Properties of Laplacians and Riesz Potentials on Manifolds with Ends

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0. Introduction

As is well known, the Laplacian on a compact Riemannian manifold (M, g) is a Fredholm operator from $H^p_{s+2}(M)$ to $H^p_s(M)$, where $H^p_t(M)$ is the tth-order L^p Sobolev space on M. Furthermore, its Fredholm inverse is a compact operator on $H^p_s(M)$. For noncompact manifolds, the situation is more complicated.

To begin with, one must be careful in defining the Sobolev spaces; unlike the compact case, not all definitions are equivalent (see [1] and [2]). In order to have any chance of having Δ_g be Fredholm, one must take $H^p_s(M)$ to be the largest domain in $L^p(M, dV_g)$ for $\Delta^{s/2}$ and take $\|u\|_{p,s} = \|u\|_p + \|\Delta^{s/2}u\|_p$. It turns out that an equivalent definition can be made in terms of Bessel potentials (see [9]).

However, even if one does this, it is not always the case that

$$\Delta_g\colon H^p_{s+2}(M) \to H^p_s(M)$$

is Fredholm. And if it is, it is not necessarily the case that the Fredholm inverse is a compact operator on $H^p_s(M)$. Obviously, therefore, one would like to find simple geometric criteria for deciding when $\Delta_g\colon H^p_{s+2}(M) \to H^p_s(M)$ has any of these properties.

In [6] an upper bound for the infimum of the essential spectrum of Δ_g, denoted $\tau_g(\rho)$, was found in terms of certain isoperimetric inequalities. (Note the details given later.) Thus $\tau_g(\rho) > 0$ is a necessary condition for Δ_g: $H^2_2(M) \to L^2(M, dV_g)$ to be Fredholm, and $\tau_g(\rho) = \infty$ is a necessary condition for the Fredholm inverse to be compact on $L^2(M, dV_g)$. As we shall see in this paper, these results can be extended to $\Delta_g\colon H^p_{s+2}(M) \to H^p_s(M)$ for $1 < p < \infty$.

A natural question to ask is whether these necessary conditions are also sufficient. As was shown in [6], the answer is negative. However it was also shown there that if M has finitely many ends then a refinement of $\tau_g(\rho)$, denoted by $\hat{\tau}_g(\rho)$, could be made. (Again, the details will be given later.)

I conjecture that, for manifolds with finitely many ends, $\hat{\tau}_g(\rho) > 0$ is both necessary and sufficient for $\Delta_g\colon H^p_{s+2}(M) \to H^p_s(M)$ to be Fredholm and that

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$\hat{\tau}_g(\rho) = \infty$ is both necessary and sufficient for the Fredholm inverse to be a compact operator on $H_s^p(M)$.

The main purpose of this paper is to offer evidence supporting this conjecture. Namely, it will be shown that the conjecture is true for metrics that are quasi-isometric to ones that are warped products on the ends of M. More precisely, let $\Omega \subset M$ be an open subset and $\rho: \Omega \to \mathbb{R}$ be a compact exhaustion of Ω. This means that ρ is C^∞ and $\rho^{-1}((-\infty, r])$ is a compact subset of Ω for each $r \in \mathbb{R}$. By Sard's theorem, $\rho^{-1}(t)$ is a smooth compact submanifold for almost all t. For such t, take $d\mu_t$ to be the measure on $\rho^{-1}(t)$ induced by g and set $P(t) = \int_{\rho^{-1}(t)} |\nabla\rho| \, d\mu_t$. Also, for $a < r$, take $V_a(r)$ to be the volume of $\rho^{-1}([a, r])$ and $V_r(\infty)$ to be the volume of $\rho^{-1}([r, \infty))$.

The isoperimetric inequalities we deal with are given in the following definition.

DEFINITION 0.1. For a complete metric g on a noncompact manifold M, Ω an open subset of M, and $\rho: \Omega \to \mathbb{R}$ a compact exhaustion:

(1) $b_{\mathrm{ex}}(\Omega, g, \rho) = \lim_{a \to \infty} \sup_r V_a(r) \int_r^\infty 1/P(t)\, dt$; and
(2) $b_{\mathrm{co}}(\Omega, g, \rho) = \lim_{a \to \infty} \sup_r V_r(\infty) \int_a^r 1/P(t)\, dt$.

In the definition the subscripts "ex" and "co" stand for expanding and contracting, respectively. The motivation for this is the example $\Omega = M = \mathbb{R} \times S^1$, $g = dr^2 + e^{kr^2} d\theta^2$, and $\rho(r, \theta) = |r|$ for $|r| > 1$. Then $b_{\mathrm{ex}}(M, g, \rho)$ is finite if and only if $k > 0$, whereas $b_{\mathrm{co}}(M, g, \rho)$ is finite if and only if $k < 0$.

DEFINITION 0.2. For a complete metric g on a noncompact manifold M and $\rho: M \to \mathbb{R}$ a compact exhaustion,

$$\tau_g(\rho) = \max(1/b_{\mathrm{ex}}(M, g, \rho), 1/b_{\mathrm{co}}(M, g, \rho)).$$

As mentioned above, one of the main results of [6] is that $\tau_g(\rho)$ is an upper bound for the infimum of the essential spectrum of Δ_g on an arbitrary complete and noncompact manifold M. However, when M contains an open submanifold M_0 with the properties (i) $\bar{M}_0$ is compact, (ii) ∂M_0 is a compact $(n-1)$-dimensional manifold, and (iii) $M \setminus M_0$ is diffeomorphic to $\mathbb{R}^+ \times \partial M_0$—that is, when M has finitely many ends—then one can improve this bound.

DEFINITION 0.3. Let M have finitely many ends and let g be a complete metric on M. For each end E of M (i.e. component of $M \setminus M_0$), let $\rho_E: E \to \mathbb{R}$ be a compact exhaustion

$$\hat{\tau}_g(\rho) = \min_E \max(1/b_{\mathrm{ex}}(E, g, \rho_E), 1/b_{\mathrm{co}}(E, g, \rho_E)),$$

where the min is taken over the set of ends.

We shall say that an end is *expanding* if $b_{\mathrm{ex}}(E, g, \rho_E)$ is finite and *contracting* if $b_{\mathrm{co}}(E, g, \rho_E)$ is finite. With this terminology, the conjecture can be stated as follows.

CONJECTURE 0.4. *Let (M,g) be a complete Riemannian manifold with finitely many ends, and let $1 < p < \infty$. Then $\Delta_g\colon H^p_{s+2}(M) \to H^p_s(M)$*

(1) *is Fredholm if and only if each end is either expanding or contracting for all ρ,*
(2) *is an isomorphism if and only if each end is either expanding or contracting for all ρ and $\mathrm{Vol}(M) = \infty$,*
(3) *has a Fredholm inverse that is a compact operator on $H^p_s(M)$ if and only if $\hat{\tau}_g(\rho) = \infty$ for all ρ.*

For $\alpha > 0$, the αth Riesz potential associated with Δ_g is the operator $\mathfrak{J}_\alpha = (\Delta_g)^{-\alpha/2}$. In other words, the αth Riesz potential is the $(\alpha/2)$th power of the Green operator for Δ_g. Whenever the conjecture is valid for a set of complete Riemannian manifolds, an important and easy consequence is the classification of the Riesz potentials associated with the set that are bounded operators.

THEOREM 0.5. *Let $\mathfrak{G}$ be a set of complete Riemannian manifolds for which the conjecture is valid. Suppose that $(M,g) \in \mathfrak{G}$. Then the Riesz potentials associated with g are bounded operators on $H^p_s(M)$ if and only if each end is either expanding or contracting and $\mathrm{Vol}(M) = \infty$. Furthermore, these potentials are compact operators if and only if, in addition, $\hat{\tau}_g(\rho) = \infty$.*

A metric $\tilde{g}$ is a warped product on the ends of M if for each component Γ of ∂M_0 we have $\tilde{g} = dr^2 + h_E^2(r)\, d\theta_\Gamma^2$ on the end $E = \mathbb{R}^+ \times \Gamma$, with h_E a positive C^∞ function and $d\theta_\Gamma^2$ a metric on Γ. The metric g is quasi-isometric to $\tilde{g}$ if there is a $c > 0$ such that $g/c \le \tilde{g} \le cg$. For example, when $\mathbb{R}^n$ is equipped with polar coordinates then the Euclidean metric is $\tilde{g} = dr^2 + r^2\, d\theta^2_{S^{n-1}}$ and so is a warped product. A metric g is quasi-isometric to $\tilde{g}$ provided the eigenvalues of $g(x)$ are bounded above and below by constants that are independent of x. The associated Laplacians are precisely the uniformly elliptic operators of the form $-(\det g)^{-1/2}\, \partial_i g^{ij} \sqrt{\det g}\, \partial_j$.

The main result of this paper is given by the following.

THEOREM 0.6. *The conjecture is true for metrics that are quasi-isometric to warped products.*

The proof will consist of two parts. The first is the reduction of the problem to the L^2 case. When M has infinite volume, either $\Delta_g\colon H^2_2(M) \to L^2(M, dV_g)$ is an isomorphism or is not Fredholm. In this situation we can reduce to the L^2 case by using results of Strichartz that are based on the Stein interpolation theorem for analytic families of operators [9; 7]. Once this is done, the result can be extended to manifolds with finite volume by using a simple surgery and a modification of a theorem of Donnelly and Li [3].

The compactness results will be proved by using Krasnosel'skii's extension of the Riesz–Thorin theorem [5]. As with the reduction of the problem to the L^2 case, this proof, which is done in the first section, holds for arbitrary complete, noncompact Riemannian manifolds—not just those with ends.

To show that the conjecture is true for $\Delta_g: H_2^2(M) \to L^2(M, dV_g)$ we need only show that a positive multiple of $\hat{r}_g(\rho)$ is a lower bound for the infimum of the essential spectrum of Δ_g, since it was already shown to be an upper bound in [6]. It follows from the result of Donnelly and Li [3] that we can do this by showing that a positive multiple of $\hat{r}_g(\rho)$ is a lower bound for the infimum of the spectrum of the Dirichlet Laplacian on $[a, \infty) \times \Gamma$, for each component Γ of ∂M_0 and each $a > 0$. Furthermore, it is a consequence of the Rayleigh-Ritz theorem that this is true for g if it is true for some metric $\tilde{g}$ that is quasi-isometric to g.

It is at this stage that we require $\tilde{g}$ to be a warped product on each end, since it is for such metrics that we can explicitly construct the minimal positive Green function for the Dirichlet Laplacian by separation of variables and estimate the norm of the associated Green operator in terms of $\hat{r}_{\tilde{g}}(\rho)$. Since this Green operator is self-adjoint, the reciprocal of this estimate will provide a lower bound for the infimum of the spectrum of the Dirichlet Laplacian. One can prove the conjecture in general if a similar estimate could be established for the Green operator associated with the general metric g.

As might be guessed, much of the analysis deals with the ordinary differential operators that arise from separation of variables in the Dirichlet problem on $[a, \infty) \times \Gamma$. We need to estimate the norm of the associated Green operators on $L^2([a, \infty), h_E^{n-1}(r)\, dr)$. It is worth noting that this estimate, which is made in the third section, is obtained with the aid of the Marcinkiewicz interpolation theorem and that it can be used because of the isoperimetric inequalities.

It seems to me that it is worth exploring what other connections there may be between isoperimetric inequalities and weak type estimates.

1. Reduction to the L^2 Case

As mentioned in the introduction, a definition of $H_s^p(M)$ that is equivalent to the one given there can be made in terms of Bessel potentials. Namely,

$$H_s^p(M) = \{u \in L^p(M, dV_g): u = (1 + \Delta_g)^{-s/2} v \text{ for some } v \in L^p(M, dV_g)\}.$$

The corresponding norm is $\|u\|_{p,s}^{\hat{}} = \|v\|_p$. It is equivalent to the one given in the introduction (see [9, §4]).

It clearly follows from this that if α and β are nonnegative, then the vertical maps in the commutative diagram

$$\begin{array}{ccc} H_{2+\alpha}^p(M) & \xrightarrow{\Delta_g} & H_\alpha^p(M) \\ {\scriptstyle (1+\Delta_g)^{(\beta-\alpha)/2}} \uparrow & & \downarrow {\scriptstyle (1+\Delta_g)^{(\alpha-\beta)/2}} \\ H_{2+\beta}^p(M) & \xrightarrow{\Delta_g} & H_\beta^p(M) \end{array}$$

are isomorphisms. Hence, either both horizontal maps are Fredholm or both are not Fredholm. Similarly, if they are Fredholm, either they both have a compact Fredholm inverse or they both do not. In other words, we have proved the following lemma.

LEMMA 1.1. *$\Delta_g: H^p_{s+2}(M) \to H^p_s(M)$ is Fredholm for all $s \geq 0$ if and only if it is for some $s \geq 0$. If it is Fredholm and G is its Fredholm inverse, then $G: H^p_s(M) \to H^p_s(M)$ is a compact operator for all $s \geq 0$ if and only if it is compact for some $s \geq 0$.*

LEMMA 1.2. *$\Delta_g: H^p_2(M) \to L^p(M, dV_g)$ is an isomorphism for all $1 < p < \infty$ if and only if it is for some p, $1 < p < \infty$.*

Proof. An equivalent statement is that Δ_g^{-1} exists and is a bounded operator on $L^p(M, dV_g)$ for all $1 < p < \infty$ if and only if it exists and is bounded for at least one such p. The proof of this is essentially the same as the proof of Theorem 5.5 in [9]. Namely, one considers the analytic family of operators $(wI + \Delta_g)^{-1}$ and uses the analytic families interpolation theorem of Stein [7]. □

With the exception of the compactness results, these two lemmas provide the sought reduction to the L^2 case for manifolds with infinite volume. In order to deal with manifolds with finite volume, we shall use the following simple modification of a result of Donnelly and Li [3]. In its statement we let $K \subset M$ be a precompact open subset with smooth boundary. We take $\Omega_1, \Omega_2, \ldots, \Omega_N$ to be the components of $M \backslash K$, and set

$$\mathfrak{B}^p(\Omega_k) = \{u \in L^p(\Omega_k, dV_g): \Delta_g u \in L^p(\Omega_k, dV_g) \text{ and } u|_{\partial\Omega_k} = 0\}. \quad (1.3)$$

(Note that the Ω_k need not be ends, since in general they are not diffeomorphic to $\mathbb{R}^+ \times \Gamma$, with Γ a component of ∂K.)

LEMMA 1.4. *$\Delta_g: H^p_2(M) \to L^p(M, dV_g)$ is Fredholm if and only if*

$$\Delta_g: \mathfrak{B}^p(\Omega_k) \to L^p(\Omega_k, dV_g)$$

is Fredholm for each Ω_k.

Proof. A simple modification of the proof of Proposition 2.1 in [3] suffices for a proof here. □

THEOREM 1.5. *Let (M, g) be a complete Riemannian manifold.*

(1) *$\Delta_g: H^p_{s+2}(M) \to H^p_s(M)$ is Fredholm for all $1 < p < \infty$ and all $s \geq 0$ if and only if it is Fredholm for $p = 2$ and $s = 0$.*
(2) *If Δ_g is Fredholm and G is its Fredholm inverse, then G is a compact operator on $H^p_s(M)$ for all $1 < p < \infty$ and all $s \geq 0$ if and only if it is compact for $p = 2$ and $s = 0$.*

Proof. We know that the first statement is a consequence of Lemmas 1.1 and 1.2 when M has infinite volume. Suppose therefore that $\mathrm{Vol}(M) < \infty$. The problem in this case, of course, is that $\Delta_g: H^p_{s+2}(M) \to H^p_s(M)$ has a nontrivial kernel consisting of the constants, and so we cannot use Lemma 1.2. What we shall do is construct a new manifold $\tilde{M}$ that has infinite volume and coincides with M outside a compact subset of M.

As before, let K be an open, precompact subset of M with a smooth boundary, and take $\Omega_1, \ldots, \Omega_N$ to be the components of $M \backslash K$. Pick $x \in K$ and $\nu > 0$ so that the ball $\bar{B}_{2\nu}(x)$ is contained in a coordinate neighborhood of x in K. Remove $\bar{B}_\nu(x)$ and smoothly patch in $\mathbb{R}^+ \times S^{n-1}_\nu(x) \equiv \Omega_0$. Now extend the metric on $M \backslash \bar{B}_{2\nu}(x)$ in a way so that on $[1, \infty) \times S^{n-1}_\nu(x)$ it is the warped product $dr^2 + e^{r^2} d\theta^2$, with $d\theta^2$ a metric on $S^{n-1}_\nu(x)$. Call the resulting manifold $\tilde{M}$ and the resulting metric $\tilde{g}$.

Since $\mathbb{R} \times S^{n-1}_\nu$ equipped with the metric $dr^2 + e^{r^2} d\theta^2$ has infinite volume, $\tilde{M}$ has infinite volume. Hence, we know that the first statement of the theorem is valid for $\tilde{M}$. It therefore follows from Lemma 1.4 that $\Delta_g: \mathfrak{B}^p(\Omega_k) \to L^p(\Omega_k, dV_g)$ is Fredholm for all $1 < p < \infty$, all $s \geq 0$ and all $k = 0, 1, \ldots, N$ if and only if it is Fredholm for $p = 2$, $s = 0$, and all $k = 0, 1, \ldots, N$.

In fact, more can be said. Namely, we shall see in the next section that $\Delta_g: \mathfrak{B}^2(\Omega_0) \to L^2(\Omega_0, dV_g)$ is Fredholm. Hence we actually can say that Δ_g: $\mathfrak{B}^p(\Omega_k) \to L^p(\Omega_k\, dV_g)$ is Fredholm for all $1 < p < \infty$, all $s \geq 0$ and $k = 1, \ldots, N$ if and only if it is Fredholm for $p = 2$, $s = 0$, and all $k = 1, \ldots, N$. By Lemma 1.4 this is all we need to show that the first statement of the theorem is true for M.

The second statement follows directly from Krasnosel'skii's extension of the Riesz–Thorin theorem [5].

2. The L^2 Case

Each end E of M is diffeomorphic to $\mathbb{R}^+ \times \Gamma$ for some component Γ of ∂M_0. Henceforth we shall assume that a diffeomorphism has been chosen and identify E with $\mathbb{R}^+ \times \Gamma$. Furthermore, it follows from the already established necessity of the conditions in Conjecture 0.4 that we need only show their sufficiency for a specific compact exhaustion in order to prove the theorem. We shall take $\rho: \mathbb{R}^+ \times \Gamma \to \mathbb{R}$ to be $\rho(r, \theta) = r$. We also take the warped product metric on E to be $g = dr^2 + h_E^2(r)\, d\theta_\Gamma^2$, with $d\theta_\Gamma^2$ a fixed metric on Γ. With these choices and with $V_\Gamma = \mathrm{Vol}(\Gamma)$, the quantities $b_{\mathrm{ex}}(E, g, \rho)$ and $b_{\mathrm{co}}(E, g, \rho)$ become

$$b_{\mathrm{ex}}(E, g, \rho) = \lim_{a \to \infty} \sup_r V_\Gamma \int_a^r h_E^{n-1}(t)\, dt \int_r^\infty h_E^{1-n}(t)\, dt$$

and

$$b_{\mathrm{co}}(E, g, \rho) = \lim_{a \to \infty} \sup_r V_\Gamma \int_r^\infty h_E^{n-1}(t)\, dt \int_a^r h_E^{1-n}(t)\, dt.$$

Notice that in this case it is impossible for an end to be both expanding and contracting. Whether this is true in general is an interesting question.

The Laplacian associated with g is $\Delta_g = h_E^{1-n}(-\partial_r h_E^{n-1}\partial_r + h_E^{n-3}\Delta_\Gamma)$, with Δ_Γ the Laplacian on Γ coming from $d\theta_\Gamma^2$. When $Y_\lambda(\theta)$ is an eigenfunction of Δ_Γ and $f(r,\theta) = u(r)Y_\lambda(\theta)$, we have $\Delta_g f(r,\theta) = Y_\lambda(\theta)\mathfrak{L}_a^\lambda u(r)$ with

$$\mathfrak{L}_a^\lambda u(r) = h_E^{1-n}(r)(-(h_E^{n-1}(r)u'(r))' + \lambda h_E^{n-3}(r)u(r)).$$

Our first task is to show that $\mathfrak{L}_a^\lambda$ is an isomorphism for the Dirichlet problem on $[a,\infty)$. More precisely, let $I_a = [a,\infty)$ and take $B_\lambda^p(I_a)$ to be the space

$$B_\lambda^p(I_a) = \{\phi \in L^p(I_a, h_E^{n-1}(t)\,dt) \colon \phi(a) = 0 \text{ and } \mathfrak{L}_a^\lambda \phi \in L^p(I_a, h_E^{n-1}(t)\,dt)\},$$

with norm $\|\phi\|_{B^p} = \|\phi\|_p + \|\mathfrak{L}_a^\lambda \phi\|_p$. Clearly $\mathfrak{L}_a^\lambda \colon B_\lambda^p(I_a) \to L^p(I_a, h_E^{n-1}(t)\,dt)$ is bounded. We shall demonstrate that if either $b_{\text{ex}}(E,g,\rho)$ or $b_{\text{co}}(E,g,\rho)$ is finite, then it has a bounded inverse T_a^λ and that the norm of T_a^λ can be estimated in terms of quantities related to $b_{\text{ex}}(E,g,\rho)$ and $b_{\text{co}}(E,g,\rho)$.

First, a basis for $\ker(\mathfrak{L}_a^\lambda)$ that has particularly nice properties will be constructed. For $\lambda = 0$ this is easy. Take $u_{00}(r) = \int_r^\infty h_E^{1-n}(t)\,dt$ and $u_{10}(r) = 1 - u_{00}(r)/u_{00}(a)$, if $\int_r^\infty h_E^{1-n}(t)\,dt < \infty$. Otherwise pick $u_{00}(r) = 1$ and $u_{10}(r) = \int_a^r h_E^{1-n}(t)\,dt$. For $\lambda > 0$ we shall use the following theorem.

THEOREM 2.1. *For each $\lambda > 0$ there is a $u_{0\lambda} \in \ker(\mathfrak{L}_a^\lambda)$ that is strictly positive, has an everywhere negative derivative, and goes to 0 at infinity. Furthermore, $u_{0\lambda}$ is unique up to constant multiple.*

Proof. Let w_λ be the solution to the initial value problem $\mathfrak{L}_a^\lambda w_\lambda = 0$ with $w_\lambda(a) = 1$ and $w_\lambda'(a) = 0$. Then

$$w_\lambda'(r)w_\lambda(r) = h_E^{1-n}(r)\int_a^r h_E^{n-1}(t)(w_\lambda'(t))^2 + \lambda h_E^{n-3}(t)(w_\lambda(t))^2\,dt,$$

and so $w_\lambda(r) > 0$ for $r > a$. However, according to [4, Chap. XI, Cor. 6.1], the existence of a strictly positive solution to $\mathfrak{L}_a^\lambda w = 0$ implies that $\mathfrak{L}_a^\lambda w = 0$ is disconjugate on I_a. This in turn, by Theorem 6.4 and Corollary 6.4 of [4, Chap. XI], assures the existence of solutions $\tilde{u}_{0\lambda}$ all of which are multiples of one another, are strictly positive, and have everywhere negative derivatives.

Obviously, w_λ is linearly independent of any such $\tilde{u}_{0\lambda}$ and so by Corollary 6.3 of [4, Chap. XI] one such $\tilde{u}_{0\lambda}$ is

$$u_{0\lambda}(r) = w_\lambda(r)\int_r^\infty w_\lambda^{-2}(t)h_E^{1-n}(t)\,dt.$$

Since w_λ is increasing for $r > a$, it is clear that $\lim_{r\to\infty} u_{0\lambda}(r) = 0$ if either w_λ is bounded or $\int_a^\infty h_E^{1-n}(t)\,dt$ is finite. Therefore, suppose neither of these is true. By L'Hôpital's rule, $\lim_{r\to\infty} u_{0\lambda}(r) = \lim_{r\to\infty} h_E^{1-n}(r)/w_\lambda'(r)$—which in turn is $(\int_a^\infty \lambda h_E^{n-3}(t)w_\lambda(t)\,dt)^{-1}$, because $\mathfrak{L}_a^\lambda w_\lambda = 0$, $w_\lambda(a) = 1$, and $w_\lambda'(a) = 0$. If this last quantity is 0 we are done. If it is $\gamma > 0$, then again by L'Hôpital's rule we have

$$\lim_{r\to\infty} \frac{w_\lambda(r)}{\int_a^r h_E^{1-n}(t)\,dt} = \lim_{r\to\infty} h_E^{n-1}(r)w_\lambda'(r) = \frac{1}{\gamma}.$$

Hence there is a number $c>0$ such that $w_\lambda(r) \geq c\int_a^r h_E^{1-n}(t)\,dt$. Thus

$$\left(\int_a^\infty \lambda h_E^{n-3}(t) w_\lambda(t)\,dt\right)^{-1} \leq \left(c\int_a^\infty \lambda h_E^{n-3}(r)\int_a^r h_E^{1-n}(t)\,dt\,dr\right)^{-1}. \tag{2.1.1}$$

Let $f(r)=\int_a^r h_E^{1-n}(t)dt$ and $\beta=(n-3)/(1-n)$. Then the integral on the right-hand side of (2.1.1) is $\int_a^\infty \lambda(f'(t))^\beta f(t)\,dt$. By assumption, $\lim_{t\to\infty} f(t)=\infty$; hence this integral is infinite if $\beta=0$ or 1. Therefore, suppose $-1<\beta<0$. Set $j(t)=(f'(t))^\beta f(t)$. Then $f'(t)(f(t))^{1/\beta}=(j(t))^{1/\beta}$. Integrating both sides of this, we obtain

$$[f(t)]^{1+1/\beta}=(1+1/\beta)\int_{a+1}^t (j(s))^{1/\beta}\,ds+(f(a+1))^{1+1/\beta}. \tag{2.1.2}$$

Since $-1<\beta<0$ and $\lim_{t\to\infty} f(t)=\infty$, we know that the integral in (2.1.2) is bounded as $t\to\infty$. Hence $j^{1/\beta}<1$ on a set of infinite measure, which means $(f')^\beta f=j>1$ on a set of infinite measure. In other words, the quantity on the right side of (2.1.1) is 0. □

Obviously the $u_{0\lambda}$ of the theorem will form half of the basis of $\ker(\mathfrak{L}_a^\lambda)$ that we are seeking. For the other half, choose

$$u_{1\lambda}(r)=u_{0\lambda}(r)\int_a^r (u_{0\lambda}(t))^{-2} h_E^{1-n}(t)\,dt. \tag{2.2}$$

Clearly $u_{1\lambda}(a)=0$. Another property of $u_{1\lambda}(a)$ that we shall utilize is that it is increasing (see [4, Chap. XI]). Finally, we shall also use the next proposition.

PROPOSITION 2.3. *If $r>a$, then $u_{0\lambda}(r)=u_{1\lambda}(r)\int_r^\infty (u_{1\lambda}(t))^{-2} h_E^{1-n}(t)\,dt$.*

Proof. It follows from (2.2) that

$$\left(\frac{u_{0\lambda}}{u_{1\lambda}}\right)'(s)=-\frac{h_E^{1-n}(s)}{u_{1\lambda}^2(s)}.$$

Upon integration of both sides from r to b, this becomes

$$\left(\frac{u_{0\lambda}}{u_{1\lambda}}\right)(r)=\int_r^b (u_{1\lambda}(t))^{-2} h_E^{1-n}(t)\,dt+\frac{u_{0\lambda}(b)}{u_{1\lambda}(b)}. \tag{2.3.1}$$

Since $\lim_{b\to\infty} u_{0\lambda}(b)=0$ and $u_{1\lambda}$ is increasing, we obtain the result by letting $b\to\infty$ in (2.3.1). □

We can now define T_a^λ, the inverse of $\mathfrak{L}_a^\lambda$. Setting $(r,t)_+=\max(r,t)$ and $(r,t)_-=\min(r,t)$, we take $K_a^\lambda(r,t)=u_{0\lambda}((r,t)_+)u_{1\lambda}((r,t)_-)$ and define T_a^λ to be the operator

$$T_a^\lambda v(r)=\int_a^\infty K_a^\lambda(r,t)v(t)h_E^{n-1}(t)\,dt.$$

THEOREM 2.4. *Let*

$$b_{\mathrm{ex}}(E_a) = \sup_r \int_a^r h_E^{n-1}(t)\,dt \int_r^\infty h_E^{1-n}(t)\,dt, \tag{2.4.1}$$

$$b_{\mathrm{co}}(E_a) = \sup_r \int_r^\infty h_E^{n-1}(t)\,dt \int_a^r h_E^{1-n}(t)\,dt. \tag{2.4.2}$$

If either $b_{\mathrm{ex}}(E_a)$ or $b_{\mathrm{co}}(E_a)$ is finite then $T_a^\lambda: L^p(I_a, h_E^{n-1}(t)\,dt) \to B_\lambda^p(I_a)$ is a bounded operator for $1<p<\infty$. Moreover, $\|T_a^\lambda\| \le c_p \min(b_{\mathrm{ex}}(E_a), b_{\mathrm{co}}(E_a))$, with c_p a constant that depends solely upon p and n.

Proof. When $b_{\mathrm{ex}}(E_a) < \infty$, we can use Proposition 2.3 and the fact that $u_{\infty\lambda}$ is decreasing to derive the estimate

$$K_a^\lambda(r,t) \le \int_{(r,t)_+}^\infty h_E^{1-n}(s)\,ds \equiv \phi_{\mathrm{ex}}((r,t)_+).$$

In this case we define a new operator T_{ex} by

$$T_{\mathrm{ex}}v(r) = \int_a^\infty v(t)\phi_{\mathrm{ex}}((r,t)_+)h_E^{n-1}(t)\,dt.$$

Alternatively, when $b_{\mathrm{co}}(E_a) < \infty$, we can use (2.2) and the fact that $u_{0\lambda}$ is decreasing to get

$$K_a^\lambda(r,t) \le \int_a^{(r,t)_-} h_E^{1-n}(s)\,ds \equiv \phi_{\mathrm{co}}((r,t)_-).$$

In this case we define T_{co} to be the operator

$$T_{\mathrm{co}}v(r) = \int_a^\infty v(t)\phi_{\mathrm{co}}((r,t)_-)h_E^{n-1}(t)\,dt.$$

Evidently, $|T_a^\lambda v(r)|$ is bounded above by $T_{\mathrm{ex}}|v(r)|$ if $b_{\mathrm{ex}}(E_a) < \infty$ and by $T_{\mathrm{co}}|v(r)|$ if $b_{\mathrm{co}}(E_a) < \infty$. Thus it suffices to show that T_{ex} and T_{co} are bounded operators on $L^p(I_a, h_E^{n-1}(t)\,dt)$ in the appropriate cases. First suppose that $b_{\mathrm{ex}}(E_a) < \infty$.

In this case we decompose T_{ex} into $T_{\mathrm{ex}} = S_{\mathrm{ex}}^0 + S_{\mathrm{ex}}^1$, with

$$S_{\mathrm{ex}}^0 v(r) = \int_a^r v(t)h_E^{n-1}(t)\,dt\,\phi_{\mathrm{ex}}(r),$$

$$S_{\mathrm{ex}}^1 v(r) = \int_r^\infty v(t)\phi_{\mathrm{ex}}(t)h_E^{n-1}(t)\,dt.$$

For $v \in L^1(I_a, h_E^{n-1}(t)\,dt)$, we have $|S_{\mathrm{ex}}^j v(r)| \le \|v\|_1 \phi_{\mathrm{ex}}(r)$. If $v \not\equiv 0$ this implies that

$$\begin{aligned} m\{r: |S_{\mathrm{ex}}^j v(r)| \ge \alpha\} &\le m\{r: \|v\|_1\phi_{\mathrm{ex}}(r) \ge \alpha\} \\ &= m\{r: \phi_{\mathrm{ex}}(r) \ge \alpha/\|v\|_1\} \equiv \mu_v(\alpha), \end{aligned}$$

where $m(A) = \int_A h_E^{n-1}(t)\,dt$ for $A \subset I_a$.

Since ϕ_{ex} decreases to 0, either $\mu_v(\alpha)=0$ or there is exactly one $r_0 \geq a$ for which $\phi_{ex}(r)=\alpha/\|v\|_1$. In the latter case we have $\mu_v(\alpha)=\int_a^r h_E^{n-1}(t)\,dt$, the measure of the interval on which $\phi_{ex}(r)\geq \alpha/\|v\|_1$. Using the definition (2.4.1) of $b_{ex}(E_a)$, we can conclude from this that

$$\mu_v(\alpha) \leq \frac{b_{ex}(E_a)}{\phi_{ex}(r_0)} = b_{ex}(E_a)\frac{\|v\|_1}{\alpha}.$$

In other words, for all α we have $m\{r: |S_{ex}^j v(r)| \geq \alpha\} \leq b_{ex}(E_a)\|v\|_1/\alpha$. Hence both S_{ex}^0 and S_{ex}^1 are weak type $(1,1)$, which implies that T_{ex} is weak type $(1,1)$ also.

It is easy to see from the definition of S_{ex}^0 that $b_{ex}(E_a)<\infty$ implies that $\|S_{ex}^0 v\|_\infty \leq b_{ex}(E_a)\|v\|_\infty$ for $v\in L^\infty(I_a)$. Thus S_{ex}^0 is both type (∞,∞) and weak type $(1,1)$. The Marcinkiewicz interpolation theorem (see [8]) can therefore be used to show that S_{ex}^0 is a bounded operator on $L^p(I_a, h_E^{n-1}(t)\,dt)$ for $1<p<\infty$ and that $\|S_{ex}^0\| \leq c_p b_{ex}(E_a)$ with c_p depending only upon p and n.

As for S_{ex}^1, suppose that $v\in L^q(I_a, h_E^{n-1}(t)\,dt)$ for some $1<q<\infty$. From Hölder's inequality we obtain

$$|S_{ex}^1 v(r)| \leq \|v\|_q \left(\int_r^\infty (\phi_{ex}(t))^{q'} h_E^{n-1}(t)\,dt\right)^{1/q'},$$

with $1/q'+1/q=1$. Since $b_{ex}(E_a)<\infty$,

$$\begin{aligned} |S_{ex}^1 v(r)| &\leq b_{ex}(E_a)\|v\|_q\left(\int_r^\infty \left(\int_a^t h^{n-1}(x)\,dx\right)^{-q'} h_E^{n-1}(t)\,dt\right)^{1/q'} \\ &= \frac{b_{ex}(E_a)}{q'-1}\|v\|_q\left(\int_a^r h_E^{n-1}(t)\,dt\right)^{-1/q}. \end{aligned}$$

Because of this, we can say

$$m\{r: |S_{ex}^1 v(r)| \geq \alpha\} \leq m\left\{r: \int_a^r h_E^{n-1}(t)\,dt \leq \left(\frac{b_{ex}(E_a)}{q'-1}\frac{\|v\|_q}{\alpha}\right)^q\right\}.$$

The set $\{r\in I_a: \int_a^r h_E^{n-1}(t)\,dt \leq \beta\}$ is an interval $[a, r_\beta]$, since the integral is an increasing function of r. Furthermore, by definition, $m([a,r_\beta]) = \int_a^{r_\beta} h_E^{n-1}(t)\,dt=\beta$. Thus

$$m\{r: |S_{ex}^1 v(r)| \geq \alpha\} \leq \left(\frac{b_{ex}(E_a)}{q'-1}\frac{\|v\|_q}{\alpha}\right)^q.$$

Hence S_{ex}^1 is weak type (q,q). Coupling this with the fact that it is also weak type $(1,1)$, we can again use the Marcinkiewicz interpolation theorem to see that S_{ex}^1 is a bounded operator on $L^p(I_a, h_E^{n-1}(t)\,dt)$ for all $1<p<q$, which means for all $1<p<\infty$ since q was arbitrary. Furthermore, $\|S_{ex}^1\|\leq c_p b_{ex}(E_a)$, with c_p depending only upon p and n. This completes the proof for the case $b_{ex}(E_a)<\infty$.

Now assume that $b_{co}(E_a)<\infty$ and decompose T_{co} into $T_{co}=S_{co}^0+S_{co}^1$, with

$$S_{co}^0 v(r) = \int_a^r v(t)\phi_{co}(t)h_E^{n-1}(t)\,dt,$$

$$S_{\mathrm{co}}^1 v(r) = \int_r^\infty v(t) h_E^{n-1}(t)\, dt\, \phi_{\mathrm{co}}(r).$$

For $v \in L^\infty$, we have $\|S_{\mathrm{co}}^1 v\|_\infty \le b_{\mathrm{co}}(E_a)\|v\|_\infty$ and so S_{co}^1 is type (∞, ∞). For $v \in L^1(I_a, h_E^{n-1}(t)\, dt)$, we have $|S_{\mathrm{co}}^j v(r)| \le \|v\|_1 \phi_{\mathrm{co}}(r)$. Thus

$$m\{r: |S_{\mathrm{co}}^j v(r)| \ge \alpha\} \le m\{r: \phi_{\mathrm{co}}(r) \ge \alpha/\|v\|_1\} \equiv \mu_v(\alpha).$$

Since in this case $\phi_{\mathrm{co}}(r)$ increases to infinity as $r \to \infty$ and $\phi_{\mathrm{co}}(a) = 0$, there is an $r_0 \ge a$ such that $\phi_{\mathrm{co}}(r_0) = \alpha/\|v\|_1$ and $\phi_{\mathrm{co}}(r) > \alpha/\|v\|_1$ for $r > r_0$. Thus $\mu_v(\alpha) = \int_{r_0}^\infty h_E^{n-1}(t)\, dt$. From this and the finiteness of $b_{\mathrm{co}}(E_a)$, we can assert that $\mu_v(\alpha) \le b_{\mathrm{co}}(E_a)/\phi_{\mathrm{co}}(r_0) = b_{\mathrm{co}}(E_a)\|v\|_1/\alpha$. In other words, S_{co}^j is weak type $(1, 1)$. As in the previous case, we arrive at the conclusion that S_{co}^1 is type (p, p) for $1 < p < \infty$ and that $\|S_{\mathrm{co}}^1\| \le c_p b_{\mathrm{co}}(E_a)$ with c_p depending only upon p and n.

Finally, suppose that $1 < q < \infty$. Then

$$|S_{\mathrm{co}}^0 v(r)| \le \|v\|_q \left(\int_a^r (\phi_{\mathrm{co}}(t))^{q'} h_E^{n-1}(t)\, dt \right)^{1/q'}.$$

The finiteness of $b_{\mathrm{co}}(E_a)$ then implies

$$|S_{\mathrm{co}}^0 v(r)| \le \frac{b_{\mathrm{co}}(E_a)}{q'-1} \|v\|_q \left(\int_r^\infty h_E^{n-1}(t)\, dt \right)^{-1/q}.$$

We may now proceed as before to conclude that S_{co}^0 is weak type (q, q) for all $1 < q < \infty$ and so is type (p, p) for $1 < p < \infty$. Moreover, $\|S_{\mathrm{co}}^0\| \le c_p b_{\mathrm{co}}(E_a)$, with the constant depending only on p and n. □

We are now in the position to construct the Green operator for the Dirichlet Laplacian on $E_a = I_a \times \Gamma$. This is the Laplace operator Δ_g with the domain $\mathfrak{B}^2(E_a)$ defined in (1.3).

Since $g = dr^2 + h_E^2(r)\, d\theta_\Gamma^2$ is a warped product on E, this space decomposes into an orthogonal direct sum. Let $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of Δ_Γ, and let $\{Y_k\}$ be a corresponding complete orthonormal set of eigenfunctions satisfying $\Delta_\Gamma Y_k = \lambda_k Y_k$. For each k define Φ_k to be the projection $\Phi_k : \mathfrak{B}^2(E_a) \to \mathfrak{B}^2(E_a)$ given by $\Phi_k w(r) = \int_\Gamma w(r, \theta) Y_k(\theta)\, d\theta$. Then $\mathfrak{B}^2(E_a) = \bigoplus \Phi_k \mathfrak{B}^2(E_a)$.

Our desired Green operator is

$$G_a v(r, \theta) = \sum_{k=0}^\infty T_a^{\lambda_k} \Phi_k v(r) \cdot Y_k(\theta).$$

THEOREM 2.5. *Suppose that M is a manifold with finitely many ends, that g is a complete metric on M that is a warped product on the ends, and that the compact exhaustion on each end is as stipulated above. Then there is a constant c, which depends only on the dimension of M, that satisfies the inequality*

$$c\hat{\tau}_g(\rho) \le \inf \sigma_{\mathrm{ess}}(\Delta_g).$$

Proof. Given Theorem 2.4, an easy calculation shows that the norm of G_a on $L^2(E_a, dV_g)$ is bounded above by $\tilde{c}\min(b_{\text{ex}}(E_a), b_{\text{co}}(E_a))$, with $\tilde{c}$ depending only upon $\dim(M)$. From this and the self-adjointness of the Dirichlet Laplacian Δ^D_{g,E_a} on E_a, we get $\inf\sigma(\Delta^D_{g,E_a}) \geq c\max(1/b_{\text{ex}}(E_a), 1/b_{\text{co}}(E_a))$.

Since the ends are disjoint, this inequality can be extended to the Dirichlet Laplacian $\Delta^D_{g,a}$ on $[a,\infty)\times\Gamma$ as

$$\inf\sigma(\Delta^D_{g,a}) \geq c\min_E \max(1/b_{\text{ex}}(E_a), 1/b_{\text{co}}(E_a)).$$

The theorem then follows from the result of Donnelly and Li mentioned above. □

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