int}(\bar{G}\hat{\ })]\backslash\alpha$. By Lemma 3.6 and [BCGJ, Thm.], $\omega_G|_\alpha \perp \omega_{\Omega_G}|_\alpha$ and so $\mu_{a_2}^0 = \mu_\alpha^0$ on α. Now argue as in the proof of Theorem 3.12 to derive a contradiction. □

COROLLARY 3.15. *Let $1 \le t < \infty$ and let G be a crescent. Define μ on G by $d\mu = w\,dm_2|_G + h\,d\omega_G$, where m_2 is 2-dimensional Lebesgue measure, $0 \le w \in L^1(m_2|_G)$, $0 \le h \in L^1(\omega_G)$, and either $\int \log(h)\,d\omega_G > -\infty$ or $w = |F|$, where $0 \not\equiv F$ is analytic on G. If* $\mathrm{Rat}(\bar{G}) \not\subseteq P^t(\mu)$, *then* $\mathrm{abpe}(P^t(\mu)) = \mathrm{int}(\bar{G}\hat{\ })$.

Proof. By Proposition 3.5, μ satisfies the segment condition for G and t at each tangent point λ of $\partial\Omega_G$ $(\lambda \ne d_G)$. Since $\mu_{a_2}^0 \equiv 0$ here, the result follows from Theorem 3.14. □

Notice that Corollary 3.15, with $h \equiv 1$ and $w \equiv 0$, completely answers the question addressed and only partially answered by [A1]. We finish this paper by giving an example that shows the necessity of the integral conditions found in the hypotheses of Proposition 3.2 and Theorems 3.7, 3.12, and 3.14.

EXAMPLE 3.16. Let Γ be the Jordan arc constructed in [A2, Thm. 7] and let $U := \mathbf{D}\backslash\Gamma$ ($\mathbf{D} := \{z: |z| < 1\}$). Then the polynomials are dense in the Hardy space $H^2(U)$, and so $\mathrm{abpe}(P^2(\omega_U)) = U$. Now extend Γ to a Jordan curve γ so that $\gamma\backslash\{1\} \subseteq \mathbf{D}$. Let $V = \mathrm{inside}(\gamma)$, $G = \mathbf{D}\backslash\bar{V}$, and $\mu = \omega_G + \omega_V$. Notice that μ is a finite, positive Borel measure with support in $\bar{G}$ (in fact, with support in ∂G), and $G \cup \mathrm{inside}(\gamma) \subseteq \mathrm{abpe}(P^2(\mu))$. Furthermore, μ satisfies the segment condition for G and 2 at each tangent point of $\partial\Omega_G$ not equal to d_G (see Proposition 3.5). Now, since $G \cup \mathrm{inside}(\gamma) \subseteq U$, there is a positive constant M such that $\|p\|_{L^2(\mu)} \le M\|p\|_{L^2(\omega_U)}$ for any polynomial p. Consequently, $G \cup \mathrm{inside}(\gamma) \subseteq \mathrm{abpe}(P^2(\mu)) \subseteq U$. Therefore, $\mathrm{Rat}(\bar{G}) \not\subseteq P^2(\mu)$ and yet $\mathrm{abpe}(P^2(\mu)) \neq \mathrm{int}(\bar{G}\hat{\ })$. Hence we cannot dispense with the integral condition in the hypothesis of Theorem 3.14 and expect the conclusion to remain true. For this example, it turns out that $\mu^0_{a_2}|_\Gamma = \omega_U|_\Gamma$. We have constructed an example for $P^2(\mu)$, but of course there is nothing special about $t = 2$ here. Similar examples exist for $P^t(\mu)$ whenever $1 \le t < \infty$. Furthermore, if the extension of Γ to γ is pathological enough (like Γ), then we have $\mathrm{abpe}(P^2(\mu)) = G \cup V$; that is, $\mathrm{Rat}(\bar{G}) \not\subseteq P^2(\mu)$ and yet $(\partial G) \cap \mathrm{abpe}(P^2(\mu)) = \emptyset$. Therefore, the integral condition in the hypothesis of Theorem 3.7 is likewise indispensable.

Appendix

The estimate given by the following proposition can be attributed to Beurling [Be]. An elementary proof based on Harnack's inequality is included.

PROPOSITION. *Let U be a bounded, simply connected region, and let $\omega_U = \omega(\cdot, U, z_0)$. If z is any point in U, then ω_U and $\omega(\cdot, U, z)$ are boundedly equivalent and*

$$\frac{d\omega(\cdot, U, z)}{d\omega_U} \le \exp\left[2 \cdot \int_\gamma \frac{1}{r(\zeta)}\, d|\zeta|\right],$$

where $r(\zeta) := \mathrm{dist}(\zeta, \partial U)$ for any rectifiable arc $\gamma: [0,1] \to U$ with $\gamma(0) = z_0$ and $\gamma(1) = z$.

Proof. Choose z in U, and let $\gamma: [0,1] \to U$ be a rectifiable arc such that $\gamma(0) = z_0$ and $\gamma(1) = z$ (as usual, we let γ denote both the arc and its trace $\gamma([0,1])$). Let n be a positive integer, and for k in $\{0, 1, 2, \ldots, n\}$ let $z_k = \gamma(k/n)$. We assume that n is large enough so that $|z_k - z_{k+1}| < \mathrm{dist}(\gamma, \partial U)$ whenever $0 \le k \le n-1$. Thus, by Harnack's inequality, $\omega(\cdot, U, z_k)$ and $\omega(\cdot, U, z_{k+1})$ are boundedly equivalent, and

$$\frac{d\omega(\cdot, U, z_{k+1})}{d\omega(\cdot, U, z_k)} \le \frac{r(z_k) + |z_k - z_{k+1}|}{r(z_k) - |z_k - z_{k+1}|}.$$

Consequently, ω_U and $\omega(\cdot, U, z)$ are boundedly equivalent, and

$$\frac{d\omega(\cdot, U, z)}{d\omega_U} \leq \prod_{k=0}^{n-1}\left(\frac{r(z_k)+|z_k - z_{k+1}|}{r(z_k)-|z_k - z_{k+1}|}\right).$$

Therefore,

$$\begin{aligned}\log\left(\frac{d\omega(\cdot, U, z)}{d\omega_U}\right) &\leq \sum_{k=0}^{n-1} \log\left(\frac{r(z_k)+|z_k - z_{k+1}|}{r(z_k)-|z_k - z_{k+1}|}\right) \\ &\leq \sum_{k=0}^{n-1}\left(\frac{r(z_k)+|z_k - z_{k+1}|}{r(z_k)-|z_k - z_{k+1}|} - 1\right) \\ &= 2\cdot \sum_{k=0}^{n-1}\left(\frac{|z_k - z_{k+1}|}{r(z_k)-|z_k - z_{k+1}|}\right),\end{aligned}$$

which can be made as close to $2\cdot\int_\gamma (1/r(\zeta))\, d|\zeta|$ as we like with a sufficiently large choice of n. □

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