Weighted Reproducing Kernels in Bergman Spaces

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1. Introduction

Let $\Delta = \{z \in \mathbf{C} \colon |z| < 1\}$ and let dA denote Lebesgue measure on Δ. For $p \geq 1$ let A^p denote the Bergman space, that is, the set of functions f which are analytic in Δ and satisfy

$$\|f\|_{A^p} = \left[\frac{1}{\pi}\int_\Delta |f(z)|^p \, dA(z)\right]^{1/p} < \infty. \tag{1}$$

More generally, for a nonnegative measure μ on Δ let $A^p(\mu)$ denote the set of functions analytic in Δ that satisfy (1) where $dA(z)$ is replaced by $d\mu(z)$. If the set of point evaluation functionals are locally uniformly bounded, then there is a reproducing kernel $K_\mu(z,w)$ which is symmetric (i.e., $\overline{K_\mu(z,w)} = K_\mu(w,z)$) and belongs to $A^p(\mu)$. In the case $d\mu(z) = dA(z)$ this kernel is the usual Bergman kernel given by $K(z,w) = (1-\bar{w}z)^{-2}$. Another important case is when μ is absolutely continuous with respect to Lebesgue measure, and specifically when $d\mu(z) = |\varphi(z)|^\alpha \, dA(z)$ and φ is a function analytic in Δ. For such measures we use the notation $K_\alpha(z,w;\varphi)$ for the reproducing kernel, and call them *weighted* reproducing kernels. These kernels are closely connected with the solutions of many extremal problems on spaces of analytic functions (see [2], [4], and [8]). For example, the solutions of point evaluation problems play the role of inner functions as shown in the recently developed factorization theory in Bergman spaces [2; 4]. Weighted kernels also play a significant role in the general theory of domination in Bergman spaces. Two references on this area are [6] and [9]. Thus the analytic properties of these kernels are very important in the theory of Bergman spaces.

In this paper we consider the case where the weight function φ has a finite number of zeros in Δ. In the special case where φ is a finite Blaschke product, we prove that the weighted reproducing kernel is a rational function. More precisely, our central result is the following theorem.

Theorem 1. *Let B be a finite Blaschke product and let α be a complex number satisfying* $\operatorname{Re}\alpha > 0$. *Then there is a weighted reproducing kernel $K_\alpha(z,w) = K_\alpha(z,w;B)$ having the following properties.*

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(i) $(1/\pi)\int_\Delta \overline{K_\alpha(z,w)} f(z)|B(z)|^\alpha\, dA(z) = f(w)$ *for* $|w|<1$ *and for every* $f \in H^\infty$.

(ii) $K_\alpha(z,w)$ *is a rational function of* $z, \bar{w}$ *of the same degree as a function of* z *or as a function of* $\bar{w}$. *This degree equals* $n+2$ *where* n *is the degree of* B.

(iii) *If* α *is real then* $\overline{K_\alpha(z,w)} = K_\alpha(w,z)$.

(iv) *for each fixed* w, *the poles of* $K_\alpha(\cdot, w)$ *are at* $1/\bar{w}$ (*a double pole*) *and at the points* $1/\bar{a}_k$ *where* $\{a_k\}$ *are the zeros of* B.

(v) $K_\alpha(z,w)$ *depends anti-analytically on* α.

In fact, we find a formula for $K_\alpha(z,w)$ involving a finite number of constants depending on α (see (13)). Theorem 1 is used to prove Theorem 2, which is stated below and characterizes finite Blaschke products in terms of rational reproducing kernels. We use the notation $\mathfrak{A}$ for the disk algebra, which consists of the functions analytic in Δ and continuous in $\bar{\Delta}$.

THEOREM 2. *A function* $f \in \mathfrak{A}$ *which does not vanish on* $\partial\Delta$ *is a constant multiplied by a finite Blaschke product if and only if the weighted reproducing kernel* $K_\alpha(z,w;f)$ *is a rational function of* $z, \bar{w}$ *for all* α *satisfying* $\operatorname{Re}\alpha > 0$.

We obtain a number of consequences of Theorem 1. Perhaps the most significant consequence is that the contractive zero-divisors in A^p which have a finite number of zeros are pseudo-rational functions (that is, a power of a rational function). We also prove that the solutions of certain integral equations related to the duality $A^p \leftrightarrow A^{p'}(1/p+1/p'=1)$ depend anti-analytically on the parameter. Finally we express the solution of a finite optimal recovery problem in terms of weighted reproducing kernels.

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2. Weights Generated by Finite Blaschke Products. Proof of Theorem 1

Let $|a_k|<1$ for $k=1,2,\ldots,n$ and let

$$B(z) = \prod_{k=1}^{n} \frac{z-a_k}{1-\bar{a}_k z}.$$

It is easily seen that the L^2 norm generated by the weighted measure $|B|^\alpha\, dA$ is equivalent to the usual norm (1) for all positive α. This implies the existence of the symmetric reproducing kernel $K_\alpha(z,w) = K_\alpha(z,w;B)$. Our aim is to find an explicit formula for $K_\alpha(z,w)$.

For $j=1,2,\ldots,n$ let

$$B_j(z) = \prod_{\substack{k=1\\k\neq j}}^{n} \frac{z-a_k}{1-\bar{a}_k z}.$$

We first consider the case $a_k \neq a_j$ for $k \neq j$. Let $|w|<1$ and let $h \in H^\infty$. The following relation can be obtained through an application of the residue theorem:

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{B(w)\bar{z}}{1-\bar{z}w}\overline{B(z)}h(z)\,dz = h(w) - \sum_{k=1}^{n} \frac{B_k(w)}{1-\bar{a}_k w}\frac{1-|a_k|^2}{B_k(a_k)}h(a_k). \tag{2}$$

The argument begins by noting that $z\bar{z}=1$ and $B(z)\overline{B(z)}=1$ for $|z|=1$. (In the case where w coincides with one of the numbers $a_1, a_2, \dots, a_n$, each side of (2) equals zero.) We can rewrite the left-hand side of (2) as

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{B(w)\bar{z}}{1-\bar{z}w}\overline{B(z)}|B(z)|^\alpha h(z)\,dz.$$

If we apply the complex form of Green's theorem to this integral and use the result in (2), we obtain

$$\frac{1}{\pi}\int_\Delta B(w)\left[\frac{\overline{B(z)}}{(1-\bar{z}w)^2}+\left(\frac{\alpha}{2}+1\right)\frac{\bar{z}\overline{B'(z)}}{1-\bar{z}w}\right]h(z)|B(z)|^\alpha\,dA(z)$$
$$= h(w) - \sum_{k=1}^{n} \frac{B_k(w)}{1-\bar{a}_k w}\frac{1-|a_k|^2}{B_k(a_k)}h(a_k). \tag{3}$$

The right-hand side of (3) also can be expressed as an integral, and since (3) holds for all $h \in H^\infty$ this yields

$$\overline{B(w)}\left[\frac{B(z)}{(1-z\bar{w})^2}+\left(\frac{\alpha}{2}+1\right)\frac{zB'(z)}{1-z\bar{w}}\right]$$
$$= K_\alpha(z,w) - \sum_{k=1}^{n} \frac{\overline{B_k(w)}}{1-a_k\bar{w}}\frac{1-|a_k|^2}{\overline{B_k(a_k)}}K_\alpha(z,a_k).$$

Hence

$$K_\alpha(z,w) = \frac{B(z)\overline{B(w)}}{(1-z\bar{w})^2}+\left(\frac{\alpha}{2}+1\right)\frac{zB'(z)\overline{B(w)}}{1-z\bar{w}}$$
$$+\sum_{k=1}^{n} \frac{\overline{B_k(w)}}{1-a_k\bar{w}}\frac{1-|a_k|^2}{\overline{B_k(a_k)}}K_\alpha(z,a_k). \tag{4}$$

The substitution $z=a_l$ in (4) gives

$$K_\alpha(a_l,w) = \left(\frac{\alpha}{2}+1\right)\frac{a_l B'(a_l)\overline{B(w)}}{1-a_l\bar{w}}$$
$$+\sum_{k=1}^{n} \frac{\overline{B_k(w)}}{1-a_k\bar{w}}\frac{1-|a_k|^2}{\overline{B_k(a_k)}}K_\alpha(a_l,a_k). \tag{5}$$

Because of the symmetry of the kernel, this yields

$$K_\alpha(z,a_l) = \left(\frac{\alpha}{2}+1\right)\frac{\overline{a_l}\overline{B'(a_l)}B(z)}{1-\bar{a}_l z}$$
$$+\sum_{k=1}^{n} \frac{B_k(z)}{1-\bar{a}_k z}\frac{1-|a_k|^2}{B_k(a_k)}K_\alpha(a_k,a_l). \tag{6}$$

If we use (6) and (4) and express B' in terms of B_k, we find that

$$K_\alpha(z,w)=\frac{B(z)\overline{B(w)}}{(1-z\bar{w})^2}+\left(\frac{\alpha}{2}+1\right)\sum_{k=1}^{n}\frac{B_k(z)\overline{B_k(w)}}{(1-\bar{a}_k z)(1-a_k\bar{w})}\frac{z\bar{w}-|a_k|^2}{1-z\bar{w}}$$
$$+\sum_{k,l=1}^{n}\frac{(1-|a_k|^2)(1-|a_l|^2)}{(1-\bar{a}_k z)(1-a_l\bar{w})}\frac{B_k(z)}{B_k(a_k)}\frac{\overline{B_l(w)}}{\overline{B_l(a_l)}}K_\alpha(a_k,a_l). \tag{7}$$

For every $f\in H^\infty$, the following relation holds:

$$\frac{1}{\pi}\int_\Delta \overline{K_{\alpha+2}(z,w)}\,\overline{B(z)}B(w)f(z)B(z)|B(z)|^\alpha\,dA(z)=f(w)B(w).$$

Since the set $\{fB: f\in H^\infty\}$ is the same as the set of functions in H^∞ which vanish at a_k $(k=1,2,\dots,n)$, this implies that for each fixed w $(|w|<1)$ there are constants $c_k(w)$ such that

$$K_{\alpha+2}(z,w)B(z)\overline{B(w)}=K_\alpha(z,w)+\sum_{k=1}^{n}c_k(w)K_\alpha(z,a_k) \tag{8}$$

[1, p. 421].

If we let $z=a_l$ $(1\le l\le n)$ in (8), we obtain

$$K_\alpha(a_l,w)=-\sum_{k=1}^{n}c_k(w)K_\alpha(a_l,a_k). \tag{9}$$

It is easy to see that the set $\{K_\alpha(z,a_k): k=1,2,\dots,n\}$ is linearly independent. Hence the determinant of the matrix formed from the inner products $\langle K_\alpha(z,a_k),K_\alpha(z,a_l)\rangle$ $(k,l=1,2,\dots,n)$ is not zero. This inner product equals $K_\alpha(a_l,a_k)$ and hence the matrix $[K_\alpha(a_l,a_k)]$ is invertible. Thus (9) implies that

$$c_k(w)=\sum_{k=1}^{n}c_{kl}K_\alpha(a_l,w), \tag{10}$$

where c_{kl} are constants depending only on α. Using (10) in (8), we have

$$K_{\alpha+2}(z,w)B(z)\overline{B(w)}=K_\alpha(z,w)+\sum_{k,l=1}^{n}c_{kl}K_\alpha(z,a_k)K_\alpha(a_l,w). \tag{11}$$

We next show the following relations, referred to collectively as (12):

$$K_\alpha(a_k,a_l)+\sum_{p,q=1}^{n}c_{pq}K_\alpha(a_k,a_p)K_\alpha(a_q,a_l)=0 \tag{12a}$$

$$\frac{a_kB_k(a_k)}{(1-|a_k|^2)(1-a_k\bar{w})}+\sum_{p,q=1}^{n}c_{pq}\frac{a_qB'(a_q)}{1-a_q\bar{w}}K_\alpha(a_k,a_p)=0 \tag{12b}$$

$$\frac{\overline{a_k}\overline{B_k(a_k)}}{(1-|a_k|^2)(1-\bar{a}_kz)}+\sum_{p,q=1}^{n}c_{pq}\frac{a_pB'(a_p)}{1-\bar{a}_pz}K_\alpha(a_q,a_k)=0 \tag{12c}$$

for $k,l=1,2,\dots,n$. Equation (12a) follows from (11) by letting $z=a_k$ and $w=a_l$. By using (5), (6), and (7) in (11) with $z=a_k$ and also (12a), we find that this yields (12b). A similar argument gives (12c). If (7) is substituted in (11) and then (12) is used, we obtain a formula for $K_{\alpha+2}(z,w)$. With $\alpha+2$ replaced by α this yields

$$K_\alpha(z,w)=\frac{1}{(1-z\bar{w})^2}+\frac{\alpha}{2}\left[\sum_{k=1}^{n}\frac{1-|a_k|^2z\bar{w}}{(1-z\bar{w})(1-\bar{a}_kz)(1-a_k\bar{w})}\right.$$
$$\left.+\sum_{k,l=1}^{n}\frac{d_{kl}}{(1-\bar{a}_kz)(1-a_l\bar{w})}\right], \tag{13}$$

where d_{kl} are constants depending only on α. Equation (13) holds for $\alpha>2$, and hence the statements (ii), (iii), and (iv) in Theorem 1 follow for $\alpha>2$.

For every $f\in A^2$ the function φ_f defined by

$$\varphi_f(\alpha,w)=f(w)-\frac{1}{\pi}\int_\Delta\left[\frac{1}{(1-\bar{z}w)^2}\right.$$
$$\left.+\frac{\alpha}{2}\sum_{k=1}^{n}\frac{1-|a_k|^2\bar{z}w}{(1-\bar{z}w)(1-a_k\bar{z})(1-\bar{a}_kw)}\right]f(z)|B(z)|^\alpha\,dA(z)$$

is analytic in $\{\alpha:\operatorname{Re}\alpha>0\}\times\Delta$. For $k=1,2,\dots,n$ and $\operatorname{Re}\alpha>0$, define ψ_f^k by

$$\psi_f^k(\alpha)=\frac{1}{\pi}\int_\Delta\frac{f(z)}{1-a_k\bar{z}}|B(z)|^\alpha\,dA(z).$$

Then ψ_f^k is analytic in $\{\alpha:\operatorname{Re}\alpha>0\}$. Since $a_k\neq a_l$ for $k\neq l$, the $n\times n$ matrix $[1/(1-\bar{a}_ka_l)]$ has a nonzero determinant [5]. Therefore a function σ_f^l is defined through the system of linear equations

$$\sum_{l=1}^{n}\frac{\sigma_f^l(\alpha)}{1-\bar{a}_la_k}=\frac{2}{\alpha}\varphi_f(\alpha,a_k)\quad(k=1,2,\dots,n), \tag{14}$$

and σ_f^l is analytic in $\{\alpha:\operatorname{Re}\alpha>0\}$. For each fixed α, the set of functionals $\{\psi_f^k(\alpha): k=1,2,\dots,n\}$ is linearly independent. To see this, note that otherwise there are constants c_k $(k=1,2,\dots,n)$ such that

$$\int_\Delta\left[\sum_{k=1}^{n}\frac{c_k}{1-a_k\bar{z}}\right]f(z)|B(z)|^\alpha\,dA(z)=0\quad\text{for all } f\in A^2.$$

In particular, letting $f(z)=\sum_{k=1}^{n}\bar{c}_k/(1-\bar{a}_kz)$, this leads to $c_k=0$ for $k=1,2,\dots,n$. The linear independence of the set of functionals implies that to each fixed α there are functions $f_1,f_2,\dots,f_n$ in A^2 which are bi-orthogonal at α. In particular, this shows that the $n\times n$ matrix $[\psi_{f_j}^k(\alpha)]$ $(j,k=1,2,\dots,n)$ satisfies

$$\det[\psi_{f_j}^k(\alpha)]\neq 0 \tag{15}$$

for the fixed value of α. The determinant in (15) defines an analytic function in $\{\alpha:\operatorname{Re}\alpha>0\}$. Thus, for every l $(l=1,2,\dots,n)$ there are constants d_{kl}^* $(k=1,2,\dots,n)$ which solve the system of linear equations

$$\sum_{k=1}^{n}\overline{d_{kl}^*}\psi_{f_j}^k(\alpha)=\sigma_{f_j}^l(\alpha)\quad(j=1,2,\dots,n). \tag{16}$$

This provides n systems of equations which determine $\overline{d_{kl}^*}$ as meromorphic functions in $\{\alpha:\operatorname{Re}\alpha>0\}$.

Let $J_\alpha(z,w)$ be defined by the right-hand side of (13), where the factor $\alpha/2$ is replaced by $\bar{\alpha}/2$ and d_{kl} is replaced by d^*_{kl} as determined by (16). Then $\overline{J_\alpha(z,w)}$ is a meromorphic function of α in $\{\alpha\colon \operatorname{Re}\alpha > 0\}$. From (13), (14), and (16), we see that $d_{kl}(\alpha) = d^*_{kl}(\alpha)$ for all real $\alpha > 2$ except possibly a discrete set. Also, for $f \in A^2$ and $|w| < 1$ we have

$$\frac{1}{\pi}\int_\Delta \overline{J_\alpha(z,w)} f(z)|B(z)|^\alpha \, dA(z) = f(w). \tag{17}$$

Since the integral on the left-hand side of (17) is a meromorphic function of α, (17) implies that the equality holds for all α where $\operatorname{Re}\alpha > 0$. We also note that $d^*_{kl}(\alpha)$ are independent of the choice of $f_1, f_2, \dots, f_n$ (for $\alpha > 2$ this follows from (13), (17), and the uniqueness of the kernel). For every α such that $\operatorname{Re}\alpha > 0$, we can find $f_1, f_2, \dots, f_n$ such that (15) holds at α, and hence $d_{ml} = d_{ml}(\alpha)$ are anti-analytic in α.

In the case of multiple zeros of the Blaschke product, we can obtain the same conclusion by using the case proved above and taking limits. This completes the proof of Theorem 1. □

We next obtain a direct consequence of Theorem 1 about the anti-analytic dependence on α of representing elements. Let $\alpha > 0$ and let A^α_B denote the Hilbert space of functions analytic in Δ with the norm

$$\|f\|^2_{A^\alpha_B} = \frac{1}{\pi}\int_\Delta |f(z)|^2 |B(z)|^\alpha \, dA(z).$$

For each $g \in A^2$, consider the bounded linear functional on A^α_B defined by

$$f \mapsto \frac{1}{\pi}\int_\Delta f(z)\overline{g(z)}\, dA(z). \tag{18}$$

By the Riesz representation theorem, there exists $g_\alpha \in A^\alpha_B$ such that (18) is given by

$$f \mapsto \frac{1}{\pi}\int_\Delta f(z)\overline{g_\alpha(z)}|B(z)|^\alpha \, dA(z).$$

Corollary 1. *$g_\alpha(z)$ can be extended to an anti-analytic function of α in $\{\alpha\colon \operatorname{Re}\alpha > 0\}$. In other words, there is a function $g(\alpha,z)$ defined on $\{\alpha\colon \operatorname{Re}\alpha > 0\} \times \Delta$ which is anti-analytic in α and analytic in z and satisfies*

$$\int_\Delta f(z)\overline{g(\alpha,z)}|B(z)|^\alpha \, dA(z) = \int_\Delta f(z)\overline{g(z)}\, dA(z) \tag{19}$$

for $f \in A^\alpha_B$ and $\operatorname{Re}\alpha > 0$.

Proof. Define $g(\alpha,z) = (1/\pi)\int_\Delta K_\alpha(z,w)g(w)\, dA(w)$, where $K_\alpha(z,w)$ is the kernel constructed in the proof of Theorem 1. By statement (v) of Theorem 1, $g(\alpha,z)$ is anti-analytic in α. The equality (19) can be verified directly. □

3. Weights Generated by Functions in the Disk Algebra. Proof of Theorem 2

First we prove a preliminary result.

PROPOSITION 1. *Suppose that* $f \in H^\infty$, $g \in \mathcal{A}$, *and* g *does not vanish in* $\bar{\Delta}$. *Then*

$$K_\alpha(z, w; fg) = \frac{K_\alpha(z, w; f)}{[g(z)]^{\alpha/2}[\overline{g(w)}]^{\alpha/2}}. \tag{20}$$

Proof. Let $J_\alpha(z, w)$ be defined by the right-hand side of (20) for $|z| < 1$ and $|w| < 1$. Then J_α is analytic in z, anti-analytic in w, and symmetric. Hence it remains to show only that J_α has the reproducing property. Suppose that $h \in H^\infty$. Then

$$\begin{aligned} &\frac{1}{\pi}\int_\Delta \overline{J_\alpha(z, w)} h(z) |f(z) g(z)|^\alpha \, dA(z) \\ &= \frac{1}{\pi [g(w)]^{\alpha/2}} \int_\Delta \overline{K_\alpha(z, w; f)} [g(z)]^{\alpha/2} h(z) |f(z)|^\alpha \, dA(z) \\ &= \frac{1}{[g(w)]^{\alpha/2}} \cdot [g(w)]^{\alpha/2} h(w) = h(w). \end{aligned}$$

□

Proof of Theorem 2. Suppose that $f \in \mathcal{A}$ and that f does not vanish on $\partial\Delta$. Then f has a finite number of zeros in Δ. Hence we can write $f = BF$, where B is the finite Blaschke product having the same zeros as f and $F \in \mathcal{A}$ and does not vanish in $\bar{\Delta}$. By Proposition 1 we have

$$K_\alpha(z, w; f) = \frac{K_\alpha(z, w; B)}{[F(z)]^{\alpha/2}[\overline{F(w)}]^{\alpha/2}}.$$

Theorem 1 shows that $K_\alpha(z, w; B)$ is a rational function (of z and $\bar{w}$). Therefore $K_\alpha(z, w; f)$ is rational if and only if $F^{\alpha/2}$ is rational. Thus $K_\alpha(z, w; f)$ is rational for all α if and only if $F^{\alpha/2}$ is rational for all α. This occurs if and only if F is constant. □

4. Contractive Zero-Divisors in A^p

Contractive zero-divisors were introduced in [2] and [4]. They are functions $G \in A^p$ ($p \geq 1$) that satisfy

$$\frac{1}{\pi}\int_\Delta [|G(z)|^p - 1] h(z) \, dA(z) = 0 \tag{21}$$

for all $h \in H^\infty$. They play the role of inner functions for factorization in Bergman spaces. It was proved in [2; 4] that if Λ is a zero set for A^p then there is a unique (up to a constant factor of modulus 1) contractive zero-divisor $G_\Lambda \in A^p$ whose zero set is Λ. If Λ is a finite subset of Δ then G_Λ is analytic in $\bar{\Delta}$ [2; 4]. In this section we apply Theorem 1 to establish further properties of G_Λ. Namely, we prove that for a finite zero set the corresponding zero-divisor is a product of a power of a nonvanishing rational function and a Blaschke product. First we obtain two propositions that establish a relation between contractive zero-divisors and weighted reproducing kernels.

Proposition 2. *Let $G \in \mathcal{Q}$. Then G is a contractive zero-divisor for A^p $(p \geq 1)$ if and only if $K_p(z,0;G)=1$ for $|z|<1$.*

Proof. Suppose that G is a contractive zero-divisor for A^p. Then (21) implies

$$\frac{1}{\pi}\int_\Delta h(z)|G(z)|^p\,dA(z) = \frac{1}{\pi}\int_\Delta h(z)\,dA(z) = h(0)$$

for all $h \in H^\infty$. Also we have

$$\frac{1}{\pi}\int_\Delta \overline{K_p(z,0;G)}h(z)|G(z)|^p\,dA(z) = h(0),$$

and hence

$$\int_\Delta [\overline{K_p(z,0;G)}-1]h(z)|G(z)|^p\,dA(z) = 0 \quad \text{for all } h \in H^\infty.$$

Therefore, $K_p(z,0;G)=1$ for $|z|<1$.

Conversely, suppose that $K_p(z,0;G)=1$ for $|z|<1$. If $h \in H^\infty$, then

$$h(0) = \frac{1}{\pi}\int_\Delta \overline{K_p(z,0;G)}h(z)|G(z)|^p\,dA(z) = \frac{1}{\pi}\int_\Delta h(z)|G(z)|^p\,dA(z).$$

Using $h(0) = (1/\pi)\int_\Delta h(z)\,dA(z)$, we obtain (21) for all $h \in H^\infty$. □

The following result was proved in [2]. It also can be deduced from a general result proved in [7, see Thm. 1], which establishes a connection between the solutions of point-evaluation-type extremal problems and weighted reproducing kernels. To make our presentation self-contained, we give a proof of this proposition which is different from that given in [2].

Proposition 3. *Let Λ be a finite subset of Δ and let B be the Blaschke product for which Λ is its zero set. Then the contractive zero-divisors for A^p are given by*

$$G_\Lambda(z) = b\frac{[K_p(z,0;B)]^{2/p}B(z)}{[K_p(0,0;B)]^{1/p}}, \tag{22}$$

where $|b|=1$.

Proof. Define F by $G_\Lambda = B\cdot F$. Then $F \in \mathcal{Q}$ and F does not vanish in $\bar{\Delta}$ [2]. By Proposition 1 we have

$$K_p(z,w;G_\Lambda) = \frac{K_p(z,w;B)}{[F(z)]^{p/2}[\overline{F(w)}]^{p/2}}. \tag{23}$$

Proposition 2 and (23) give

$$\frac{K_p(z,0;B)}{[\overline{F(0)}]^{p/2}} = [F(z)]^{p/2}.$$

Hence $K_p(z,0;B)$ does not vanish in $\bar{\Delta}$, and

$$F(z) = \frac{[K_p(z,0;B)]^{2/p}}{\overline{F(0)}}.$$

This yields $|F(0)|^2 = [K_p(0,0;B)]^{2/p}$ and thus

$$F(z) = \frac{b[K_p(z,0;B)]^{2/p}}{[K_p(0,0;B)]^{1/p}},$$

where $|b| = 1$. This proves (22). □

The following result is a direct consequence of Theorem 1 and Proposition 3.

THEOREM 3. *If* Λ *is a finite subset of* Δ *and* $1 \le p < \infty$, *then the contractive zero-divisor* G_Λ *in* A^p *is a product of a nonvanishing pseudo-rational function and the Blaschke product determined by the zero set* Λ.

By combining (22) and (13), we obtain a somewhat explicit expression for the contractive zero-divisors for A^p for a finite zero set. In the case of one or two zeros there are exact formulas for these zero-divisors [2; 3; 8]. The arguments yielding such formulas are different from the general approach developed in this paper.

5. Optimal Recovery in A^p

Let $1 \le p < \infty$ and let $a_0, a_1, \dots, a_n \in \Delta$, where $a_k \ne a_0$ for $k = 1, 2, \dots, n$. Consider the extremal problem

$$E(p,\Lambda) = \inf_{S} \sup_{\substack{f \in A^p \\ \|f\|_{A^p} \le 1}} |f(a_0) - S(f(a_1), \dots, f(a_n))|. \tag{24}$$

Here $\Lambda = \{a_k\}_{k=0}^n$ and S varies over all complex-valued functions of n complex variables. This problem is called the *optimal recovery problem,* and the function S is called an *algorithm*. An algorithm S_* is called optimal if

$$E(p,\Lambda) = \sup\{|f(a_0) - S_*(f(a_1), \dots, f(a_n)|: f \in A^p, \|f\|_{A^p} = 1\}.$$

A function f_* is called a *worst function* if

$$E(p,\Lambda) = |f_*(a_0) - S_*(f_*(a_1), \dots, f_*(a_n))|.$$

Let $B(z) = \prod_{k=1}^n [(z - a_k)/(1 - \bar{a}_k z)]$. The following result is a special case of a general theorem proved in [8] although not explicitly stated there.

Theorem A. *If $K_p(z, a_0; B)$ does not vanish in Δ, then*

$$f_*(z) = \frac{[K_p(z, a_0; B)]^{2/p} B(z)}{[K_p(a_0, a_0; B)]^{1/p}} \tag{25}$$

is a worst function for the problem (24) *for some optimal algorithm.*

To apply Theorem A, we must prove that $K_p(z, a_0; B)$ does not vanish in Δ. This was shown in Proposition 3 in the case $a_0 = 0$. The following proposition reduces the general problem to this case.

Proposition 4. *Suppose that $|b| < 1$, $\varphi(z) = (b - z)/(1 - \bar{b}z)$ for $|z| < 1$, and the function f is analytic in $\bar{\Delta}$. Then*

$$K_p(z, w; f \circ \varphi) = \left[\frac{1 - |b|^2}{(1 - \bar{b}z)(1 - b\bar{w})}\right]^2 K_p(\varphi(z), \varphi(w); f). \tag{26}$$

Proof. Note that φ is an automorphism of Δ and $\varphi^{-1} = \varphi$. Let $g = f \circ \varphi$. Suppose that $h \in H^\infty$ and let $k = h \circ \varphi$. Then

$$\int_\Delta \overline{K_p(z, w; f)} k(z) |f(z)|^p \, dA(z) = k(w) \tag{27}$$

for $|w| < 1$. Under the change of variables $z = \varphi(\zeta)$, the left-hand side of (27) becomes

$$\int_\Delta \overline{K_p(\varphi(\zeta), w; f)} l(\zeta) \overline{\varphi'(\zeta)} |g(\zeta)|^p \, dA(\zeta),$$

where $l = h \cdot \varphi'$. Hence (27) yields

$$\int_\Delta \overline{\varphi'(\zeta) K_p(\varphi(\zeta), \varphi(\sigma); f) \overline{\varphi'(\sigma)}} l(\zeta) |g(\zeta)|^p \, dA(\zeta) = l(\sigma), \tag{28}$$

where $\sigma = \varphi(w)$. Since φ' is analytic in $\bar{\Delta}$ and does not vanish there, l is an arbitrary function in H^∞. Thus (28) holds for all $l \in H^\infty$ and for all σ ($|\sigma| < 1$). Therefore

$$K_p(\zeta, \sigma; g) = \varphi'(\zeta) K_p(\varphi(\zeta), \varphi(\sigma); f) \overline{\varphi'(\sigma)}$$

for $|\zeta| < 1$ and $|\sigma| < 1$. This is equivalent to (26). □

The following statement is a consequence of Proposition 4 and the proof of Proposition 3.

Corollary 2. *Let $1 \le p < \infty$. Then $K_p(z, w; B) \ne 0$ for $|z| < 1$ and $|w| < 1$ and for all finite Blaschke products B.*

Thus we can apply Theorem A, and the combination of Theorem 1 and Theorem A yields the following result.

Theorem 4. *Let $\Lambda = \{a_k\}_{k=0}^n \subset \Delta$, $1 \le p < \infty$, and*

$$B(z)=\prod_{k=1}^{n}\frac{z-a_k}{1-\bar{a}_k z}.$$

Then the intrinsic error of optimal recovery (24) *is given by*

$$E(p,\Lambda)=\left\{\frac{1}{(1-|a_0|^2)^2}+\frac{p}{2}\left[\sum_{k=1}^{n}\frac{1-|a_0|^2|a_n|^2}{(1-|a_0|^2)|1-a_0\bar{a}_k|^2}\right.\right.$$
$$\left.\left.+\sum_{k,l=1}^{n}\frac{d_{kl}}{(1-a_0\bar{a}_k)(1-a_l\bar{a}_0)}\right]\right\}^{1/p}\cdot|B(a_0)|,$$

where d_{kl} *are defined by* (16) *and* $\alpha=p$.

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