Regularity of the Dirichlet Problem in Convex Domains in the Plane

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Then $\partial\Omega$ is Lipschitz but generally not C^1. Consider the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

The regularity of u is limited by the regularity of $\partial\Omega$. In this paper, it is shown that for sufficiently smooth f, u has almost three derivatives in $L^1(\Omega)$. Specifically, we will establish the following theorem concerning the operator G defined by $Gf = u$.

THEOREM 1.1. *If $\Omega \subset \mathbb{R}^2$ is bounded and convex, $0 < \epsilon < 1$, $1 < p < 2/(2-\epsilon)$, and $f \in L^p_{1-\epsilon}(\Omega)$, then for some C,*

$$Gf \in L^p_{3-\epsilon}(\Omega) \quad \textit{and} \quad \|Gf\|_{L^p_{3-\epsilon}(\Omega)} \le C\|f\|_{L^p_{1-\epsilon}(\Omega)}. \tag{2}$$

The following theorem, stated in [7], summarizes some previous regularity results for problem (1) in bounded convex domains in $\mathbb{R}^n$, $n \ge 2$.

THEOREM 1.2. *If $\Omega \subset \mathbb{R}^n$ is bounded and convex and $n \ge 2$, and either $1 \le \epsilon \le 2$ and $1 < p < 2/(2-\epsilon)$ or $\epsilon = 1$ and $p = 2$, then (2) holds. The Dirichlet boundary condition is satisfied in the sense that $Gf \in W^{1,p}_0(\Omega)$.*

The case $\epsilon = 1$ and $p = 2$ follows from integration by parts and is due to Kadlec [12]; an accessible exposition may be found in the third chapter of [9] or [10]. The case $\epsilon = 1$ and $1 < p < 2$ is due to Dahlberg, Verchota, and Wolff (see [7]) and independently to Adolfsson [1]. The proof of the result for $\epsilon = 2$ may be found in [7].

In particular, even if $f \in C^\infty(\bar{\Omega})$, the best one can say about ∇u and $\nabla^2 u$ is that $\nabla u \in L^\infty(\Omega)$ and $\nabla^2 u \in L^2(\Omega)$. In addition, by considering a family of truncated convex sectors in $\mathbb{R}^2$ with opening angles increasing to π, it

Received April 19, 1993. Revision received April 29, 1994.
The author was supported in part by NSF grant DMS 91-06507. The work described here formed part of the author's doctoral dissertation, [6].
Michigan Math. J. 41 (1994).

follows that given any fixed $0<\epsilon<2$ and $p>2/(2-\epsilon)$, estimate (2) cannot hold simultaneously for all convex Ω.

Theorem 1.2 and the counterexample led the author to conjecture Theorem 1.1. This was also motivated by the following heuristic principle: It is often the case that for sufficiently smooth f, $u|_{\partial\Omega}$ should be as smooth as $\partial\Omega$ but no smoother. Since Ω is convex, the curvature of $\partial\Omega$ is a measure, and thus the submanifold $\partial\Omega$ is almost twice differentiable in L^1. Therefore, $u|_{\partial\Omega}$ is likely to have almost two gradients in $L^1(\partial\Omega)$. Usual extension theorems then show that u should have almost three derivatives in $L^1(\Omega)$.

The hypothesis of convexity is necessary in that both theorems fail completely if Ω is only Lipschitz. Furthermore, in terms of the integrability of the third gradient of the solution u, a convex domain Ω that is additionally C^2 is no better behaved than a generic convex domain (see Proposition 2.2).

In the next section, the required estimate for the kernel is derived. The proof relies on conformal mapping and so is applicable only to Ω in $\mathbb{R}^2$, so that Theorem 1.1 (unlike Theorem 1.2) is restricted here to the case $\Omega\subset\mathbb{R}^2$. This technical restriction to the case $n=2$ was overcome in [8] using new methods. In Section 3, the atomic decomposition of function spaces is used to pass from the estimate on the kernel to (2); this section is written for Ω in $\mathbb{R}^2$, but the proofs generalize to higher-dimensional domains with no additional work.

Hereafter, $G(x,y)$ will denote the kernel of G; $N(x-y)=(1/2\pi)\log|x-y|$ is the fundamental solution for Δ in $\mathbb{R}^2$, and $U(x,y)=N(x-y)-G(x,y)$. By $\operatorname{dist}(\cdot,\cdot)$ we denote the (Euclidean) distance between points and/or sets, with $\mathrm{d}(\cdot)$ denoting the distance between a point or set and $\mathbb{R}^2\backslash\Omega$. By $|\cdot|$ we denote 2-dimensional Lebesgue measure as well as the Euclidean metric on $\mathbb{R}^2$ and $\mathbb{C}$. $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Banach space of functions with one derivative in $L^p(\Omega)$. $L_\alpha^p(\mathbb{R}^2)$ and $B_\alpha^{p,q}(\mathbb{R}^2)$ denote (respectively) the Bessel potential and Besov spaces on $\mathbb{R}^2$; $L_\alpha^p(\Omega)$ and $B_\alpha^{p,q}(\Omega)$ denote the restrictions of such spaces to Ω. (For a study of these spaces, see [15].) C will denote various constants that depend on at most ϵ, p, and Ω, unless otherwise stated.

2. The Estimate on the Green Function

Central to Theorem 1.1 is the corresponding estimate on the kernel $G(x,y)$, stated here for $U(x,y)$. That the weighted Sobolev-type bound below is really an $L_{3-\epsilon}^p$-type bound is seen from the proof of Lemma 3.2.

PROPOSITION 2.1. *Let $\Omega\subset\mathbb{R}^2$ be convex and bounded. Let $0<\epsilon<1$ and $1<p<2/(2-\epsilon)$. Then*

$$\left(\int_\Omega |\mathrm{d}(x)^\epsilon \nabla_x^3 U(x,y)|^p\,dx\right)^{1/p} \le C\mathrm{d}(y)^{\epsilon-2/p'-1}, \tag{3}$$

where C is independent of Ω.

Proof. First we show that the estimate

$$\left(\int_{\Omega\setminus B_r(y)} |\mathrm{d}(x)^\epsilon \nabla_x^3 G(x,y)|^p\,dx\right)^{1/p} \le C\mathrm{d}(y)^{\epsilon-2/p'-1} \quad \text{where } r=\tfrac{1}{2}\mathrm{d}(y) \tag{4}$$

implies estimate (3). Since Ω is convex, $|\nabla_x G(x,y)| \le C\mathrm{d}(y)|x-y|^{-2}$ (see [7]). Moreover, $|\nabla_x N(x-y)| = C|x-y|^{-1}$. Hence

$$|\nabla_x U(x,y)| \le |\nabla_x N| + |\nabla_x G(x,y)| \le C\mathrm{d}(y)^{-1}$$

for $x \in \partial B_{(3/4)\mathrm{d}(y)}(y)$. By interior estimates for harmonic functions, it follows that $|\nabla_x^3 U(x,y)| \le C\mathrm{d}(y)^{-3}$ for $x \in B_r(y)$. So

$$\left(\int_{B_r(y)} |\mathrm{d}(x)^\epsilon \nabla_x^3 U(x,y)|^p\,dx\right)^{1/p} \le C\mathrm{d}(y)^{\epsilon-2/p'-1}. \tag{5}$$

Since $|\nabla_x^3 N(x-y)| = C|x-y|^{-3}$ and $\mathrm{d}(x) \le 3|x-y|$ (when $x \in \Omega\setminus B_r(y)$),

$$\left(\int_{\Omega\setminus B_r(y)} |\mathrm{d}(x)^\epsilon \nabla_x^3 N(x-y)|^p\,dx\right)^{1/p} \le C\mathrm{d}(y)^{\epsilon-2/p'-1}. \tag{6}$$

By (5) and (6) and the definition $U = N-G$, (3) follows from (4).

It remains to show (4). For the remainder of the proof, view subsets of $\mathbb{R}^2$ simultaneously as subsets of $\mathbb{C}$. For $x=(x_1,x_2)$ and $y=(y_1,y_2)$ in $\mathbb{R}^2$ as above, define the complex variables $w = x_1+ix_2$ and $w_0 = y_1+iy_2$. Let $B = B_1(0) \subset \mathbb{C}$ and $\Psi:\Omega\to B$ be a conformal bijection with $\Psi(w_0)=0\in B$. Write $\Psi(w) = z \in B$ and define $\Phi = \Psi^{-1}: B \to \Omega$. It is well known that

$$G(x,y) = \frac{1}{2\pi}\log|\Psi(w)|.$$

By the Cauchy-Riemann equations, $|\nabla_x^3 G(x,y)| = (1/\pi)|(d/dw)^3 \log\Psi(w)|$. Using the chain rule and writing $(d/dw)^3\log\Psi$ in terms of z and $\Phi'(z)$,

$$\frac{d}{dw}\log\Psi(w) = \frac{1}{\Psi(w)}\Psi'(w) = \frac{1}{z\Phi'(z)}$$

and

$$\begin{aligned}
\frac{d^3}{dw^3}\log\Psi(w) &= \frac{d^2}{dw^2}\left(\frac{1}{z}\frac{1}{\Phi'(z)}\right)\\
&= \left(\Phi'^{-1}\frac{d}{dz}\right)^2\left(\frac{1}{z}\frac{1}{\Phi'}\right)\\
&= z^{-3}\Phi^{-3}\left\{2\left(1+z\frac{\Phi''}{\Phi'}\right)^2 - z\left(1+z\frac{\Phi''}{\Phi'}\right)'\right\}.
\end{aligned}$$

Set $F(z) = 1+z\Phi''(z)/\Phi'(z)$. Then

$$|\nabla_x^3 G(x,y)| = C|z\Phi'|^{-3}|2F^2 - zF'|. \tag{7}$$

By [2], since Φ maps B univalently and conformally onto the convex region Ω, $\operatorname{Re}F(z) \ge 0$. Moreover, every nonnegative harmonic function is the harmonic extension of a finite nonnegative measure on ∂B. Hence

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z}\, d\mu(\phi) + ic.$$

Since $F(0) = 1$, c vanishes and the positive measure μ has total mass 2π.

Set $z = re^{i\theta}$ and $(P+iQ)(r,\theta) = (1/2\pi)(1+re^{i\theta})/(1-re^{i\theta})$ for $0 < r < 1$ and $0 \le \theta < 2\pi$. Then

$$F(re^{i\theta}) = \int_0^{2\pi} (P+iQ)(r, \theta-\phi)\, d\mu(\phi).$$

We now assert that $|z| = |\Psi(w)|$ is bounded below when x is far from y or (equivalently) when w is far from w_0. Because Ω is convex, $|G(x,y)| \le C\mathrm{d}(y)|x-y|^{-1}$ (see [7]). Thus, if $|x-y| \ge (1/100)\mathrm{d}(y)$ then $|G(x,y)| \le C$. Hence

$$|z| = |\Psi(w)| = e^{2\pi G(x,y)} \ge e^{-2\pi C}. \tag{8}$$

Next, the quantity $\mathrm{d}(x)$ in (4) will be estimated in terms of $\mathrm{dist}(z, \partial B)$, assuming that $|x-y| \ge (1/100)\mathrm{d}(y)$. As in the preceding paragraph, $|G(x,y)| \le C_1$. Therefore, $\mathrm{dist}(z,\partial B) = 1-|z| = 1-e^{-2\pi|G(x,y)|} \ge C|G|$, where $C > 0$ depends only on C_1. When $|x-y| \ge (1/100)\mathrm{d}(y)$, the interior gradient estimate for harmonic functions shows that $|\nabla_x G(x,y)| \le C\mathrm{d}(x)^{-1}|G(x,y)|$. Moreover,

$$|\nabla_x G(x,y)| = \frac{1}{2\pi}\left|\frac{d}{dw}\log\Psi(w)\right| = \frac{1}{2\pi}|z\Phi'(z)|^{-1}.$$

Therefore, $\mathrm{dist}(z,\partial B) \ge C\mathrm{d}(x)|z\Phi'(z)|^{-1}$, and hence

$$\mathrm{d}(x) \le C|z\Phi'(z)|\,\mathrm{dist}(z,\partial B) \quad \text{when } |x-y| \ge \frac{1}{100}\mathrm{d}(y). \tag{9}$$

Consider the left-hand side of (4). When x lies in the region of integration $\Omega\setminus B_{(1/2)\mathrm{d}(y)}(y)$, $|z| = |\Psi(w)|$ is bounded below by a positive constant by (8), and $\mathrm{d}(x) \le C|z\Phi'|\mathrm{dist}(z,\partial B)$ by (9). Since the area elements $dA(w)$ and $dA(z)$ are related by $dA(w) = |\Phi'(z)|^2 dA(z)$, changing variables and using (7) yields

$$\left(\int_{\Omega\setminus B_r(y)} |\mathrm{d}(x)^\epsilon \nabla_x^3 G(x,y)|^p\, dx\right)^{1/p}$$
$$\le C\left(\int_{\Psi(\Omega\setminus B_r(y))} \{[\mathrm{dist}(z,\partial B)|z\Phi'|]^\epsilon |z\Phi'|^{-3}|2F^2 - zF'|\}^p |\Phi'|^2\, dA(z)\right)^{1/p}$$
$$\le C\left(\int_{\Psi(\Omega\setminus B_r(y))} |\Phi'|^{(\epsilon-3)p+2}\{\mathrm{dist}(z,\partial B)^\epsilon[|F|^2 + |F'|]\}^p\, dA(z)\right)^{1/p}.$$

By the convexity of Ω, $|\nabla_x G(x,y)| \le C|x-y|^{-1}$ (see [7]). Since $|x-y| \ge \frac{1}{2}\mathrm{d}(y)$ in the region of integration above, $|\Phi'|^{-1} \le |z\Phi'|^{-1} = 2\pi|\nabla_x G(x,y)| \le C\mathrm{d}(y)^{-1}$. Since $(\epsilon-3)+2/p = \epsilon - 2/p' - 1 < 0$, $|\Phi'|^{\epsilon-3+2/p} \le C\mathrm{d}(y)^{\epsilon-2/p'-1}$. Therefore,

$$\left(\int_{\Omega\setminus B_r(y)} |d(x)^\epsilon \nabla_x^3 G(x,y)|^p\,dx\right)^{1/p}$$
$$\le C d(y)^{\epsilon-2/p'-1}\left(\int_B \{\operatorname{dist}(z,\partial B)^\epsilon[|F|^2+|F'|]\}^p\,dA(z)\right)^{1/p}.$$

The proof of the proposition will be complete once it is shown that there is a $C=C(\epsilon,p)$ independent of Ω such that

$$\left(\int_B \{\operatorname{dist}(z,\partial B)^\epsilon[|F^2|+|F'|]\}^p\,dA(z)\right)^{1/p}\le C.$$

By Minkowski's integral inequality,

$$\left(\int_B [\operatorname{dist}(z,\partial B)^{\epsilon/2}|F|]^{2p}\,dA(z)\right)^{1/2p}$$
$$=\left[\int_B\left|\int_0^{2\pi}\operatorname{dist}(z,\partial B)^{\epsilon/2}(P+iQ)(r,\theta-\phi)\,d\mu(\phi)\right|^{2p}dA(z)\right]^{1/2p}$$
$$\le\int_0^{2\pi}\left[\int_B|\operatorname{dist}(z,\partial B)^{\epsilon/2}(P+iQ)(r,\theta-\phi)|^{2p}\,dA(z)\right]^{1/2p}d\mu(\phi).$$

Since $z=re^{i\theta}$ in the preceding line,

$$|(P+iQ)(r,\theta-\phi)|=\frac{1}{2\pi}\left|\frac{1+re^{i(\theta-\phi)}}{1-re^{i(\theta-\phi)}}\right|=\frac{1}{2\pi}\left|\frac{e^{i\phi}+z}{e^{i\phi}-z}\right|\le C\frac{1}{|z-e^{i\phi}|}.$$

Moreover, since $e^{i\phi}\in\partial B$, $\operatorname{dist}(z,\partial B)\le|z-e^{i\phi}|$. Hence

$$\operatorname{dist}(z,\partial B)^{\epsilon/2}|(P+iQ)(r,\theta-\phi)|\le C|z-e^{i\phi}|^{\epsilon/2-1}.$$

We now conclude that

$$\left(\int_B [\operatorname{dist}(z,\partial B)^{\epsilon/2}|F|]^{2p}\,dA(z)\right)^{1/2p}$$
$$\le C\int_0^{2\pi}\left[\int_B|z-e^{i\phi}|^{(\epsilon/2-1)2p}\,dA(z)\right]^{1/2p}d\mu(\phi)$$
$$\le C\int_0^{2\pi}d\mu(\phi)\le C\operatorname{mass}\mu=2\pi C,$$

since the inner integral over $z\in B$ is uniformly bounded in $e^{i\phi}\in\partial B$ for $(\epsilon/2-1)2p>-2$, which holds since $\epsilon/2>1/p'$.

For the term involving F', differentiating under the integral sign gives

$$F'(z)=\frac{d}{dz}\frac{1}{2\pi}\int_{\partial B}\left(\frac{1+ze^{-i\phi}}{1-ze^{-i\phi}}\right)d\mu(\phi)=\frac{1}{\pi}\int_{\partial B}\frac{e^{i\phi}}{(z-e^{i\phi})^2}\,d\mu(\phi),$$

so

$$\left(\int_B |\mathrm{dist}(z,\partial B)^\epsilon F'(z)|^p\,dA(z)\right)^{1/p}$$
$$=\left(\int_B \left|\frac{1}{\pi}\int_0^{2\pi} \mathrm{dist}(z,\partial B)^\epsilon \frac{e^{i\phi}}{(z-e^{i\phi})^2}\,d\mu(\phi)\right|^p dA(z)\right)^{1/p}.$$

Next, one applies the Minkowski inequality and then follows the treatment of the $|F^2|$ term, finishing the proof of Proposition 2.1. □

REMARK. When $n\geq 3$, a homogeneity argument leads to the conjecture that (3) holds with the exponent on the right-hand side changed to $\epsilon - n/p' - 1$. The proof of Proposition 2.1 relies on conformal mapping and cannot be adapted to the case $n\geq 3$. Consideration of the object $\log|\nabla_x G(x,y)|$, superharmonic when Ω is convex, leads to a proof of (3) when $n\geq 3$ and $1<p<3/(3-\epsilon)$ (see [8]). This range of p is sufficient for proving Theorem 1.1.

The methods of this section show that the conclusions of Theorem 1.1 fail when $\epsilon=0$ for any $p\geq 1$, even if some additional smoothness of Ω is assumed.

PROPOSITION 2.2. *There exists a bounded, strictly convex and C^2 domain $\Omega\subset\mathbb{R}^2$ and an $f\in C_0^\infty(\Omega)$ such that $\nabla^3 Gf\notin L^1(\Omega)$.*

Proof. We first show that there exists a bounded, strictly convex and C^2 domain $\Omega\subset\mathbb{R}^2$ such that

$$\int_{\Omega\setminus B_\delta(y)} |\nabla_x^3 G(x,y)|\,dx=\infty \tag{10}$$

for some $y\in\Omega$ and $\delta>0$, where $B_\delta(y)\subset\bar{B}_{2\delta}(y)\subset\Omega$.

Rudin [13] shows that in the unit disk B, there is a harmonic function $u\in C(\bar{B})$ satisfying $\int_B|\nabla u(z)|\,dA(z)=\infty$, where $\nabla u=(\partial/\partial z_1 u,\partial/\partial z_2 u)$ for $z=z_1+iz_2$.

By adding a constant if necessary, it may be assumed that $u\geq c>0$ in B. Multiplying by a positive constant, we may take $u(0)=1$. Let $\tilde{u}$ be the harmonic conjugate of u. Since $\tilde{u}$ is arbitrary up to an additive constant, assume $\tilde{u}(0)=0$. Then $F(z)=(u+i\tilde{u})(z)$ is analytic in B and equals 1 at the origin. Define an analytic function h in B by $1+zh(z)=(u+i\tilde{u})(z)$. Define $H(z)=\int_0^z h(\zeta)\,d\zeta$, that is, $H'(z)=h(z)$ and $H(0)=0$. Define Φ by $\Phi(z)=\int_0^z e^{H(\zeta)}\,d\zeta$.

Note that $\Phi'\neq 0$ in B, so Φ is conformal. Also,

$$\mathrm{Re}\left\{1+\frac{z\Phi''(z)}{\Phi'(z)}\right\}=\mathrm{Re}\,F(z)=u(z)>0.$$

By [2], an analytic function Φ in B is *convex univalent* – that is, has convex image and is injective and conformal – if and only if $\mathrm{Re}\,z\Phi''(z)/\Phi'(z)\geq -1$. Thus $\Phi: B\to\Omega$ bijectively and conformally, where Ω is convex.

Since $u|_{\partial B} \in L^2(\partial B)$, $\tilde{u}|_{\partial B} \in L^2(\partial B)$; hence $h(z)|_{\partial B} \in L^2(\partial B)$. From this it follows that H extends to a function continuous on $\bar{B}$. Because $\log|\Phi'(z)| = \operatorname{Re} H(z)$, $|\Phi'(z)|$ and $|\Phi'(z)|^{-1}$ are continuous and bounded. The boundedness of $|\Phi'|$ shows that Ω is in fact bounded.

As before, define $\Psi = \Phi^{-1}: \Omega \to B$, let G be the Green function of Ω, and let $w = \Phi(z)$ with $x = (\operatorname{Re} w, \operatorname{Im} w) \in \mathbb{R}^2$. As before, $G(x,0) = (2\pi)^{-1} \log|\Psi(x)|$ since $\Psi(0) = 0$, and

$$|\nabla_x^3 G(x,0)| = \frac{1}{\pi}\left|\frac{d^3}{dw^3}\log \Psi(w)\right| = \frac{1}{\pi}|z^{-3}\Phi'^{-3}(2F^2 - zF')|.$$

Now, in (10), let $y = 0$ and pick $\delta > 0$ small enough so that $\bar{B}_{2\delta}(0) \subset \Omega$ and $\Phi(B_1(0)\setminus\bar{B}_{1/10}(0)) \subset \Omega\setminus B_\delta(0)$. (Recall that $\Phi(0) = 0$.) Then

$$\int_{\Omega\setminus B_\delta(0)} |\nabla_x^3 G(x,0)|\,dx \geq \frac{1}{\pi}\int_{1/10<|z|<1} |z^{-3}\Phi'(z)^{-3}||2F^2 - zF'||\Phi'(z)|^2\,dA(z)$$

$$\geq C\int_{1/10<|z|<1} |2F^2 - zF'|\,dA(z); \tag{11}$$

the last inequality follows since $|\Phi'| \geq C > 0$.

By the Cauchy-Riemann equations, $|F'| = |\nabla u|$. Therefore

$$\int_B |F'|\,dA(z) = \int_B |\nabla u|\,dA(z) = \infty$$

and hence $\int_{1/10<|z|<1} |z||F'|\,dA(z) = \infty$. From $(u + i\tilde{u})|_{\partial B} \in L^2(\partial B)$ it follows that $\int_{|z|<1} |F|^2\,dA(z) < \infty$. Thus, (10) follows from (11).

Next we show that Ω is strictly convex and C^2. Let s be arclength on $\partial\Omega$ and let τ represent the angle of inclination of the tangent vector to $\partial\Omega$ at s. The curvature is $\kappa = d\tau/ds$. If θ is arclength on $\partial B = \Psi(\partial\Omega)$ (i.e., $\theta = \arg z$ when $|z| = 1$), then

$$\kappa = \frac{d\tau}{d\theta}\frac{d\theta}{ds}.$$

The tangent vector at the point $z \in \partial B$ corresponding to $s \in \partial\Omega$ has inclination $\pi/2 + \theta$; and since Φ is angle-preserving, $\tau = (\pi/2 + \theta) + \arg \Phi'(z)$. Thus

$$\frac{d\tau}{d\theta} = 1 + \frac{d}{d\theta}\operatorname{Im}\log\Phi'(e^{i\theta}) = 1 + \operatorname{Im}\frac{\Phi''(e^{i\theta})}{\Phi'(e^{i\theta})} ie^{i\theta}$$

$$= \operatorname{Re}\left\{1 + z\frac{\Phi''(z)}{\Phi'(z)}\right\} = u(z).$$

Moreover, $d\theta/ds = |\Psi'|$, since lengths are multiplied by $|\Psi'|$ in going from Ω to B. Therefore, $\kappa = |\Psi'|(u\circ\Psi)$. It was shown above that $|\Psi'|$ is continuous. Hence $\Psi|_{\partial\Omega} \in C(\partial\Omega)$. Since $u \in C(\bar{B})$, it follows that $\kappa = |\Psi'|(u\circ\Psi)$ is continuous.

Since a Lipschitz domain in the plane whose boundary has continuous curvature is a C^2 domain, Ω is C^2. Furthermore, since both u and $|\Psi'|$ are bounded away from zero, κ is bounded below by a positive constant and hence Ω is also strictly convex.

Finally, choose $\eta \in C^\infty(\bar{\Omega})$ satisfying $\eta = 1$ in $\Omega \backslash B_\delta(y)$ and $\eta = 0$ in $\bar{B}_{\delta/2}(y)$. Let $f(x) = \Delta_x(\eta(x)G(x,y))$. Clearly

$$f(x) = G(x,y)\Delta\eta(x) + 2\nabla\eta(x)\cdot\nabla_x G(x,y)$$

is in $C_0^\infty(\Omega)$ since both $\nabla\eta$ and $\Delta\eta$ are in $C_0^\infty(\Omega)$ and vanish near $x = y$. Finally, $Gf(x) = \eta(x)G(x,y)$, since $\eta(\cdot)G(\cdot,y)$ vanishes on the boundary. Because $Gf = \eta(\cdot)G(\cdot,y)$, $\nabla^3(Gf) \notin L^1(\Omega)$ follows from (10). □

3. The Atomic Decomposition

Throughout this section, $0 < \epsilon < 1$, $1 < p < 2/(2-\epsilon)$, and Ω will denote a fixed bounded and convex domain in $\mathbb{R}^2$.

The rest of this paper is devoted to using estimate (3) on the kernel $U(x,y)$ to derive estimate (2) on the operator G. An important technical tool will be the atomic decomposition of function spaces.

Before proceeding to the proof of Theorem 1.1, we present some facts about the atomic decompositions of the function spaces $B_\alpha^{p,q}$ and L_α^p for $\alpha > 0$ and $1 < p < \infty$. Fix an arbitrarily large integer M. Using the normalization given in [4], we say that $a(x)$ is an *(α,p)-atom* if, for some closed cube $Q \subset \mathbb{R}^2$, $\operatorname{supp} a \subseteq 3Q$ and

$$\left|\frac{\partial^{\gamma_1+\gamma_2}}{\partial x_1^{\gamma_1}\partial x_2^{\gamma_2}} a(x)\right| \le |Q|^{\alpha/2-1/p-(\gamma_1+\gamma_2)/2} \quad \text{if } \gamma_1+\gamma_2 \le M.$$

Here, $3Q$ denotes the 3-fold dilation of Q about its center.

Denote an atom a associated with a cube Q by a_Q. Only *dyadic* cubes, those whose vertices are of the form $2^{-\nu}m$ with ν and m in $\mathbb{Z}$, are needed for the results stated below. Moreover, since only the *inhomogeneous* spaces $B_\alpha^{p,q}(\mathbb{R}^2)$ and $L_\alpha^p(\mathbb{R}^2)$ appear here, we may restrict consideration to those Q with $|Q| \le 1$, that is, $\nu \ge 0$. Given such a cube, let $\ell(Q)$ be the length of a side of Q. If a_Q is a $(1-\epsilon,p)$-atom, then

$$\|a_Q\|_{L^\infty(\mathbb{R}^2)} \le |Q|^{(1-\epsilon)/2-1/p} = \ell(Q)^{-1-\epsilon+2/p'}. \tag{12}$$

Let $1 < p < \infty$ and $1 \le q \le \infty$, and let $\{s_Q\}$ denote a sequence of real numbers indexed by the set of dyadic cubes in $\mathbb{R}^2$. We define a Banach space of sequences, $b^{p,q}$, by defining a norm

$$\|\{s_Q\}\|_{b^{p,q}} = \left[\sum_{\nu=0}^{\infty}\left(\sum_{\ell(Q)=2^{-\nu}} |s_Q|^p\right)^{q/p}\right]^{1/q}.$$

Then, given $1 < p < \infty$, $\alpha > 0$, and $f \in B_\alpha^{p,q}(\mathbb{R}^2)$, there are (α,p)-atoms a_Q and a sequence $\{s_Q\} \in b^{p,q}$ for which

$$f = \sum_{\nu=0}^{\infty} \sum_{\ell(Q)=2^{-\nu}} s_Q a_Q \quad \text{with } \|\{s_Q\}\|_{b^{p,q}} \leq C\|f\|_{B_\alpha^{p,q}(\mathbb{R}^2)}.$$

Such a double sum will often be abbreviated as $\sum_Q s_Q a_Q$.

The previous inequality has a converse that is usually stated for *molecules,* which are more general than atoms: Given (α, p)-atoms $\{a_Q\}$,

$$\left\|\sum_Q s_Q a_Q\right\|_{B_\alpha^{p,q}(\mathbb{R}^2)} \leq C\|\{s_Q\}\|_{b^{p,q}}. \tag{13}$$

For $L_\alpha^p(\mathbb{R}^2)$, where $1 < p < \infty$ and $\alpha > 0$, there is a similar decomposition and converse, in which $b^{p,q}$ is replaced by $l^{p,2}$. This is the sequence space with norm

$$\|\{s_Q\}\|_{l^{p,2}} = \left\|\left[\sum_Q (|s_Q|\tilde{1}_Q(x))^2\right]^{1/2}\right\|_{L^p(dx)}, \tag{14}$$

where $\tilde{1}_Q$ is the indicator function 1_Q normalized to have L^p-norm 1. (See [5]; note that the normalization used here is that of [4].)

We will need the following facts:

$$\begin{aligned} B_\alpha^{p,1}(\Omega) &\hookrightarrow B_\alpha^{p,p}(\Omega) \text{ is a continuous inclusion;} \\ B_\alpha^{p,p}(\Omega) &\hookrightarrow L_\alpha^p(\Omega) \text{ is a continuous inclusion if } p \leq 2. \end{aligned} \tag{15}$$

Vanishing moment conditions on atoms, which are optional for $\alpha > 0$ and $p > 1$, have been omitted since they are not needed here.

Finally, we make two remarks on notation. Given a countable set J such as some set of cubes, we will let l^p denote $L^p(J)$, with J assigned the counting measure. If a function g is not supported in Ω, which is possibly the case for $g = a_Q$, the notations Ug and Gg will denote $U(g|_\Omega)$ and $G(g|_\Omega)$.

We first use (3) to obtain an estimate on the third gradient of Uf, where f is a linear combination of atoms. For technical reasons, it will be assumed that all the atoms are at the same scale and are supported close to $\partial\Omega$. It may be assumed that $\operatorname{supp} a_Q \cap \Omega \neq \emptyset$ for any atom a_Q, for otherwise $Ua_Q = 0$.

LEMMA 3.1. *Fix an integer* $\nu \geq 0$. *Suppose* $\{a_Q\}$ *is a collection of* $(1-\epsilon, p)$-*atoms, where the cubes* Q *satisfy the restrictions* $\ell(Q) = 2^{-\nu}$ *and* $\mathrm{d}(3Q) \leq \ell(Q)$. *If* $\{s_Q\} \in \ell^p$, *then*

$$\|\mathrm{d}(x)^\epsilon \nabla_x^3 U(\sum s_Q a_Q)\|_{L^p(\Omega)} \leq C\|\{s_Q\}\|_{l^p}, \tag{16}$$

with C *not depending on the choice of* ν.

Proof. A simple rescaling argument shows that we need only consider the case $\nu = 0$ (i.e. $\ell(Q) = 1$) as long as the constant C in (16) is shown not to depend on the diameter of Ω. A careful inspection of the proof below shows that the resulting C can indeed be taken to be independent of Ω. It should be noted that the constants C in the pointwise estimates on $G(x, y)$ and $\nabla_x G(x, y)$ used below are independent of Ω.

With $\ell(Q)=1$, $|a_Q|\le 1$ (by (12)) and $|\operatorname{supp} a_Q|\le |3Q|\le 9$. In what follows, let $\{Q'\}$ denote the set of all dyadic cubes Q' with $\ell(Q')=1$ and $Q'\cap\Omega\ne\emptyset$. (Note that this is *not* the condition $3Q'\cap\Omega\ne\emptyset$.) Since $\Omega=\bigcup_{Q'}(Q'\cap\Omega)$,

$$\begin{aligned}\|\mathrm{d}(\cdot)^\epsilon\nabla^3U\textstyle\sum s_Qa_Q\|^p_{L^p(\Omega)}&=\sum_{Q'}\Big\|\sum_Q s_Q\mathrm{d}(\cdot)^\epsilon\nabla^3Ua_Q\Big\|^p_{L^p(Q'\cap\Omega)}\\&\le\sum_{Q'}\Big\{\sum_Q|s_Q|\,\|\mathrm{d}(\cdot)^\epsilon\nabla^3Ua_Q\|_{L^p(Q'\cap\Omega)}\Big\}^p\\&=\sum_{Q'}\Big\{\sum_Q|s_Q|k_{Q',Q}\Big\}^p,\end{aligned}\tag{17}$$

where $k_{Q',Q}=\|\mathrm{d}(\cdot)^\epsilon\nabla^3Ua_Q\|_{L^p(Q'\cap\Omega)}$.

Because sequences are simply functions on countable sets and since summation is integration with respect to a counting measure, $k_{Q',Q}$ is an integral kernel from functions on $\{Q\}$ to functions on $\{Q'\}$. The proof of Lemma 3.1 is complete once it is shown that k is continuous from $l^p(\{Q\})$ to $l^p(\{Q'\})$. Suppose that there is a $C<\infty$ such that

$$\sup_{Q'}\sum_Q k_{Q',Q}\le C\tag{18}$$

and

$$\sup_{Q}\sum_{Q'} k_{Q',Q}\le C.\tag{19}$$

Then, by the generalized Young's inequality (see [3]), the rightmost member of (17) is controlled by $\|s_Q\|^p_{\ell^p}$, completing the proof of Lemma 3.1. Therefore, it is enough to show (18) and (19).

Given Q' and Q, denote the distance between Q' and $3Q$ by r. Those pairs (Q',Q) for which $r\le 100$ are few; for such (Q',Q), the summand $k_{Q',Q}$ is bounded by $\sup_{Q',Q}k_{Q',Q}$. Appropriate control of $k_{Q',Q}$ for $r\ge 100$ is obtained by first passing from U to G by getting bounds for N and then breaking the case $r\ge 100$ for G into two subcases, $\mathrm{d}(Q')<1$ and $\mathrm{d}(Q')\ge 1$.

We will make use of the following fact, whose proof is elementary and will be omitted. Let $D\subset\mathbb{R}^2$ be bounded and convex. If $\mathrm{d}(\cdot)$ denotes $\operatorname{dist}(\cdot,\mathbb{R}^2\setminus D)$, then for $\eta>0$ we have

$$\int_D \mathrm{d}(y)^{\eta-1}\,dy\le C\cdot(\operatorname{diam}D)^{\eta+1},\tag{20}$$

where C does not depend on D.

For convenience, the remainder of the proof of Lemma 3.1 will be broken into several steps.

Step 1: We first show that, for some C,

$$\sup_{Q',Q}k_{Q',Q}\le C<\infty.\tag{21}$$

By using the Minkowski integral inequality, estimate (3) on the kernel U, and the fact that $\operatorname{supp} a_Q\subseteq 3Q$, we see that

$$\begin{aligned} k_{Q',Q} &\le \left\| \int_\Omega \mathrm{d}(x)^\epsilon \nabla_x^3 U(x,y)\, a_Q(y)\, dy \right\|_{L^p(\Omega;\, dx)} \\ &\le \int_\Omega \|\mathrm{d}(x)^\epsilon \nabla_x^3 U(x,y)\|_{L^p(\Omega;\, dx)} |a_Q(y)|\, dy \\ &\le C\|a_Q\|_{L^\infty(\mathbb{R}^2)} \int_{3Q\cap\Omega} \mathrm{d}(y)^{\epsilon-2/p'-1}\, dy. \end{aligned}$$

Inequality (21) now follows by using the bound (12) and applying (20) to $D = 3Q\cap\Omega$, while making the additional observation that

$$\mathrm{d}(y) = \operatorname{dist}(y, \mathbb{R}^2\backslash\Omega) \ge \operatorname{dist}(y, \mathbb{R}^2\backslash(3Q\cap\Omega)).$$

Step 2: Suppose $r \ge 100$. Then

$$\|\mathrm{d}(\cdot)^\epsilon \nabla^3 N(a_Q|_\Omega)\|_{L^p(Q'\cap\Omega)} \le Cr^{-3+\epsilon}. \tag{22}$$

If $x \in Q'$, $y \in \operatorname{supp} a_Q \subseteq 3Q$, and $r \ge 100$, then $|x-y| \approx r$. Since $|\nabla_x^3 N(x-y)| = C|x-y|^{-3}$, it follows that for $x \in Q'$,

$$|\nabla_x^3 N(a_Q|_\Omega)(x)| = \left| \int_\Omega \nabla_x^3 N(x-y)\, a_Q(y)\, dy \right| \le Cr^{-3}.$$

Since $\ell(Q') = \ell(Q) = 1$, $\operatorname{dist}(Q', 3Q) = r$, $\mathrm{d}(3Q) \le 1$, and $r \ge 100$, it follows that $\mathrm{d}(x) \le Cr$ for $x \in Q'$. Hence $\|\mathrm{d}(\cdot)^\epsilon \nabla^3 N(a_Q|_\Omega)\|_{L^p(Q'\cap\Omega)} \le Cr^\epsilon r^{-3}$.

Step 3: If $r \ge 100$ and $\mathrm{d}(Q') < 1$, then

$$\|\mathrm{d}(\cdot)^\epsilon \nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)} \le Cr^{-2}. \tag{23}$$

Pick $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $\phi = 1$ on $2Q'$ and $\phi = 0$ off $4Q'$, where $2Q'$ and $4Q'$ denote the dilates of Q' about its center by factors of 2 and 4, respectively. Then

$$\|\mathrm{d}(\cdot)^\epsilon \nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)} = \|\mathrm{d}(\cdot)^\epsilon \nabla^3 [\phi G a_Q]\|_{L^p(Q'\cap\Omega)}.$$

Let $w = \phi G a_Q$. Since $\phi \in C_0^\infty(\mathbb{R}^2)$, $w|_{\partial\Omega} = 0$ in the sense that $Ga_Q|_{\partial\Omega} = 0$. Therefore, $w = G\Delta w$, and

$$\Delta w = \Delta(\phi G a_Q) = \Delta\phi G a_Q + 2\nabla\phi\cdot\nabla G a_Q + \phi\Delta G a_Q.$$

Since $\operatorname{dist}(Q', 3Q) \ge 100$, $\phi\Delta G a_Q = \phi\cdot a_Q|_\Omega = 0$. Because Ω is convex,

$$|G(x,y)| \le C\mathrm{d}(x)\mathrm{d}(y)|x-y|^{-2}$$

(see [7]). Also, $\mathrm{d}(x) \le C$ when $x \in \operatorname{supp}\phi$, $\mathrm{d}(y) \le C$ when $y \in \operatorname{supp} a_Q$, and $|x-y| \approx r$ for $r \gg 1$. Thus

$$|Ga_Q(x)| \le \int_{3Q\cap\Omega} \mathrm{d}(x)\mathrm{d}(y)|x-y|^{-2}|a_Q(y)|\, dy \le Cr^{-2}$$

for $x \in \operatorname{supp}\phi$. Similarly, because $|\nabla_x G(x,y)| \le C\mathrm{d}(y)|x-y|^{-2}$ (see [7]),

$$|\nabla_x G a_Q(x)| = \left| \int_\Omega \nabla_x G(x,y)\, a_Q(y)\, dy \right| \le Cr^{-2}$$

for $x \in \operatorname{supp} \phi$. Hence $|\Delta w| \le Cr^{-2}$. Now,

$$\begin{aligned}\|d(\cdot)^\epsilon \nabla^3 w\|_{L^p(Q'\cap\Omega)} &= \|d(\cdot)^\epsilon \nabla^3 G\Delta w\|_{L^p(Q'\cap\Omega)} \\ &\le \|d(\cdot)^\epsilon \nabla^3 N\Delta w\|_{L^p(Q'\cap\Omega)} + \|d(\cdot)^\epsilon \nabla^3 U\Delta w\|_{L^p(Q'\cap\Omega)}. \end{aligned} \tag{24}$$

Since $\nabla\phi = 0$ on $2Q'$, $\Delta w = 0$ there. Thus $|x-y| \ge C$ if $x \in Q'$ and $y \in \operatorname{supp} \Delta w$. For such x,

$$\begin{aligned}|\nabla_x^3 N\Delta w(x)| &\le \int |\nabla_x^3 N(x-y)||\Delta w(y)|\,dy \\ &\le C\|\Delta w\|_{L^\infty(\Omega)} |\operatorname{supp} \Delta w| \le Cr^{-2}.\end{aligned}$$

Since $x \in Q'$ implies $d(x) \le C$,

$$\|d(\cdot)^\epsilon \nabla^3 N\Delta w\|_{L^p(Q'\cap\Omega)} \le Cr^{-2}. \tag{25}$$

To bound the other term in the third member of (24), note that the proof of step 1 shows that

$$\|d(\cdot)^\epsilon \nabla^3 U\Delta w\|_{L^p(Q'\cap\Omega)} \le C\|\Delta w\|_{L^\infty(\Omega)} \le Cr^{-2}. \tag{26}$$

By (24), (25), and (26), $\|d(\cdot)^\epsilon \nabla^3(\phi G a_Q)\|_{L^p(Q'\cap\Omega)} \le r^{-2}$.

Step 4: $r \ge 100$ and $d(Q') \ge 1$ imply that

$$\|d(\cdot)^\epsilon \nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)} \le Cr^{-2} d(Q')^{\epsilon-2}. \tag{27}$$

Let $c_{Q'}$ denote the center of Q', and let Q'' be a dilate of Q' about $c_{Q'}$ with diameter equal to $d(c_{Q'})$. We have $d(c_{Q'}) \ge 1/2 + d(Q') \ge 3/2$, so

$$\frac{\operatorname{diam} Q'}{\operatorname{diam} Q''} = \frac{2^{1/2}}{d(c_{Q'})} \le \frac{2^{3/2}}{3} < 1. \tag{28}$$

Furthermore, it is easily seen that $\operatorname{dist}(Q'', 3Q) \ge Cr$. Since $d(3Q) \le 1$, $x \in Q''$ implies

$$|\nabla_x G a_Q(x)| = \left| \int_\Omega \nabla_x G(x,y) a_Q(y)\,dy \right| \le C \int_{3Q} d(y)|x-y|^{-2}\,dy \le Cr^{-2}.$$

By the interior gradient estimate for Δ, (28) implies that for $x \in Q'$,

$$|\nabla_x^3 G a_Q(x)| \le C(\operatorname{diam} Q'')^{-2} \sup_{x' \in Q''} |\nabla_{x'} G a_Q(x')| \le C(\operatorname{diam} Q'')^{-2} r^{-2}.$$

But since $d(Q') \ge 1$, $d(c_{Q'}) \approx d(Q')$. Then $\operatorname{diam} Q'' = d(c_{Q'}) \approx d(Q')$, whence $|\nabla_x^3 G a_Q(x)| \le C d(Q')^{-2} r^{-2}$ for $x \in Q'$. Since $x \in Q'$ and $d(Q') \ge 1$ imply $d(x) \le C d(Q')$, it follows that $\|d(\cdot)^\epsilon \nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)} \le C d(Q')^\epsilon d(Q')^{-2} r^{-2}$.

Step 5: Verification of (18). Given a fixed Q', the number of Q for which $r = \operatorname{dist}(3Q, Q') \le 100$ is bounded above by a constant depending only on the dimension $n = 2$. By step 1, then,

$$\sup_{Q'} \sum_{Q : r \le 100} k_{Q',Q} \le C.$$

So, to establish (18) it suffices to consider Q with $r \ge 100$ and $d(3Q) \le 1$. When $r \ge 100$, step 2 shows that

$$k_{Q',Q} \leq \|\mathrm{d}(\cdot)^{\epsilon}\nabla^3 N(a_Q|_\Omega)\|_{L^p(Q'\cap\Omega)} + \|\mathrm{d}(\cdot)^{\epsilon}\nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)}$$
$$\leq C r^{\epsilon-3} + \|\mathrm{d}(\cdot)^{\epsilon}\nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)}. \tag{29}$$

The bound

$$\sup_{Q'} \sum_{Q:\, r\geq 100} r^{\epsilon-3} \leq C < \infty$$

follows by comparing the sum with the integral $\int_{|x|\geq 1}|x|^{\epsilon-3}\,dx$. For the other term in (29),

$$\sup_{Q'} \sum_{Q:\, r\geq 100} \|\mathrm{d}(\cdot)^{\epsilon}\nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)} \leq C \sup_{Q':\, Q'\cap\Omega\neq\emptyset} \sum_{Q:\, r\geq 100} r^{-2}$$

follows from step 3 and step 4. This last quantity is finite, with a bound not depending on Ω. We omit the calculation, but note that it can be modeled by first replacing Ω by the half-plane $\mathbb{R}^2_+ = \{(x_1, x_2) : x_2 > 0\}$ and then passing from sums over cubes to integrals. The result follows by comparison with the quantity

$$\sup_{t\in\mathbb{R}} \int_{s\in\mathbb{R},\, |(s,0)-(0,t)|\geq 1} |(s,0)-(0,t)|^{-2}\,ds,$$

which is finite.

Step 6: Verification of (19). Again, step 1 and step 2 show that it is enough to bound

$$\sup_{Q} \sum_{Q':\, r\geq 100,\, \mathrm{d}(Q')<1} \|\mathrm{d}(\cdot)^{\epsilon}\nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)}$$
$$+ \sup_{Q} \sum_{Q':\, r\geq 100,\, \mathrm{d}(Q')\geq 1} \|\mathrm{d}(\cdot)^{\epsilon}\nabla^3 G a_Q\|_{L^p(Q'\cap\Omega)}. \tag{30}$$

By step 3, the first term is bounded by

$$C \sup_{Q} \sum_{Q':\, r\geq 100,\, \mathrm{d}(Q')<1,\, Q'\cap\Omega\neq\emptyset} r^{-2}.$$

This expression is similar to the one encountered in step 5, with Q and Q' interchanged and the role of the constraint $\mathrm{d}(3Q)\leq 1$ played by $\mathrm{d}(Q')<1$. It has a bound independent of Ω for the same reasons.

By step 4, the second term of (30) is bounded by

$$C \sup_{Q} \sum_{Q':\, r\geq 100,\, \mathrm{d}(Q')\geq 1} r^{-2}\mathrm{d}(Q')^{\epsilon-2}.$$

This quantity is also bounded by a constant independent of Ω. This calculation is also omitted, but we note that if Ω is modeled by $\mathbb{R}^2_+$, a comparison can be made with the integral

$$\int_{y=(y_1,y_2),\, y_2\geq 1} |y|^{-2} y_2^{\epsilon-2}\,dy,$$

which is finite for $\epsilon < 2$.

Having shown (18) and (19), the proof of Lemma 3.1 is complete. □

Next, the constraint that the atoms are at the same scale is lifted. Also, the following estimate will be written in terms of G, not U. Finally, the size of the $L^p_{3-\epsilon}(\Omega)$ norm is estimated, instead of the weighted norm appearing in (16).

LEMMA 3.2. *Suppose* $\{a_Q\}$ *is a collection of* $(1-\epsilon, p)$*-atoms, where the cubes* Q *satisfy the restriction* $\mathrm{d}(3Q) \le \ell(Q)$. *If* $\{s_Q\} \in b^{p,1}$, *then*

$$\left\| G \sum_Q s_Q a_Q \right\|_{L^p_{3-\epsilon}(\Omega)} \le C \|\{s_Q\}\|_{b^{p,1}}. \tag{31}$$

Proof. Because

$$\left\| G \sum_{\nu=0}^{\infty} \sum_{\ell(Q)=2^{-\nu}} s_Q a_Q \right\|_{L^p_{3-\epsilon}(\Omega)} \le \sum_{\nu=0}^{\infty} \left\| G \sum_{\ell(Q)=2^{-\nu}} s_Q a_Q \right\|_{L^p_{3-\epsilon}(\Omega)}$$

and

$$\sum_{\nu=0}^{\infty} \left(\sum_{\ell(Q)=2^{-\nu}} |s_Q|^p \right)^{1/p} = \|\{s_Q\}\|_{b^{p,1}},$$

it is enough to show (31) for Q restricted to a single fixed scale - that is, $\ell(Q) = 2^{-\nu}$ for some fixed $\nu \ge 0$. In that case, the right-hand side of (31) may be replaced by the equivalent expression $\|\{s_Q\}\|_{l^p}$.

Let $f = \sum s_Q a_Q|_\Omega$. Since $\|g\|_{L^p_{3-\epsilon}(\Omega)} \approx \|g\|_{L^p_2(\Omega)} + C\|\nabla^2 g\|_{L^p_{1-\epsilon}(\Omega)}$ (see e.g. [15]),

$$\begin{aligned} \|Gf\|_{L^p_{3-\epsilon}(\Omega)} &\le C\|Gf\|_{L^p_2(\Omega)} + C\|\nabla^2 Gf\|_{L^p_{1-\epsilon}(\Omega)} \\ &\le C\|Gf\|_{L^p_2(\Omega)} + C\|\nabla^2 Nf\|_{L^p_{1-\epsilon}(\Omega)} + C\|\nabla^2 Uf\|_{L^p_{1-\epsilon}(\Omega)}. \end{aligned}$$

Next, if Ω is bounded and Lipschitz and $0 < \epsilon < 1$,

$$\|v\|_{B^{p,p}_{1-\epsilon}(\Omega)} \le C\|\mathrm{d}(\cdot)^\epsilon v\|_{L^p(\Omega)} + C\|\mathrm{d}(\cdot)^\epsilon \nabla v\|_{L^p(\Omega)}.$$

(The essentials of the proof are given in [16, pp. 884-885].) By (15),

$$\|v\|_{L^p_{1-\epsilon}(\Omega)} \le C\|v\|_{B^{p,p}_{1-\epsilon}(\Omega)}.$$

Hence

$$\|\nabla^2 Uf\|_{L^p_{1-\epsilon}(\Omega)} \le C\|\nabla^2 Uf\|_{L^p(\Omega)} + C\|\mathrm{d}(\cdot)^\epsilon \nabla^3 Uf\|_{L^p(\Omega)}.$$

Since

$$\|\nabla^2 Uf\|_{L^p(\Omega)} \le \|Gf\|_{L^p_2(\Omega)} + \|\nabla^2 Nf\|_{L^p(\Omega)}$$

and

$$\|\nabla^2 Nf\|_{L^p(\Omega)} \le C\|\nabla^2 Nf\|_{L^p_{1-\epsilon}(\Omega)},$$

$$\|Gf\|_{L^p_{3-\epsilon}(\Omega)} \le C\|Gf\|_{L^p_2(\Omega)} + C\|\nabla^2 Nf\|_{L^p_{1-\epsilon}(\Omega)} + \|\mathrm{d}(\cdot)^\epsilon \nabla^3 Uf\|_{L^p(\Omega)}. \tag{32}$$

By using Theorem 1.2, the inclusion $B^{p,1}_{1-\epsilon}(\Omega) \hookrightarrow L^p(\Omega)$, the restriction $B^{p,1}_{1-\epsilon}(\mathbb{R}^2) \to B^{p,1}_{1-\epsilon}(\Omega)$, and (13), it follows that $\|Gf\|_{L^p_2(\Omega)}$ is bounded by $C\|\{s_Q\}\|_{l^p}$. By Lemma 3.1, the third term on the right side of (32) is also bounded by $C\|\{s_Q\}\|_{l^p}$. Finally, the middle term of the right side of (32) is appropriately bounded by noting that N is smoothing of degree 2 and that the injection $B^{p,1}_{1-\epsilon}(\Omega) \hookrightarrow L^p_{1-\epsilon}(\Omega)$ is continuous. □

The restriction that the cubes Q be close to $\partial\Omega$ (i.e. $d(3Q) \le \ell(Q)$) is removed in the next lemma. This permits consideration of general $f \in B_{1-\epsilon}^{p,1}(\Omega)$.

LEMMA 3.3. *If $f \in B_{1-\epsilon}^{p,1}(\Omega)$, then $Gf \in L_{3-\epsilon}^{p}(\Omega)$ with*

$$\|Gf\|_{L_{3-\epsilon}^{p}(\Omega)} \le C\|f\|_{B_{1-\epsilon}^{p,1}(\Omega)}.$$

Proof. Let Ef denote the extension of f by 0 to $\mathbb{R}^2$. Since Ω is convex, $f \mapsto Ef$ is bounded from $L_{1-\epsilon}^{p}(\Omega)$ to $L_{1-\epsilon}^{p}(\mathbb{R}^2)$ by [14]; real interpolation shows that the corresponding statement for $B_{1-\epsilon}^{p,1}$ is also true. Since N is smoothing of degree 2, the map $f \mapsto \psi(\cdot)NEf(\cdot)$ is bounded from $B_{1-\epsilon}^{p,1}(\Omega)$ to $B_{3-\epsilon}^{p,1}(\mathbb{R}^2)$, where $\psi \in C_0^{\infty}(\mathbb{R}^2)$ with $\psi = 1$ in a neighborhood of $\bar{\Omega}$. By the atomic decomposition, there exist $(3-\epsilon, p)$-atoms $\tilde{a}_Q$ and a sequence $\{s_Q\} \in b^{p,1}$ with

$$\psi(\cdot)NEf(\cdot) = \sum_{Q} s_Q \tilde{a}_Q$$

and $\|\{s_Q\}\|_{b^{p,1}} \le C\|\psi(\cdot)NEf(\cdot)\|_{B_{3-\epsilon}^{p,1}(\mathbb{R}^2)} \le C\|f\|_{B_{1-\epsilon}^{p,1}(\Omega)}$. Because distributions in $\mathbb{R}^2$ restrict to Ω and since $\nabla\psi$ and $\Delta\psi$ vanish on Ω,

$$f = \sum s_Q(\Delta\tilde{a}_Q)|_{\Omega}.$$

Define

$$J = \{Q: d(3Q) > \ell(Q)\}.$$

Since the atoms are as smooth as desired and $Q \in J$ implies $\operatorname{supp} \tilde{a}_Q \subset \Omega$, $G(\Delta\tilde{a}_Q) = \tilde{a}_Q$. Furthermore, since the $\Delta\tilde{a}_Q$ are $(1-\epsilon, p)$-atoms (up to a bounded multiplicative constant), $\sum_{Q\in J} s_Q \Delta\tilde{a}_Q$ converges in $B_{1-\epsilon}^{p,1}(\mathbb{R}^2)$ and hence the sum converges in $B_{1-\epsilon}^{p,1}(\Omega)$. Then, using a previous and convenient definition of G on (say) $L^p(\Omega)$, which contains $B_{1-\epsilon}^{p,1}(\Omega)$, we have

$$G\sum_{J} s_Q\Delta\tilde{a}_Q = \sum_{J} s_Q G\Delta\tilde{a}_Q = \sum_{J} s_Q\tilde{a}_Q,$$

since G is continuous on $L^p(\Omega)$ for $1 < p < \infty$.

By the continuity of the restriction map $L_{3-\epsilon}^{p}(\mathbb{R}^2) \to L_{3-\epsilon}^{p}(\Omega)$, (15), and (13),

$$\left\|G\sum_{Q\in J} s_Q\Delta\tilde{a}_Q\right\|_{L_{3-\epsilon}^{p}(\Omega)} = \left\|\sum_{J} s_Q\tilde{a}_Q\right\|_{L_{3-\epsilon}^{p}(\Omega)} \le C\left\|\sum_{J} s_Q\tilde{a}_Q\right\|_{L_{3-\epsilon}^{p}(\mathbb{R}^2)}$$
$$\le C\left\|\sum_{J} s_Q\tilde{a}_Q\right\|_{B_{3-\epsilon}^{p,1}(\mathbb{R}^2)} \le C\|\{s_Q\}\|_{b^{p,1}}.$$

To control $G\sum_{Q\notin J} s_Q\Delta\tilde{a}_Q$, note that $\Delta\tilde{a}_Q$ is a $(1-\epsilon, p)$-atom (up to a bounded multiplicative constant) and $\|\{s_Q\}_{Q\notin J}\|_{b^{p,1}} \le \|\{s_Q\}\|_{b^{p,1}} \le C\|f\|_{B_{1-\epsilon}^{p,1}(\Omega)}$. Hence Lemma 3.2 may be applied to the atoms $a_Q = \Delta\tilde{a}_Q$ to yield

$$\left\|G\sum_{Q\notin J} s_Q\Delta\tilde{a}_Q\right\|_{L_{3-\epsilon}^{p}(\Omega)} \le C\|f\|_{B_{1-\epsilon}^{p,1}(\Omega)}.$$

This completes the proof of Lemma 3.3. □

REMARK. If $d(3Q) \le \ell(Q)$ were not needed in Lemma 3.2, then $Ef \in B_{1-\epsilon}^{p,1}(\mathbb{R}^2)$ could be directly decomposed into a sum $\sum s_Q a_Q$ of $(1-\epsilon, p)$-atoms a_Q.

The previous lemma is technically weaker than Theorem 1.1 because of the inclusion $B_{1-\epsilon}^{p,1}(\Omega) \hookrightarrow L_{1-\epsilon}^p(\Omega)$ (see (15)). The space $B_{1-\epsilon}^{p,1}(\Omega)$ was used above for technical reasons: it is simple to start with a decomposition in which the cubes Q are all at the same scale. Given the resulting estimate (16), it is then straightforward to pass to an estimate for an atomic decomposition of the data with coefficients in $b^{p,1}$, which corresponds to the space $B_{1-\epsilon}^{p,1}(\Omega)$.

We now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Given functions u_1 and u_2 on Ω, let the equation $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$ denote the condition $u_1 - u_2 \in W_0^{1,p}(\Omega)$.

Let f be in $L_{1-\epsilon}^p(\Omega)$, and let $v = Nf - Gf$. The inclusion (15) implies that $\|v\|_{L_{3-\epsilon}^p(\Omega)} \le C\|v\|_{B_{3-\epsilon}^{p,p}(\Omega)}$, so

$$\|Gf\|_{L_{3-\epsilon}^p(\Omega)} \le C\|v\|_{B_{3-\epsilon}^{p,p}(\Omega)} + \|Nf\|_{L_{3-\epsilon}^p(\Omega)}. \tag{33}$$

Note that v is determined by the properties $\Delta v = 0$ and $v|_{\partial\Omega} = Nf|_{\partial\Omega}$.

Since $Nf \in L_{3-\epsilon}^p(\Omega)$, there exists a $g \in B_{3-\epsilon}^{p,p}(\Omega)$ with

$$g|_{\partial\Omega} = Nf|_{\partial\Omega}, \qquad \|g\|_{B_{3-\epsilon}^{p,p}(\Omega)} \le C\|Nf\|_{L_{3-\epsilon}^p(\Omega)}. \tag{34}$$

This follows from results in [11] concerning the restrictions of Besov and Bessel potential spaces to $\partial\Omega$ for rather general domains Ω. A proof using the atomic decomposition is given in [6]; it closely follows the case $\Omega = \mathbb{R}_+^n$ given in [5, §3].

We now claim that $\Delta v = 0$ and $v|_{\partial\Omega} = g|_{\partial\Omega}$ imply

$$\|v\|_{B_{3-\epsilon}^{p,p}(\Omega)} \le C\|g\|_{B_{3-\epsilon}^{p,p}(\Omega)}.$$

Assuming the claim, $\|v\|_{B_{3-\epsilon}^{p,p}(\Omega)} \le C\|Nf\|_{L_{3-\epsilon}^p(\Omega)}$. Inequality (33) then shows that

$$\|Gf\|_{L_{3-\epsilon}^p(\Omega)} \le C\|Nf\|_{L_{3-\epsilon}^p(\Omega)}.$$

The theorem follows, since $f \mapsto Nf$ is continuous from $L_{1-\epsilon}^p(\Omega)$ to $L_{3-\epsilon}^p(\Omega)$.

To show the claim, note that by Lemma 3.3 $G\colon B_{1-\epsilon}^{p,1}(\Omega) \to L_{3-\epsilon}^p(\Omega)$ is continuous. Real interpolation shows that $G\colon B_{1-\epsilon}^{p,p}(\Omega) \to B_{3-\epsilon}^{p,p}(\Omega)$ is continuous. Let $h = g - v$; then $h|_{\partial\Omega} = 0$ and $\Delta h = \Delta g \in B_{1-\epsilon}^{p,p}(\Omega)$. Since $h = G\Delta h$, $\|h\|_{B_{3-\epsilon}^{p,p}(\Omega)} \le C\|\Delta h\|_{B_{1-\epsilon}^{p,p}(\Omega)} \le C\|g\|_{B_{3-\epsilon}^{p,p}(\Omega)}$ and

$$\|v\|_{B_{3-\epsilon}^{p,p}(\Omega)} \le \|h\|_{B_{3-\epsilon}^{p,p}(\Omega)} + \|g\|_{B_{3-\epsilon}^{p,p}(\Omega)} \le C\|g\|_{B_{3-\epsilon}^{p,p}(\Omega)}.$$

This completes the proof of Theorem 1.1. □

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